The Groebner basis of a polynomial system related to the Jacobian conjecture

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Abstract

We compute the Groebner basis of a system of polynomial equations related to the Jacobian conjecture using a recursive formula for the Catalan numbers.

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1. Introduction

In this paper K is a characteristic zero field and $K[y]((x^{-1}))$ is the algebra of Laurent series in x^{-1} with coefficients in $K[y]$. In a recent article the following theorem was proved [3, Theorem 1.9].

Theorem 1.1. *The Jacobian conjecture in dimension two is false if and only if there exist*

- $P, Q \in K[x,y]$ *and* $C, F \in K[y]((x^{-1}))$,
- $n, m \in \mathbb{N}$ *such that* $n \nmid m$ *and* $m \nmid n$,
- $\nu_i \in K$ $(i = 0, \dots, m+n-2)$ *with* $\nu_0 = 1$,

such that

- C *has the form*

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \qquad \text{with each } C_{-i} \in K[y],$$

- $gr(C) = 1$ *and* $gr(F) = 2 - n$, *where* gr *is the total degree,*
- $F_+ = x^{1-n}y$, *where* F_+ *is the term of maximal degree in* x *of* F,
- $C^n = P$ *and* $Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F$.

Furthermore, under these conditions (P, Q) *is a counterexample to the Jacobian conjecture.* □

Motivated by this result, the authors consider the following slightly more general situation. Let D be a K-algebra (in Theorem 1.1 we take $D = K[y]$), n, m positive integers such that $n \nmid m$ and $n \nmid m$, $(\nu_i)_{0 \le i \le n+m-2}$ a family of elements in K with $\nu_0 = 1$, and $F_{1-n} \in D$ (in Theorem 1.1 we take $F_{1-n} = y$). A Laurent series in x^{-1} of the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \qquad \text{with } C_{-i} \in D,$$

is **a solution of the system** $S(n, m, (\nu_i), F_{1-n})$ if there are $P, Q \in D[x]$ and $F \in D[[x^{-1}]]$, such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + F_{-1-n}x^{-1-n} + \cdots ,$$
$$P = C^n, \qquad \text{and} \qquad Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F.$$

For example, if $n = 2$, then

$$\begin{aligned} P(\mathbf{x}) = C^2 = \ & \mathbf{x}^2 + 2C_{-1} + 2C_{-2}\ \mathbf{x}^{-1} + (C_{-1}^2 + 2C_{-3})\ \mathbf{x}^{-2} \\ & + (2C_{-1}C_{-2} + 2C_{-4})\ \mathbf{x}^{-3} + (C_{-2}^2 + 2C_{-1}C_{-3} + 2C_{-5})\ \mathbf{x}^{-4} \\ & + (2C_{-2}C_{-3} + 2C_{-1}C_{-4} + 2C_{-6})\ \mathbf{x}^{-5} + \dots , \end{aligned}$$

and the condition $C^2 \in K[x]$ translates into the following conditions on C_{-k}:

$$\begin{aligned} 0 &= (C^2)_{-1} = 2C_{-2}, \\ 0 &= (C^2)_{-2} = C_{-1}^2 + 2C_{-3}, \\ 0 &= (C^2)_{-3} = 2C_{-1}C_{-2} + 2C_{-4}, \\ 0 &= (C^2)_{-4} = C_{-2}^2 + 2C_{-1}C_{-3} + 2C_{-5}, \\ 0 &= (C^2)_{-5} = 2C_{-2}C_{-3} + 2C_{-1}C_{-4} + 2C_{-6}, \\ 0 &= (C^2)_{-6} = C_{-3}^2 + 2C_{-2}C_{-4} + 2C_{-1}C_{-5} + 2C_{-7}, \\ 0 &= (C^2)_{-7} = 2C_{-3}C_{-4} + 2C_{-2}C_{-5} + 2C_{-1}C_{-6} + 2C_{-8}, \\ 0 &= (C^2)_{-8} = C_{-4}^2 + 2C_{-3}C_{-5} + 2C_{-2}C_{-6} + 2C_{-1}C_{-7} + 2C_{-9}, \\ & \ \ \vdots \end{aligned}$$

In general, the condition $P(x) = C^n \in K[x]$ yields $(C^n)_{-k} = 0$, whereas $Q(x) = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F \in K[x]$ handles us equations $\left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F\right)_{-k} = 0$, with $F_{-k} = 0$ for $k = 1, \dots, n-2$.

It is easy to see (e.g. [3, Remark 1.13]) that the first $m + n - 2$ coefficients determine the others, i.e., the coefficients $C_{-1}, \dots, C_{-m-n+2}$ determine univocally the coefficients C_{-k} for $k > m + n - 2$. Moreover, the F_{-k} for $k > n-1$ depend only on F_{1-n} and C. Consequently, having a solution C to the system $S(n, m, (\nu_i), F_{1-n})$ is the same as having a solution $(C_{-1}, \ldots, C_{-m-n+2})$ to the system

$$\begin{aligned} E_k &= (C^n)_{-k} = 0, && \text{for } k = 1, \ldots, m-1, \\ E_{m-1+k} &= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} \right)_{-k} = 0, && \text{for } k = 1, \ldots, n-2, \\ E_{m+n-2} &= \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} \right)_{1-n} + F_{1-n} = 0, && (1.1) \end{aligned}$$

with $m + n - 2$ equations $E_k = 0$ and $m + n - 2$ unknowns C_{-k}.

In order to understand the solution set of this system, it would be very helpful to find a Groebner basis for the ideal generated by the polynomials E_k in $D[C_{-1}, \ldots, C_{m+n-2}]$. In this paper we compute such a Groebner basis of (1.1) in a very particular case: we assume $n = 2$, $m = 2r + 1$ for some integer $r > 0$, and $\nu_i = 0$ for $i > 0$. Moreover, we consider $D = C[y]$ and $F_{1-n} = y$, as in Theorem 1.1.

2. Computation of a Groebner basis for I_{2r}

Assume $n = 2$, $m = 2r + 1$ for some integer $r > 0$, and $\nu_i = 0$ for $i > 0$. Set also $D = \mathbb{C}[y]$ and $F_{1-n} = y$.

Then the system (1.1) reads

$$E_i = \begin{cases} (C^2)_{-i}, & i = 1, \ldots, 2r \\ (C^{2r+1})_{-1} + y, & i = 2r+1, \end{cases} \qquad (2.1)$$

where $(C^2)_{-i}$ denotes the coefficient of x^{-i} in the Laurent series C^2.

Explicitly, the polynomials E_i are given by

$$\begin{aligned}
E_1 &= 2C_{-2}, \\
E_2 &= 2C_{-3} + (C_{-1})^2, \\
E_3 &= 2C_{-4} + 2C_{-2}C_{-1}, \\
E_4 &= 2C_{-5} + 2C_{-3}C_{-1} + (C_{-2})^2, \\
E_5 &= 2C_{-6} + 2C_{-2}C_{-3} + 2C_{-4}C_{-1}, \\
E_6 &= 2C_{-7} + 2C_{-5}C_{-1} + 2C_{-4}C_{-2} + (C_{-3})^2, \\
&\vdots \\
E_{2r-1} &= 2C_{-2r} + 2C_{-2}C_{-2r+3} + 2C_{-4}C_{-2r+5} + \cdots + 2C_{-2r+4}C_{-3} + \\
&\quad 2C_{-2r+2}C_{-1}, \\
E_{2r} &= 2C_{-2r-1} + 2C_{-2r+1}C_{-1} + 2C_{-2r+2}C_{-2} + \cdots + C_{-r}^2, \\
E_{2r+1} &= (C^{2r+1})_{-1} + y.
\end{aligned} \tag{2.2}$$

Each E_i is a polynomial in the ring $\mathbb{C}[C_{-1}, C_{-2}, \ldots, C_{-2r-1}, y]$, and the $2r+1$ polynomials generate the ideal

$$I = \langle E_1, \ldots, E_{2r}, E_{2r+1} \rangle.$$

Our goal is to find a Groebner basis for this I. However, in this section we will only compute a Groebner basis $(\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r-1}, \widetilde{E}_{2r})$ for the ideal $I_{2r} = \langle E_1, E_2, \ldots, E_{2r-1}, E_{2r} \rangle$.

Note that for $i = 1 \ldots, 2r$ we have

$$E_i = 2C_{-i-1} + \sum_{k=1}^{i-1} C_{-k}C_{k-i}. \tag{2.3}$$

We replace the odd numbered polynomials $E_1, E_3, E_5, E_7, \ldots, E_{2r-1}$

by new polynomials $\widetilde{E}_1, \widetilde{E}_3, \widetilde{E}_5, \widetilde{E}_7, \ldots, \widetilde{E}_{2r-1}$ defined by

$$\begin{aligned}
\widetilde{E}_1 &= C_{-2} = \frac{1}{2}E_1, \\
\widetilde{E}_3 &= C_{-4} = \frac{1}{2}E_3 - \widetilde{E}_1 C_{-1}, \\
\widetilde{E}_5 &= C_{-6} = \frac{1}{2}E_5 - \widetilde{E}_1 C_{-3} - \widetilde{E}_3 C_{-1}, \\
\widetilde{E}_7 &= C_{-8} = \frac{1}{2}E_7 - \widetilde{E}_1 C_{-5} - \widetilde{E}_3 C_{-3} - \widetilde{E}_5 C_{-1}, \\
\widetilde{E}_9 &= C_{-10} = \frac{1}{2}E_9 - \widetilde{E}_1 C_{-7} - \widetilde{E}_3 C_{-5} - \widetilde{E}_5 C_{-3} - \widetilde{E}_7 C_{-1}, \\
&\vdots \\
\widetilde{E}_{2r-1} &= C_{-2r} = \frac{1}{2}E_{2r-1} - \sum_{i=1}^{r-1} \widetilde{E}_{2i-1} C_{-2(r-i)+1}.
\end{aligned} \tag{2.4}$$

Remark 2.1. We have

$$\langle E_1, E_3, \ldots, E_{2r-1} \rangle = \langle \widetilde{E}_1, \widetilde{E}_3, \ldots, \widetilde{E}_{2r-1} \rangle.$$

In fact, if we define $\widetilde{I}_k^{odd} = \langle \widetilde{E}_1, \widetilde{E}_3, \ldots, \widetilde{E}_{2k-1} \rangle$, then (2.4) clearly implies

$$E_{2i+1} - 2\widetilde{E}_{2i+1} \in \widetilde{I}_i^{odd}, \tag{2.5}$$

and so we get $\langle E_1, E_3, \ldots, E_{2i+1} \rangle \subset \langle \widetilde{E}_1, \widetilde{E}_3, \ldots, \widetilde{E}_{2i+1} \rangle$ for $i = 0, 1, \ldots, r-1$. Using induction one sees that we also have $\langle \widetilde{E}_1, \widetilde{E}_3, \ldots, \widetilde{E}_{2r-1} \rangle \subset \langle E_1, E_3, \ldots, E_{2r-1} \rangle$, as desired.

The next proposition deals with $E_2, E_4, E_6, \ldots, E_{2r}$, the first r even numbered polynomials.

Proposition 2.2. *For all $j \in \mathbb{N}$ there exists λ_j such that for $\widetilde{E}_{2j} = C_{-2j-1} + \lambda_j C_{-1}^{j+1}$ we have*

$$C_{-2j-1} + \lambda_j C_{-1}^{j+1} - \frac{1}{2}E_{2j} \in \widetilde{I}_{2j-1} = \langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2j-2}, \widetilde{E}_{2j-1} \rangle. \tag{2.6}$$

Moreover, if we set $\lambda_0 = -1$*, then for* $j > 0$*,* λ_j *is given by*

$$\lambda_j = \frac{1}{2}\left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1}\right). \tag{2.7}$$

Proof. We proceed by induction on j. For $j = 0$ we set $\widetilde{E}_0 = 0$. Then we have

$$\widetilde{E}_0 \in \widetilde{I}_{2j-1} \quad \text{for all } j \geq 1, \quad \text{and} \quad \widetilde{E}_0 = C_{-1} + \lambda_0 C_{-1}. \tag{2.8}$$

For $j = 1$, with $\lambda_1 = \dfrac{1}{2}$ calculated by (2.7), we have

$$C_{-3} + \frac{1}{2}C_{-1}^2 - \frac{1}{2}E_2 = 0 \in \langle \widetilde{E}_1 \rangle,$$

as desired.

From (2.3) we have

$$\begin{aligned} E_{2j} &= 2C_{-2j-1} + \sum_{k=1}^{2j-1} C_{-k} C_{k-2j} \\ &= 2C_{-2j-1} + \sum_{k=0}^{j-1} C_{-2k-1} C_{2k+1-2j} + \sum_{k=1}^{j-1} C_{-2k} C_{2k-2j}, \end{aligned}$$

which clearly implies $\sum_{k=1}^{j-1} C_{-2k} C_{2k-2j} \in \widetilde{I}_{2j-1}$. Therefore we get

$$C_{-2j-1} - \frac{1}{2}E_{2j} \in -\frac{1}{2}\left(\sum_{k=0}^{j-1} C_{-2k-1} C_{2k+1-2j}\right) + \widetilde{I}_{2j-1}. \tag{2.9}$$

By the induction hypothesis and (2.8), for $0 \leq k \leq j-1$, there exist λ_k and λ_{j-k-1} such that

$$C_{-2k-1} = -\lambda_k C_{-1}^{k+1} + \widetilde{E}_{2k} \quad \text{and} \quad C_{2k+1-2j} = -\lambda_{j-k-1} C_{-1}^{j-k} + \widetilde{E}_{2(j-k-1)};$$

and hence

$$C_{-2k-1} C_{2k+1-2j} \in \lambda_k \lambda_{j-k-1} C_{-1}^{j+1} + \widetilde{I}_{2j-1}.$$

From (2.9) we obtain

$$C_{-2j-1} - \frac{1}{2}E_{2j} \in -\frac{1}{2}\left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1}\right) C_{-1}^{j+1} + \widetilde{I}_{2j-1},$$

from which Relation (2.6) follows with $\lambda_j = \frac{1}{2}\left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1}\right)$, as claimed. □

Corollary 2.3. *We have*

$$\langle E_1, E_2, \ldots, E_{2r}\rangle = \langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r}\rangle.$$

Proof. In fact, if we define $\widetilde{I}_k = \langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_k\rangle$, then (2.5) and Proposition 2.2 imply

$$E_{k+1} - 2\widetilde{E}_{k+1} \in \widetilde{I}_k,$$

and so we get $\langle E_1, E_2, \ldots, E_{k+1}\rangle \subset \langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{k+1}\rangle$ for all k. Since we have $\langle E_1\rangle = \langle \widetilde{E}_1\rangle$, using induction one also obtains $\langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_k\rangle \subset \langle E_1, E_2, \ldots, E_k\rangle$, as claimed. □

The bottom line of this corollary is that we can replace the system (2.2) with the following set of equations.

$$\begin{array}{rcl} \widetilde{E}_1 &=& C_{-2} = 0, \\ \widetilde{E}_3 &=& C_{-4} = 0, \\ & \vdots & \\ \widetilde{E}_{2r-1} &=& C_{-2r} = 0, \end{array} \tag{2.10}$$

$$\begin{aligned}
\widetilde{E}_2 &= C_{-3} + \lambda_1 C_{-1}^2 = 0,\\
\widetilde{E}_4 &= C_{-5} + \lambda_2 C_{-1}^3 = 0,\\
&\vdots\\
\widetilde{E}_{2r} &= C_{-2r-1} + \lambda_r C_{-1}^{r+1} = 0,\\
\\
\widetilde{E}_{2r+1} &= (C^{2r+1})_{-1} + y = 0.
\end{aligned}$$

Proposition 2.4. *If we fix the lex order with* $C_{-2r-1} > C_{-2r} > \cdots > C_{-3} > C_{-2} > C_{-1} > y$*, then* $G_{2r} = (\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r-1}, \widetilde{E}_{2r})$ *is a Groebner basis of the ideal*

$$\widetilde{I}_{2r} = \langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r-1}, \widetilde{E}_{2r} \rangle$$

Proof. We first compute the S-polynomials of G_{2r}, and prove that they satisfy $\overline{S(\widetilde{E}_i, \widetilde{E}_j)}^{G_{2r}} = 0$ for $1 \le i, j \le 2r$.

Consider first the S-polynomial of an even-numbered polynomial and an odd-numbered polynomial, say $\widetilde{E}_{2s-1}$ and $\widetilde{E}_{2t}$, with $1 \le s, t \le r$. We have then

$$\begin{aligned}
S(\widetilde{E}_{2s-1}, \widetilde{E}_{2t}) &= C_{-2t-1}C_{-2s} - C_{-2s}(C_{-2t-1} + \lambda_t C_{-1}^{t+1})\\
&= -\lambda_t C_{-1}^{t+1} C_{-2s}\\
&= -\lambda_t C_{-1}^{t+1} \widetilde{E}_{2s-1},
\end{aligned}$$

and so $\overline{S(\widetilde{E}_{2s-1}, \widetilde{E}_{2t})}^{G_{2r}} = 0$, for all $1 \le s, t \le r$.

In case both i, j are odd, we take $\widetilde{E}_{2s-1}, \widetilde{E}_{2t-1}$, with $1 \le s, t \le r$. Then we have

$$S(\widetilde{E}_{2s-1}, \widetilde{E}_{2t-1}) = C_{-2t}C_{-2s} - C_{-2s}C_{-2t} = 0,$$

and trivially we get $\overline{S(\widetilde{E}_{2s-1}, \widetilde{E}_{2t-1})}^{G_{2r}} = 0$, for all $1 \le s, t \le r$.

In the last case, when i, j are even, consider $\widetilde{E}_{2s}, \widetilde{E}_{2t}$, with $1 \leq s, t \leq r$. Then we have

$$\begin{aligned} S(\widetilde{E}_{2s}, \widetilde{E}_{2t}) &= C_{-2t-1}(C_{-2s-1} + \lambda_s C_{-1}^{s+1}) - C_{-2s-1}(C_{-2t-1} + \lambda_t C_{-1}^{t+1}) \\ &= \lambda_s C_{-1}^{s+1} C_{-2t-1} - \lambda_t C_{-1}^{t+1} C_{-2s-1}. \end{aligned}$$

Now we divide $S(\widetilde{E}_{2s}, \widetilde{E}_{2t})$ by G_{2r}. If $C_{-2t-1} > C_{-2s-1}$, then the leading term is

$$lt(S(\widetilde{E}_{2s}, \widetilde{E}_{2t})) = \lambda_s C_{-1}^{s+1} C_{-2t-1},$$

and the first division step yields

$$S(\widetilde{E}_{2s}, \widetilde{E}_{2t}) = \lambda_s C_{-1}^{s+1} \widetilde{E}_{2t} + R_1,$$

with $R_1 = -\lambda_s \lambda_t C_{-1}^{s+t+2} - \lambda_t C_{-1}^{t+1} C_{-2s-1}$. By continuing the division algorithm we obtain

$$R_1 = -\lambda_t C_{-1}^{t+1} \widetilde{E}_{2s} + 0,$$

and hence $\overline{S(\widetilde{E}_{2s}, \widetilde{E}_{2t})}^{G_{2r}} = 0$ in this case. The case $C_{-2s-1} > C_{-2t-1}$ is similar, so we get $\overline{S(\widetilde{E}_{2t}, \widetilde{E}_{2s})}^{G_{2r}} = 0$ for $1 \leq s, t \leq r$. □

From Corollary 2.3 and Proposition 2.4 we are able conclude that $(\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r-1}, \widetilde{E}_{2r})$ is a Groebner basis for $\langle E_1, E_2, \ldots, E_{2r-1}, E_{2r} \rangle$.

3. A recursive formula for the Catalan numbers and a Groebner basis for the ideal

In this last section we will determine a Groebner basis for the ideal I given by the complete system (2.1). In order to achieve this we need to establish additional properties of the λ_j's which are closely related to the ubiquitous Catalan numbers.

Lemma 3.1. *For all $j \geq 0$ the equality*

$$c_j = (-1)^{j+1} 2^j \lambda_j \tag{3.1}$$

holds, where c_j are the Catalan numbers given by $c_j = \dfrac{1}{j+1}\dbinom{2j}{j}$.

Proof. The Catalan numbers are uniquely determined (see e.g. [4, p.117 (5.6)]) by $c_0 = 1$ and the recursive relation

$$c_r = \sum_{j=0}^{r-1} c_j c_{r-1-j}.$$

Set $d_j = (-1)^{j+1} 2^j \lambda_j$. Then $d_0 = 1$, since $\lambda_0 = -1$, and so equality (2.7) gives us

$$\begin{aligned} d_j =& (-1)^{j+1} 2^j \lambda_j \\ =& (-1)^{j+1} 2^j \frac{1}{2} \left(\sum_{k=0}^{j-1} \lambda_k \lambda_{j-k-1} \right) \\ =& \sum_{k=0}^{j-1} \left((-1)^{k+1} 2^k \lambda_k \right) \left((-1)^{j-k} 2^{j-1-k} \lambda_{j-k-1} \right) \\ =& \sum_{k=0}^{j-1} d_k d_{j-1-k}, \end{aligned}$$

and hence $d_j = c_j$ for all j, as desired. □

Now we prove a recursive formula for the Catalan numbers.

Proposition 3.2. *The Catalan numbers satisfy the following formula*

$$(2r+1)\frac{c_r}{2^{2r}} = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \frac{c_j}{2^{2j}}. \tag{3.2}$$

Consequently, λ_r satisfies

$$(2r+1)(-1)^{r+1}\lambda_r = \sum_{j=0}^{r} \left(\begin{array}{c} r \\ j \end{array} \right) 2^{r-j}(-\lambda_j). \tag{3.3}$$

Proof. Replacing c_j in (3.2), and using (3.1) yields (3.3). Hence, it suffices to prove only (3.2). For that, we replace c_j by $\frac{1}{j+1}\binom{2j}{j}$ on the righthand side of (3.2) and use the equalities

$$\binom{-1/2}{j} = \frac{(-1)^j}{2^{2j}}\binom{2j}{j} \quad \text{and} \quad \binom{r+1/2}{r} = \frac{(2r+1)}{2^{2r}}\binom{2r}{r}.$$

Then we have

$$\begin{aligned}
\sum_{j=0}^{r}(-1)^j\binom{r}{j}\frac{c_j}{2^{2j}} &= \sum_{j=0}^{r}\frac{(-1)^j}{2^{2j}}\binom{2j}{j}\cdot\frac{1}{(j+1)}\binom{r}{j} \\
&= \sum_{j=0}^{r}\binom{-1/2}{j}\frac{1}{r+1}\binom{r+1}{j+1} \\
&= \frac{1}{(r+1)}\sum_{j=0}^{r}\binom{-1/2}{j}\cdot\binom{r+1}{r-j} \\
&= \frac{1}{(r+1)}\binom{r+1/2}{r} \\
&= \frac{1}{(r+1)}\frac{(2r+1)}{2^{2r}}\binom{2r}{r} \\
&= (2r+1)\frac{c_r}{2^{2r}}.
\end{aligned}$$

The second equality follows from $\dfrac{1}{j+1}\dbinom{r}{j} = \dfrac{1}{(r+1)}\dbinom{r+1}{j+1}$ and the fourth from $\dbinom{\alpha+\beta}{r} = \displaystyle\sum_{j=0}^{r}\dbinom{\alpha}{j}\dbinom{\beta}{r-j}$, relations valid for all $\alpha, \beta \in \mathbb{C}$. The last equality is known as the Chu–Vandermonde identity or Vandermonde convolution [1, p. 44, 13c′]. □

Proposition 3.3. *Let* $I_{2r} = \langle E_1, E_2, \ldots, E_{2r}\rangle$. *Then we have*

$$(C^{2r+1})_{-1} \in \mu_r C_{-1}^{r+1} + I_{2r},$$

for $\mu_r = \dfrac{2r+1}{(r+1)2^r}\dbinom{2r}{r}$.

Proof. By definition we have

$$(C^{2r+1})_{-1} = [(C^2)^r C]_{-1} = \sum_{j=-2}^{2r} [(C^2)^r]_j C_{-j-1},$$

since $C_{-j-1} = 0$ for $j < -2$ and $[(C^2)^r]_j = 0$ for $j > 2r$.

But we also have $[(C^2)^r]_j = \sum_{i_1+\cdots+i_r=j} (C^2)_{i_1} \dots (C^2)_{i_r}$. We claim that if $i_1+\cdots+i_r = j$, then $i_k \geq -2r$ for $k = 1, \dots, r$. In fact, as $i_j \leq 2$, then so we have

$$i_1 + \cdots + i_{k-1} + i_{k+1} + \cdots + i_r \leq 2(r-1),$$

and $j = i_k + (i_1 + \cdots + i_{k-1} + i_{k+1} + \cdots + i_r) \leq 2(r-1) + i_k$ as well. Therefore we get $i_k \geq j - 2r + 2 \geq -2r$, since $j \geq -2$.

By definition we have $E_i = (C^2)_{-i}$ for $i = 1, \dots, 2r$. Consequently we obtain

$$(C^2)_{i_1} \dots (C^2)_{i_r} \in I_{2r}, \quad \text{if some } i_k \text{ is negative.}$$

It follows that

$$[(C^2)^r]_j \in \sum_{\substack{i_1+\cdots+i_r=j \\ i_k \geq 0}} (C^2)_{i_1} \dots (C^2)_{i_r} + I_{2r} = [(x^2 + 2C_{-1})^r]_j + I_{2r}$$

holds, since $C^2 = x^2 + 2C_{-1} + (C^2)_{-1}x^{-1} + (C^2)_{-2}x^{-2} + (C^2)_{-3}x^{-3} + \dots$. But we also have

$$(x^2 + 2C_{-1})^r = \sum_{k=0}^{r} \binom{r}{k} (2C_{-1})^{r-k} x^{2k},$$

and so

$$[(x^2 + 2C_{-1})^r]_j = \begin{cases} \binom{r}{k}(2C_{-1})^{r-k} & \text{if } j = 2k \\ 0, & \text{if } j = 2k+1. \end{cases}$$

We arrive at

$$(C^{2r+1})_{-1} \in \sum_{k=0}^{r} \binom{r}{k} (2C_{-1})^{r-k} C_{-2k-1} + I_{2r}.$$

Note that by Proposition 2.2 we have

$$C_{-2k-1} = \widetilde{E}_{2k} - \lambda_k C_{-1}^{k+1} \in -\lambda_k C_{-1}^{k+1} + I_{2r},$$

so we obtain

$$\begin{aligned}(C^{2r+1})_{-1} &\in \sum_{k=0}^{r} \binom{r}{k} (2C_{-1})^{r-k}(-\lambda_k C_{-1}^{k+1}) + I_{2r} \\ &= \left(\sum_{k=0}^{r} \binom{r}{k} 2^{r-k}(-\lambda_k)\right)(C_{-1})^{r+1} + I_{2r},\end{aligned}$$

and the formula for μ_r follows now from (3.1) and (3.3). □

Corollary 3.4. *For* $\widetilde{E}_{2r+1} = \mu_r(C_{-1})^{r+1} + y$ *we have*

$$\langle E_1, E_2, \ldots, E_{2r-1}, E_{2r}, E_{2r+1}\rangle = \langle \widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r-1}, \widetilde{E}_{2r}, \widetilde{E}_{2r+1}\rangle.$$

Proof. By Proposition 3.3 we have $E_{2r+1} - \widetilde{E}_{2r+1} = (C^{2r+1})_{-1} - \mu_r C_{-1}^{r+1} \in I_{2r}$. The result follows now from Corollary 2.3. □

Now we can state our main result.

Theorem 3.5. *If we fix the lex order with* $C_{-2r-1} > C_{-2r} > \cdots > C_{-3} > C_{-2} > C_{-1} > y$, *then* $G_{2r+1} = (\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r}, \widetilde{E}_{2r+1})$ *is a Groebner basis for the ideal*

$$I = \langle E_1, E_2, \ldots, E_{2r-1}, E_{2r}, E_{2r+1}\rangle.$$

Proof. By Corollary 3.4 it suffices to prove that the division of the S-polynomials $S(\widetilde{E}_i, \widetilde{E}_j)$ by G_{2r+1} is zero. If $i, j \leq 2r$, then the division algorithm yields the same quotients and remainders as in Proposition 2.4, since the remainders become zero before one has to divide by $\widetilde{E}_{2r+1}$. Note that $lt(\widetilde{E}_{2r+1}) = \mu_r(C_{-1})^{r+1}$, since $\mu_r \neq 0$. It remains to divide the S-polynomials $S(\widetilde{E}_i, \widetilde{E}_{2r+1})$ by G_{2r+1}. We first consider the case $i = 2t-1$ for some $t = 1, \ldots, r$. We get

$$\begin{aligned}S(\widetilde{E}_{2t-1}, \widetilde{E}_{2r+1}) &= \frac{C_{-2t}C_{-1}^{r+1}}{C_{-2t}}(C_{-2t}) - \frac{C_{-2t}C_{-1}^{r+1}}{\mu_r C_{-1}^{r+1}}(\mu_r C_{-1}^{r+1} + y) \\ &= -\frac{1}{\mu_r} y C_{-2t},\end{aligned}$$

for all $t = 1, \ldots, r$. The first division step yields $S(\widetilde{E}_{2t-1}, \widetilde{E}_{2r+1}) = -\frac{1}{\mu_r} y\widetilde{E}_{2t-1}$, hence we obtain $\overline{S(\widetilde{E}_{2t-1}, \widetilde{E}_{2r+1})}^{G_{2r+1}} = 0$, for all $t = 1, \ldots, r$.

Now for the S-polynomials of $\widetilde{E}_{2t}$ and $\widetilde{E}_{2r+1}$, for some $t = 1, \ldots, r$, we have

$$\begin{aligned} S(\widetilde{E}_{2t}, \widetilde{E}_{2r+1}) &= \frac{C_{-2t-1}C_{-1}^{r+1}}{C_{-2t-1}}(C_{-2t-1} + \lambda_t C_{-1}^{t+1}) - \\ &\qquad \frac{C_{-2t-1}C_{-1}^{r+1}}{\mu_r C_{-1}^{r+1}}(\mu_r C_{-1}^{r+1} + y) \\ &= \lambda_t C_{-1}^{r+t+2} - \frac{1}{\mu_r} C_{-2t-1} y. \end{aligned}$$

with leading term

$$lt(S(\widetilde{E}_{2t}, \widetilde{E}_{2r+1})) = -\frac{1}{\mu_r} C_{-2t-1} y.$$

We divide $S(\widetilde{E}_{2t}, \widetilde{E}_{2r+1})$ by G_{2r+1}, and the first division step gives us

$$S(\widetilde{E}_{2t}, \widetilde{E}_{2r+1}) = -\frac{1}{\mu_r} y\widetilde{E}_{2t} + R_1$$

with $R_1 = \lambda_t C_{-1}^{r+t+2} + \frac{\lambda_t}{\mu_r} y C_{-1}^{t+1}$. Finally we take note of the equality $R_1 = \frac{\lambda_t}{\mu_r} C_{-1}^{t+1} \widetilde{E}_{2r+1}$, in order to obtain $\overline{S(\widetilde{E}_{2t}, \widetilde{E}_{2r+1})}^{G_{2r+1}} = 0$, for all $t = 1, \ldots, r$. This concludes the proof. □

In brief, we give the Groebner basis $G_{2r+1} = (\widetilde{E}_1, \widetilde{E}_2, \ldots, \widetilde{E}_{2r}, \widetilde{E}_{2r+1})$ of I explicitly as

$$\begin{aligned}
\widetilde{E}_1 &= C_{-2},\\
\widetilde{E}_3 &= C_{-4},\\
&\vdots\\
\widetilde{E}_{2r-1} &= C_{-2r},\\
\\
\widetilde{E}_2 &= C_{-3} + \lambda_1 C_{-1}^2,\\
\widetilde{E}_4 &= C_{-5} + \lambda_2 C_{-1}^3,\\
&\vdots\\
\widetilde{E}_{2r} &= C_{-2r-1} + \lambda_r C_{-1}^{r+1},\\
\\
\widetilde{E}_{2r+1} &= \mu_r (C_{-1})^{r+1} + y.
\end{aligned}$$

with

$$\mu_r = \frac{2r+1}{(r+1)2^r}\binom{2r}{r} \quad \text{and} \quad \lambda_j = \frac{(-1)^{j+1}}{(j+1)2^j}\binom{2j}{j}.$$

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Resumen

En este artículo calculamos la base de Groebner de un sistema polinomial de ecuaciones relacionada con la conjetura del jacobiano utilizando una formula recursiva para los números de Catalan.

Palabras clave: Jacobiano, bases de Groebner, números de Catalan

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