Poincaré duality in equivariant intersection theory

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Abstract

We study the Poincaré duality map from equivariant Chow cohomology to equivariant Chow groups in the case of torus actions on complete, possibly singular, varieties with isolated fixed points. Our main results yield criteria for the Poincaré duality map to become an isomorphism in this setting. The methods rely on the localization theorem for equivariant Chow cohomology and the notion of algebraic rational cell. We apply our results to complete spherical varieties and their generalizations.

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1. Introduction

Let T be a complex algebraic torus of dimension d. Let X be a compact complex algebraic variety where T acts with isolated fixed points. If X has no rational cohomology in odd degrees (i.e., if X is *equivariantly formal* [16]), then the equivariant cohomology ring $H_T^*(X)$ with rational coefficients is a commutative positively graded algebra; it is a free module of finite rank over the equivariant cohomology ring of a point (the latter is a polynomial ring in d variables). Examples include Schubert varieties and rationally smooth projective varieties where a complex reductive group acts with finitely many orbits (a complex variety of pure dimension n is rationally smooth if the local cohomology at any point is the same as the local cohomology of $\mathbb{C}^n$). In [6], Brion has shown that several topological invariants of a T-variety X can be read off $H_T^*(X)$. For instance, if X is equivariantly formal and $n = \dim X$, then equivariant Kronecker duality holds, i.e., the dualizing module of $H_T^*(X)$ is the equivariant homology $H_*^T(X)$. Furthermore, the following conditions are equivalent: (i) Poincaré duality, (ii) the algebra $H_T^*(X)$ is Gorenstein, (iii) the Betti numbers of X satisfy $b_q(X) = b_{2n-q}(X)$, for $0 \leq q \leq n$, and all equivariant multiplicities are nonzero (these are certain functionals on $H_*^T(X)$). See [6, Theorem 4.1]. Finally, Brion obtains some Morse inequalities for the Betti numbers of X, assuming all equivariant multiplicities are nonzero [6, Theorem 4.2].

The purpose of this article is to generalize Brion's results to the purely algebraic and more delicate setting of equivariant Chow groups and equivariant operational Chow groups (or Chow cohomology); we refer to Sections 2 and 3 for appropriate definitions and notation. In order to achieve our goal, first we find a suitable class of varieties that resemble equivariantly formal varieties from the viewpoint of equivariant intersection theory. This is done by combining two classes of varieties considered in previous work, namely T-linear varieties [13] and $\mathbb{Q}$-filtrable varieties [14]. Let us quickly mention some of their relevant features. In [13] we show that projective T-linear varieties satisfy equivariant Kronecker duality. This property is rather strong, and does not hold for arbitrary T-varieties. On the other hand, in [14] we introduced the class of $\mathbb{Q}$-filtrable varieties (Definition 2.12). A remarkable property of these schemes is that their equivariant Chow groups are free modules of finite rank over the equivariant Chow ring of a point. Hence, it is quite natural to consider the class of $\mathbb{Q}$-filtrable T-linear schemes as a suitable replacement for the notion of equivariant formality in equivariant intersection theory (cf. Theorem 3.5, Theorem 3.7). This expectation is confirmed in Section 4, where we obtain criteria for Poincaré duality on projective $\mathbb{Q}$-filtrable T-linear varieties. Our main results (Theorems 4.1, 4.3, and 4.5, and Corollary 4.7) yield the sought-after generalizations of Brion's results, and open the way for further work in this direction.

2. Definitions and basic properties

2.1 Conventions and notation

Throughout this paper, we fix an algebraically closed field $\Bbbk$ of characteristic zero. All schemes and algebraic groups are assumed to be defined over $\Bbbk$. By a scheme we mean a separated scheme of finite type. A variety is a reduced scheme. Observe that varieties need not be irreducible. A subvariety is a closed subscheme which is a variety. A point on a scheme will always be a closed point.

We denote by T an algebraic torus. A scheme X provided with an algebraic action of T is called a **T-scheme**. For a T-scheme X, we denote by X^T the fixed point subscheme and by $i_T : X^T \to X$ the natural inclusion. If H is a closed subgroup of T, we similarly denote by $i_H : X^H \to X$ the inclusion of the fixed point subscheme. When comparing X^T and X^H we write $i_{T,H} : X^T \to X^H$ for the natural (T-equivariant) inclusion. We denote by Δ the character group of T, and by S the symmetric algebra over $\mathbb{Q}$ of the abelian group Δ. The quotient field of S is denoted by $\mathcal{Q}$.

In this paper, torus actions are assumed to be **locally linear**, i.e., the schemes we consider are covered by invariant affine open subsets. This assumption is fulfilled for instance by T-stable subschemes of normal T-schemes [30]. A T-scheme is called **T-quasiprojective** if it has an ample T-linearized invertible sheaf. This assumption is fulfilled, among others, by T-stable subschemes of normal quasiprojective T-schemes [30].

Let G be a connected reductive group. Recall that a normal G-variety X is called **spherical** if a Borel subgroup B of G has a dense orbit in X. Then it is known that G and B have finitely many orbits in X. It follows that X contains only finitely many fixed points of a maximal torus $T \subset B$, see for example [31].

Equivariant Chow groups and equivariant operational Chow groups are considered with rational coefficients.

2.2 The Bialynicki-Birula decomposition

The material in this subsection is due to Bialynicki-Birula [2], [3] (in the smooth case) and Konarski [20] (in the general case).

Let X be a T-scheme with isolated fixed points. Then X^T is finite. We write $X^T = \{x_1, \ldots, x_m\}$. A one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ is called **generic** if $X^{\mathbb{G}_m} = X^T$, where $\mathbb{G}_m$ acts on X via λ. Generic one-parameter subgroups always exist due to the local linearity of the action. Fix a generic $\lambda : \mathbb{G}_m \to T$. For each i, define the subset

$$X_+(x_i, \lambda) = \{x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x = x_i\}.$$

Then $X_+(x_i, \lambda)$ is a locally closed T-invariant subscheme of X. The (disjoint) union of the $X_+(x_i, \lambda)$'s might not cover all of X, but when it does (e.g., when X is complete), the decomposition $\{X_+(x_i, \lambda)\}_{i=1}^m$ is called the Bialynicki-Birula decomposition, or **BB-decomposition**, of X associated to λ. Each $X_+(x_i, \lambda)$ is called a **cell** of the decomposition.

Definition 2.1. Let X be a T-scheme with finitely many fixed points. Let $\{X_+(x_i, \lambda)\}_{i=1}^m$ be the BB-decomposition associated to some generic $\lambda : \mathbb{G}_m \to T$. The decomposition $\{X_+(x_i, \lambda)\}$ is said to be **filtrable** if

there is a finite increasing sequence $\Sigma_0 \subset \Sigma_1 \subset \ldots \subset \Sigma_m$ of T-invariant closed subschemes of X such that:
a) $\Sigma_0 = \emptyset$, $\Sigma_m = X$,
b) $\Sigma_j \setminus \Sigma_{j-1}$ is a cell of the decomposition $\{X_+(x_i, \lambda)\}$, for each $j = 1, \ldots, m$.

In this context, it is common to say that X is **filtrable**, and refer to Σ_j as the **j-th filtered piece** of X. If, moreover, the cells $X_+(x_i, \lambda)$ are isomorphic to affine spaces $\mathbb{A}^{n_i}$, then X is called **T-cellular**.

Theorem 2.2 ([2], [3])**.** *Let X be a complete T-scheme with isolated fixed points, and let λ be a generic one-parameter subgroup. If X admits an ample T-linearized invertible sheaf, then the associated BB-decomposition $\{X_+(x_i, \lambda)\}$ is filtrable. Furthermore, if X is smooth, then X is T-cellular.* $\square$

2.3 Equivariant Chow groups for torus actions

Let X be a T-scheme of dimension n (not necessarily equidimensional). Let V be a finite dimensional T-module, and let $U \subset V$ be an invariant open subset so that a principal bundle quotient $U \to U/T$ exists. Then T acts freely on $X \times U$ and the quotient scheme $X_T = (X \times U)/T$ exists. Following Edidin and Graham [8], we define the i-th **equivariant Chow group** $A_i^T(X)$ by $A_i^T(X) = A_{i+\dim U - \dim T}(X)$ if $V \setminus U$ has codimension more than $n - i$. The definition is independent of the choice of (V, U), see [8] for details. Set $A_*^T(X) = \oplus_i A_i^T(X)$. If X is a T-scheme and $Y \subset X$ is a T-stable closed subscheme, then Y defines a class $[Y]$ in $A_*^T(X)$. If X is smooth, then so is X_T, and $A_*^T(X)$ admits an intersection pairing; in this case, denote by $A_T^*(X)$ the corresponding ring graded by codimension. The equivariant Chow ring $A_T^*(pt)$ is isomorphic to S, and $A_*^T(X)$ is a S-module, where Δ acts on $A_*^T(X)$ by homogeneous maps of degree -1. This module structure is induced by pullback through the flat map $p_{X,T} : X_T \to U/G$. Restriction to a fiber of $p_{X,T}$ gives $i^* : A_*^T(X) \to A_*(X)$. If X is complete, we denote by $\int_X(\alpha) \in S$ the proper pushforward to a point of a class $\alpha \in A_*^T(X)$.

Next we state Brion's description [4] of the equivariant Chow groups in terms of invariant cycles. It also shows how to recover the usual Chow groups from equivariant ones.

Theorem 2.3. *Let X be a T-scheme. Then the S-module $A_*^T(X)$ is defined by generators $[Y]$, where Y is an invariant irreducible subvariety of X, and relations $[\mathrm{div}_Y(f)] - \chi[Y]$, where f is a rational function on Y which is an eigenvector of T of weight χ. Furthermore, the map $A_*^T(X) \to A_*(X)$ vanishes on $\Delta A_*^T(X)$, and it induces an isomorphism $A_*^T(X)/\Delta A_*^T(X) \to A_*(X)$.* □

The following is a slightly more general version of the localization theorem for equivariant Chow groups [4, Corollary 2.3.2]. For a proof, see [13, Proposition 2.15].

Theorem 2.4. *Let X be a T-scheme, let $H \subset T$ be a closed subgroup, and let $i_H : X^H \to X$ be the inclusion of the fixed point subscheme. Then the induced morphism of equivariant Chow groups*

$$i_{H*} : A_*^T(X^H) \to A_*^T(X)$$

becomes an isomorphism after inverting finitely many characters of T that restrict non-trivially to H. □

2.4 T-linear schemes

We introduce our main class of testing spaces.

Definition 2.5. Let T be an algebraic torus and let X be a T-scheme.

1. We say that X is **T-equivariantly 0-linear** if it is either empty or isomorphic to $\mathrm{Spec}\,(\mathrm{Sym}(V^*))$, where V is a finite-dimensional rational representation of T.

2. For a positive integer n, we say that X is **T-equivariantly n-linear** if either one of the following conditions hold.

(i) There is a T-scheme Y, which contains X as a T-invariant open subscheme, so that Y and $Z = Y \setminus X$ are T-equivariantly $(n-1)$-linear.

(ii) There exists a T-invariant closed subscheme $Z \subseteq X$, with complement U, so that Z and U are T-equivariantly $(n-1)$-linear.

3. We say that X is **T-equivariantly linear** (or simply, **T-linear**) if it is T-equivariantly n-linear for some $n \geq 0$. T-linear varieties are varieties that are T-linear schemes.

Clearly, if $T \to T'$ is a morphism of algebraic tori, then every T'-linear scheme is also T-linear. On the other hand, if X is T-equivariantly n-linear, then the fixed point subscheme X^H of any subtorus $H \subset T$ is T-equivariantly n-linear. Observe that T-linear schemes are *linear schemes* in the sense of [17], [32], and [18].

It is known that if X is a T-linear scheme, then $A_*^T(X)$ is a finitely generated S-module and $A_*(X)$ is a finitely generated abelian group (see e.g. [13, Lemma 2.7]). The next theorem provides some concrete examples. For a proof of items (i)-(ii) see [19, Proposition 3.6], for item (iii) see [13, Theorem 2.5].

Theorem 2.6. *Let T be an algebraic torus. Then the following holds.*

(i) A T-cellular scheme is T-linear.

(ii) Every T-scheme with finitely many T-orbits is T-linear. In particular, a toric variety with dense torus T is T-linear.

(iii) Let B be a connected solvable linear algebraic group with maximal torus T. Let X be a B-scheme. If B acts on X with finitely many orbits, then X is T-linear. In particular, spherical varieties are T-linear. □

2.5 Equivariant multiplicities at nondegenerate fixed points

Let X be a T-scheme. A fixed point $x \in X$ is called **nondegenerate** if all weights of T in the tangent space T_xX are non-zero. A fixed point in a nonsingular T-variety is nondegenerate if and only if it is isolated. To study singular schemes, Brion [4] developed a notion of equivariant multiplicity at nondegenerate fixed points. The main features of this concept are outlined below, for details see [4, Section 4].

Theorem 2.7. *Let X be a T-scheme with an action of T, let $x \in X$ be a nondegenerate fixed point and let $\chi_1, \ldots, \chi_n$ be the weights of T_xX (counted with multiplicity).*

(i) There exists a unique S-linear map

$$e_{x,X} : A_*^T(X) \longrightarrow \frac{1}{\chi_1 \cdots \chi_n} S$$

such that $e_{x,X}[x] = 1$ and that $e_{x,X}[Y] = 0$ for any T-invariant irreducible subvariety $Y \subset X$ which does not contain x.

(ii) For any T-invariant irreducible subvariety $Y \subset X$, the rational function $e_{x,X}[Y]$ is homogeneous of degree $-\dim(Y)$ and coincides with $e_{x,Y}[Y]$.

(iii) The point x is nonsingular in X when $e_x[X] = \dfrac{1}{\chi_1 \cdots \chi_n}$. □

For a T-stable irreducible subvariety $Y \subset X$, set $e_{x,X}[Y] = e_x[Y]$, and call $e_x[Y]$ the **equivariant multiplicity of Y at x**.

Proposition 2.8. *Let X be a T-scheme such that all fixed points in X are nondegenerate, and let $\alpha \in A_*^T(X)$. Then, in $A_*^T(X) \otimes_S \mathcal{Q}$, we have $\alpha = \sum_{x \in X^T} e_x(\alpha)[x]$.* □

Next we describe a special class of nondegenerate fixed points. Let X be a T-variety. Call a fixed point $x \in X$ **attractive** if all weights in the tangent space T_xX are contained in an open half-space of $\Delta_{\mathbb{R}} = \Delta \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 2.9. *Let X be a T-variety with a fixed point x. The following conditions are equivalent.*

(i) The point x is attractive.

(ii) There exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ such that, for all y in a neighborhood of x, we have $\lim_{t\to 0} \lambda(t)y = x$.

If (i) or (ii) holds, then x admits a unique open affine T-stable neighborhood in X, denoted X_x, and X_x admits a closed equivariant embedding into T_xX. Moreover, $e_x[X]$ is non-zero. □

2.6 $\mathbb{Q}$-filtrable varieties and equivariant Chow groups

Here we recall some of the main results from [14].

Definition 2.10. Let X be an affine T-variety with an attractive fixed point x, and let $n = \dim X$. We say that (X, x), or simply X, is an **algebraic rational cell** when it satisfies

$$A_k(X) = \begin{cases} \mathbb{Q} & \text{if} \quad k = n \\ 0 & \text{if} \quad k \neq n. \end{cases}$$

In particular, if (X, x) is an algebraic rational cell, then it is irreducible.

Let X be an affine T-variety with an attractive fixed point x. Then there exists a generic one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ for which $X = X_+(x, \lambda)$ and X admits a closed T-equivariant embedding into T_xX (Theorem 2.9). Since all the weights of the $\mathbb{G}_m$-action on T_xX (via λ) are positive, the geometric quotient $\mathbb{P}_\lambda(X) = (X \setminus \{x\})/\mathbb{G}_m$ exists and is a projective variety. In fact, it is a closed subvariety of the weighted projective space $\mathbb{P}_\lambda(T_xX)$. Remarkably, X is an algebraic rational cell if and only if

$$A_k(\mathbb{P}_\lambda(X)) = \begin{cases} \mathbb{Q} & \text{if} \quad 0 \leq k \leq n-1, \\ 0 & \text{otherwise} \end{cases}$$

holds. See [14] for details.

Example 2.11. Let $\Bbbk = \mathbb{C}$. Algebraic rational cells are naturally found on rationally smooth spherical varieties. Indeed, let X be a G-spherical variety with an attractive fixed point $x \in X$. Let X_x be the unique open affine T-stable neighborhood of x. If X is rationally smooth at x, then (X_x, x) is an algebraic rational cell [14, Theorem 7.2].

Using algebraic rational cells as building blocks, one can study the global geometry of T-varieties equipped with a paving by algebraic rational cells.

Definition 2.12. Let X be a T-variety. We say that X is **$\mathbb{Q}$-filtrable** if the following hold:

1. the fixed point set X^T is finite, and
2. there exists a generic one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ for which the associated BB-decomposition of X is filtrable (Definition 2.1) and consists of T-invariant **algebraic rational cells**.

In particular, we have $X = \bigsqcup_j X_+(x_j, \lambda)$. Also, observe that the fixed points $x_j \in X^T$ need not be attractive in X, but they are so in their corresponding algebraic rational cells $X_+(x_j, \lambda)$. The key result is stated below.

Theorem 2.13 ([14, Theorem 4.4])**.** *Let X be a $\mathbb{Q}$-filtrable T-variety. Then the T-equivariant Chow group of X is a free S-module of rank $|X^T|$. In fact, it is freely generated by the classes of the closures of the cells $X_+(x_i, \lambda)$. Consequently, $A_*(X)$ is also freely generated by the classes of the cell closures $\overline{X_+(x_i, \lambda)}$.* □

Next we compute equivariant multiplicities of algebraic rational cells and $\mathbb{Q}$-filtrable varieties. Recall that a primitive character χ of T is called **singular** if it satisfies $X^{\ker(\chi)} \neq X^T$.

Theorem 2.14 ([14, Corollary 3.8])**.** *Let X be an irreducible T-variety with attractive fixed point x. Let X_x be the unique open affine T-stable neighborhood of x. If (X_x, x) is an algebraic rational cell, then the following hold.*

(i) $e_x[X]$ *is the inverse of a polynomial. In fact, we have*

$$e_x[X] = \frac{d}{\chi_1 \cdots \chi_r},$$

here the χ_i*'s are singular characters,* $r = \dim X$*, and* d *is a positive rational number.*

(ii) *Moreover, if the number of closed irreducible* T*-stable curves through* x *is finite, say* $\ell(x)$*, then* $\dim X = \ell(x)$*. Furthermore, we may take for* $\chi_1, \ldots, \chi_r$ *the characters associated to these curves.* □

In general, if X is an affine T-variety with an attractive fixed point x, and $\ell(x)$ as above is finite, then $\dim X \leq \ell(x)$ [5, Corollary 1.4.2].

Example 2.15. Let $\Bbbk = \mathbb{C}$. Let X be a G-spherical variety. Recall that X^T is finite. If there is a generic $\lambda : \mathbb{G}_m \to T$ so that $\{X_+(x_i, \lambda)\}$ is a filtrable BB-decomposition and each cell is rationally smooth, then X is $\mathbb{Q}$-filtrable and both the equivariant and non-equivariant cycle maps are isomorphisms [14, Theorem 7.3]. In particular, this holds for rationally smooth projective group embeddings [14, Corollary 5.10]; see below for a definition of this important class of spherical varieties.

2.7 Applications to group embeddings

We recall some results and notation on group embeddings that will be used freely throughout the paper. Here G denotes a connected reductive group with Borel subgroup B and maximal torus $T \subset B$.

An irreducible algebraic variety is called an **embedding** of G, or a **group embedding**, if it is a normal $G \times G$-variety containing an open orbit isomorphic to G itself, where $G \times G$ acts on G by left and right multiplication. When G is a torus, we recover the notion of toric varieties. Group embeddings are spherical $G \times G$-varieties (by the Bruhat decomposition). Affine embeddings of G are nothing but reductive monoids having G as group of units [29]. Recall that an **algebraic monoid** is an algebraic variety equipped with an associative product map, which is a morphism of varieties and admits an identity element. An affine algebraic monoid is called **reductive** if it is irreducible, normal, and its unit group is a reductive algebraic group.

Let M be a reductive monoid with zero and unit group G. Then there exists a central one-parameter subgroup $\epsilon : \mathbb{G}_m \to T$ that satisfies $\lim_{t\to 0} \epsilon(t) = 0$, see [7, Lemma 1.1.1]. Moreover, the quotient space $\mathbb{P}_\epsilon(M) = (M \setminus \{0\})/\epsilon(\mathbb{G}_m)$ is a projective embedding of the quotient group $G/\epsilon(\mathbb{G}_m)$. In fact, projective embeddings of connected reductive groups are exactly the projectivizations of reductive monoids [24].

Let M be a reductive monoid with zero and unit group G. It is worth noting that 0 is the unique attractive $T\times T$-fixed point of M (see e.g., [7, Lemma 1.1.1]). Let $\overline{T} \subset M$ be the Zariski closure of T in M. Then $\overline{T}$ is a normal affine toric variety [25, Theorem 5.4]. We denote by $E(M)$ the idempotent set of M, that is, $E(M) = \{e \in M \,|\, e^2 = e\}$. Likewise, $E(\overline{T})$ denotes the idempotent set of $\overline{T}$. One defines a partial order on $E(M)$ (and thus on $E(\overline{T})$) by declaring $f \leq e$ if and only if $fe = f = ef$. Write W for the Weyl group of (G, T), and denote by $\mathfrak{S}$ its set of simple reflections. Then W acts on $E(\overline{T})$ by conjugation, and the corresponding set of W-conjugacy classes can be identified with $\Lambda = \{e \in E(\overline{T}) \,|\, Be = eBe\}$. See [25] for details. The lattice Λ is called the **cross-section lattice** of M. Notably, Λ can also be identified with the (finite) set $G \times G$-orbits of M [25, Theorem 4.5]. For $e \in E(M)$, set $M_e = \overline{\{g \in G \,|\, ge = eg = e\}}$. Then M_e is a reductive monoid with e as its zero element [7]. Finally, let $\Lambda_k = \{x \in \Lambda \,|\, \dim Tx = k\}$ be the set of elements of rank k in Λ.

Definition 2.16. A reductive monoid M with zero element is called **quasismooth** if, for any minimal non-zero idempotent $e \in E(M)$, the submonoid M_e is an algebraic rational cell.

This definition agrees with that of [26]. For details, we refer to [14, Definition 5.8]. Over the complex numbers, M is quasismooth if and only if $\mathbb{P}_\epsilon(M)$ is rationally smooth [28, Theorem 2.5]. For a combinatorial classification of quasismooth monoids, see [28]. The main result in this context is stated next. See [14, Theorem 5.9] for a proof.

Theorem 2.17. *If M is a quasismooth monoid, then the projective group embedding $\mathbb{P}_\epsilon(M)$ is $\mathbb{Q}$-filtrable.* □

3. Equivariant Chow cohomology: localization and equivariant Kronecker duality

Let X be a T-scheme. The i-th **T-equivariant operational Chow group** of X, denoted $\mathrm{op}A^i_T(X)$, is defined as follows: an element $c \in \mathrm{op}A^i_T(X)$ is a collection of homomorphisms $c_f^{(m)} : A^T_m(Y) \to A^T_{m-i}(Y)$, written $z \mapsto f^*c \cap z$, for every T-map $f : Y \to X$ and all integers m. (The underlying category is the category of T-schemes.) These homomorphisms must satisfy certain compatibility conditions, see [9, Chapter 17] and [8] for details. For any X, the ring structure on $\mathrm{op}A^*_T(X) = \oplus_i \mathrm{op}A^i_T(X)$ is given by composition of such homomorphisms. The ring $\mathrm{op}A^*_T(X)$ is graded, and $\mathrm{op}A^i_T(X)$ can be non-zero for any $i \geq 0$. The basic properties we need are summarized below.

(i) The cup product $\mathrm{op}A^p_T(X) \otimes \mathrm{op}A^q_T(X) \to \mathrm{op}A^{p+q}_T(X)$, $a \otimes b \mapsto a \cup b$, is well defined and makes $\mathrm{op}A^*_T(X)$ into a graded associative ring. Note that this ring is commutative since $\mathrm{char}(\Bbbk) = 0$, and so all T-schemes admit equivariant resolution of singularities.

(ii) There are contravariant graded maps $f^* : \mathrm{op}A^i_T(X) \to \mathrm{op}A^i_T(Y)$, for arbitrary equivariant morphisms $f : Y \to X$.

(iii) The cap product $\mathrm{op}A^i_T(X) \otimes A^T_m(X) \to A^T_{m-i}(X)$, $c \otimes z \mapsto c \cap z$, is well defined and makes $A^T_*(X)$ into an $\mathrm{op}A^*_T(X)$-module satisfying the projection formula.

(iv) For any T-scheme X of pure dimension n, there is an **equivariant Poincaré duality map:**

$$\mathcal{P}_T : \mathrm{op}A^k_T(X) \to A^T_{n-k}(X), \quad z \mapsto z \cap [X].$$

If X is nonsingular, then $\mathcal{P}_T$ is an isomorphism, and the ring structure on $\mathrm{op}A_T^*(X)$ is that determined by intersection products of cycles on the mixed spaces X_T. In particular, by (iii) and (iv), $\mathrm{op}A_T^*(X)$ is a graded S-algebra. We say that X **satisfies equivariant Poincaré duality** if $\mathcal{P}_T$ is an isomorphism. Similar remarks apply to the non-equivariant Poincaré duality map (denoted $\mathcal{P}$).

Now we state the localization theorem for equivariant Chow cohomology. It is applicable to possibly singular complete T-schemes, regardless of whether $\mathrm{op}A_T^*(X)$ is a free S-module or not.

Theorem 3.1 ([13, Theorem A.6]). *Let X be a complete T-scheme and let $i_T : X^T \to X$ be the inclusion of the fixed point subscheme. Then the pull-back map*

$$i_T^* : \mathrm{op}A_T^*(X) \to \mathrm{op}A_T^*(X^T)$$

is injective, and its image is exactly the intersection of the images of

$$i_{T,H}^* : \mathrm{op}A_T^*(X^H) \to \mathrm{op}A_T^*(X^T),$$

where H runs over all subtori of codimension one in T. □

Theorem 3.1 makes equivariant Chow cohomology more computable. For instance, a version of GKM theory also holds [13, Theorem A.9], and there is a description of the equivariant operational Chow groups of spherical varieties [13, Section 4], which generalizes [4, Theorem 7.3].

Let X be a T-scheme, and let (V, U) be as in Subsection 2.2. By [8, Corollary 2], there is an isomorphism $\mathrm{op}A_T^j(X) \simeq \mathrm{op}A^j(X \times U/T)$, provided $V \setminus U$ has codimension more than j. Thus there is a canonical map $i^* : \mathrm{op}A_T^*(X) \to \mathrm{op}A^*(X)$ induced by restriction to a fiber of $p_{X,T} : X_T \to U/T$. But, unlike the case of equivariant Chow groups, this map is not surjective in general, and its kernel is not necessarily generated in degree one, not even for toric varieties [22]. This becomes an issue when trying to translate results from equivariant to non-equivariant Chow cohomology. Nevertheless, for certain $\mathbb{Q}$-filtrable varieties the map i^* is surjective. Before studying them, let us recall a definition from [13].

Definition 3.2. Let X be a complete T-scheme. We say that X **satisfies T-equivariant Kronecker duality** if the following conditions hold.

(i) The S-module $A_*^T(X)$ is finitely generated.

(ii) The equivariant Kronecker duality map

$$\mathcal{K}_T : \mathrm{op}A_T^*(X) \longrightarrow \mathrm{Hom}_S(A_*^T(X), S), \quad \alpha \mapsto (\beta \mapsto \int_X (\beta \cap \alpha)).$$

is an isomorphism of S-modules.

If, in addition, the ordinary Kronecker duality map $\mathcal{K}$ is also an isomorphism, then we say that X satisfies the **strong** T-equivariant Kronecker duality.

Example 3.3. By [13, Lemma 3.3], a nonsingular projective T-variety satisfies the T-equivariant Kronecker duality if and only if it satisfies ordinary Kronecker duality. In particular, a projective smooth curve of positive genus (with any T-action) does not satisfy T-equivariant Kronecker duality, for the kernel of $\mathcal{K}$ in degree one is the Jacobian of the curve [10].

The main result on equivariant Kronecker duality needed here is a consequence of [10], [32], and [13, Theorem 3.6].

Theorem 3.4. *Let X be a complete T-linear scheme. If X has an ample T-linearized invertible sheaf (e.g., if X is a nonsingular projective T-variety with isolated fixed points or X is a possibly singular projective spherical variety), then X satisfies the strong T-equivariant Kronecker duality.* □

The next result makes $\mathbb{Q}$-filtrations relevant to the study of the equivariant Chow cohomology of T-schemes. The proof is an easy adaptation of [13, Corollary 3.9].

Theorem 3.5. *Let X be a complete T-scheme. If X satisfies the strong T-equivariant Kronecker duality and $A_*^T(X)$ is S-free, then the S-module* $\mathrm{op}A_T^*(X)$ *is free, and the map*

$$\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \to \mathrm{op}A^*(X)$$

is an isomorphism. □

Corollary 3.6. *Let X be a complete T-linear variety having an ample T-linearized invertible sheaf. If X is $\mathbb{Q}$-filtrable, then the S-module* $\mathrm{op}A_T^*(X)$ *is free, and we have* $\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \simeq \mathrm{op}A^*(X)$. □

To conclude this section, we present two results which motivate our quest for conditions guaranteeing (equivariant) Poincaré duality.

Theorem 3.7. *Let G be a complex connected reductive group with maximal torus T. Let X be a projective complex G-spherical variety. If X is equivariantly formal, then there is a natural isomorphism* $\mathrm{op}A_T^*(X) \simeq H_T^*(X)$ *of S-algebras, and we get* $\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \simeq H^*(X)$. *In particular, the S-module* $\mathrm{op}A_T^*(X)$ *is free. If, moreover, X is $\mathbb{Q}$-filtrable, then we get* $\mathrm{op}A^*(X) \simeq H^*(X)$.

Proof. By Theorem 3.1, the pullback $i_T^* : \mathrm{op}A_T^*(X) \to \mathrm{op}A_T^*(X^T)$ is injective, and its image $\mathrm{im}(i_T^*)$ is described explicitly in [13, Theorem 4.8]. On the other hand, because X^T is finite, we get a canonical identification $\mathrm{op}A_T^*(X^T) \simeq H_T^*(X^T)$. Since X is equivariantly formal, the pullback $\widetilde{i_T}^* : H_T^*(X) \to H_T^*(X^T)$ is also injective. Moreover, X^H is equivariantly formal for any codimension-one subtorus $H \subset T$. Hence, using [13, Subsection 4.2] and the localization theorem for equivariant cohomology [16], one easily checks the equality $\mathrm{im}(i_T^*) = \mathrm{im}(\widetilde{i_T}^*)$. Consequently, we obtain $\mathrm{op}A_T^*(X) \simeq H_T^*(X)$. This, together with equivariant formality, yields $\mathrm{op}A_T^*(X)/\Delta\mathrm{op}A_T^*(X) \simeq H_T^*(X)/\Delta H_T^*(X) \simeq H^*(X)$. Finally, the last assertion follows from Theorem 3.5. □

Proposition 3.8. *Let M be a reductive monoid with zero. If M is quasismooth (Definition 2.16), then* $\mathrm{op}A_{T\times T}^*(\mathbb{P}_\epsilon(M))$ *is a free S-module,* *and it is isomorphic, as an S-algebra, to the ring of piecewise polynomial functions $PP_{T\times T}(\mathbb{P}_\epsilon(M))$ associated to the GKM graph of $\mathbb{P}_\epsilon(M)$. Furthermore, if $\Bbbk = \mathbb{C}$, then we have* $\mathrm{op}A^*_{T\times T}(\mathbb{P}_\epsilon(M)) \simeq H^*_{T\times T}(\mathbb{P}_\epsilon(M))$ *and thus also* $\mathrm{op}A^*(X) \simeq H^*(X)$.

Proof. The first part follows from Theorem 2.17, Corollary 3.6 and [13, Theorem A.9]. For the second one, use Theorem 3.7. □

For a description of the GKM graph of $\mathbb{P}_\epsilon(M)$ see [12].

4. Equivariant Poincaré duality and Chow homology Betti numbers

The goal of this section is to show that $\mathbb{Q}$-filtrable T-linear varieties are analogues of the equivariantly formal spaces of Goresky, Kottwitz, and MacPherson [16] from the viewpoint of equivariant operational Chow groups.

Let X be a projective T-variety of pure dimension n. Suppose that X is $\mathbb{Q}$-filtrable. Then, by Theorem 2.13, $A^T_*(X)$ is a free S-module of finite rank and $A_*(X)$ is a free $\mathbb{Q}$-vector space of finite dimension. Now set $b_k = \dim_{\mathbb{Q}} A_k(X)$, and call it the k-th **Chow homology Betti number** of X. It follows from Theorem 2.13 that b_k equals the number of k-dimensional algebraic rational cells. When X is smooth, these cells are actually affine spaces, and we get $b_k = b_{n-k}$ [2, Corollary 1]. Moreover, Poincaré duality holds, and all equivariant multiplicities are non-zero (Theorem 2.7). In the singular case this is not necessarily true, and our motivation for this section is to determine in which cases the identity $b_k = b_{n-k}$ holds. Is this equivalent to Poincaré duality for the Chow cohomology of X? Could it be studied via equivariant multiplicities? Notice that these multiplicities played a fundamental role in Section 2. In equivariant cohomology and for equivariantly formal varieties these questions have been answered in [6]. Below we provide some analogues of the results of [6] in equivariant Chow cohomology. Our methods rely on Theorem 3.1, Theorem 3.5, and the notion of algebraic rational cells. No comparison via the cycle map is needed.

A first approximation to Poincaré duality via equivariant multiplicities is given next. For the corresponding statement in equivariant cohomology, see [6, Theorem 4.1].

Theorem 4.1. *Let X be a complete equidimensional T-scheme with only finitely many fixed points. If all equivariant multiplicities are non-zero, then the equivariant Poincaré duality map is injective.*

Proof. In view of Theorem 3.1, the argument is the same as that of [6, Theorem 4.1]. We include it for convenience. Let $\alpha \in \mathrm{op}A_T^*(X)$ and suppose that $\alpha \cap [X] = 0$. Then we have

$$\int_X (\alpha \cup \beta) \cap [X] = 0$$

for all $\beta \in A_T^*(X)$. Thus, in $\mathcal{Q}$, we obtain

$$\sum_{x \in X^T} \alpha_x \beta_x e_T(x, X) = 0.$$

By the localization theorem, the identity holds for all sequences $(\beta_x)_{x\in X^T}$ in $\mathcal{Q}$. Since, by assumption, no $e_x[X]$ vanishes, we must have $\alpha_x = 0$ for all $x \in X^T$. Thus we get $\alpha = 0$ (for the map $i_T^* : \mathrm{op}A_T^*(X) \to \mathrm{op}A_T^*(X^T)$ is injective). $\square$

Remark 4.2. Theorem 4.1 applies to: (i) projective nonsingular T-varieties with isolated fixed points, for then the equivariant multiplicities are all inverses of polynomials (Theorem 2.7); (ii) Schubert varieties and toric varieties, as they have only attractive fixed points, so Theorem 2.9 implies that the corresponding equivariant multiplicities are non-zero; (iii) simple projective embeddings of a connected reductive group G, as they have only one closed $G \times G$-orbit, and $W \times W$ acts transitively on the $T \times T$-fixed points (at least one of these is attractive, hence so are all of them).

We now combine our previous results to produce a criterion for Poincaré duality. For equivariantly formal varieties and equivariant cohomology this was done in [6, Theorem 4.1].

Theorem 4.3. *Let X be a complete equidimensional T-variety with isolated fixed points. Suppose*

(a) X is $\mathbb{Q}$-filtrable and

(b) X satisfies the strong T-equivariant Kronecker duality.

Then the following conditions are equivalent.

(i) X satisfies Poincaré duality.

(ii) X satisfies T-equivariant Poincaré duality.

(iii) The Chow homology Betti numbers of X satisfy $b_q(X) = b_{n-q}(X)$, for $0 \leq q \leq n$, and all equivariant multiplicities are nonzero.

If any of these conditions holds, then all equivariant multiplicities are inverses of polynomial functions.

Proof. Assumptions (a) and (b) imply that the S-modules $A_*^T(X)$ and $\mathrm{op}A_T^*(X)$ are free. So the equivalence of (i) and (ii) follows readily from Theorem 3.5 and the graded Nakayama lemma.

We prove that (ii) implies (iii). It only remains to show that all the equivariant multiplicities are nonzero. For this, let $\{[\overline{W_1}], \ldots, [\overline{W_m}]\}$ be the basis of $A_*^T(X)$ consisting of the closures of the algebraic rational cells. Fix $j \in \{1, \ldots, m\}$, and let x_j be the unique attractive fixed point of W_j. By (ii) there is a unique $\alpha \in \mathrm{op}A_T^*(X)$ for which we have

$$\alpha \cap [X] - [\overline{W_j}] = 0.$$

But then, arguing as in the proof of Theorem 4.1, the identity

$$\sum_{x_i \in X^T} \beta_{x_i}(\alpha_{x_i} e_{x_i}[X] - e_{x_i}[\overline{W_j}]) = 0$$

holds for all sequences $(\beta_{x_i})_{x_i \in X^T}$ in $\mathcal{Q}$. In particular, we have

$$\alpha_{x_j} e_{x_j}[X] - e_{x_j}[\overline{W_j}] = 0.$$

Since x_j is an attractive fixed point of $\overline{W_j}$, we get $e_{x_j}[\overline{W_j}] \neq 0$. This yields $\alpha_{x_j} \neq 0$ and $e_{x_j}[X] \neq 0$, so that $e_{x_j}[X]$ is the inverse of a polynomial. Indeed, we have $e_{x_j}[\overline{W_j}] = \frac{d}{\prod_{s=1}^{\ell} \chi_s}$ (Theorem 2.14) and $\alpha_{x_j} \in S$.

Now is the turn for (iii) implies (i). In view of Theorem 4.1, it remains to show that

$$\mathcal{P}_T : \mathrm{op}A^q_T(X) \to A^T_{n-q}(X)$$

is surjective for all $q \in \mathbb{Z}$. For this, it suffices to show that the dimension of $\mathrm{op}A^q_T(X)$ matches that of $A^T_{n-q}(X)$. But this follows from the assumption on the Chow homology Betti numbers combined with the isomorphisms

$$\mathrm{op}A^*_T(X) \simeq \mathrm{op}A^*(X) \otimes_{\mathbb{Q}} S \quad \text{and} \quad A^T_*(X) \simeq A_*(X) \otimes_{\mathbb{Q}} S,$$

where the first one is granted by Theorem 3.5. □

Remark 4.4. It is worth noting that Kronecker duality does not imply Poincaré duality. For instance, consider the following example from [10, page 184]. Let X be the closure of a generic torus orbit in the Grassmannian $G(2,4)$. Then X is a toric variety with Chow homology groups $\mathbb{Q}$, $\mathbb{Q}$, $\mathbb{Q}^5$, and $\mathbb{Q}$ in dimensions 0, 1, 2, and 3. By Kronecker duality, the Chow cohomology groups are $\mathbb{Q}$, $\mathbb{Q}$, $\mathbb{Q}^5$, and $\mathbb{Q}$ in codimensions 0, 1, 2, and 3. Clearly, the Poincaré duality maps $A^k \to A_{3-k}$ are not isomorphisms.

It stems from Theorem 4.3 that the class of $\mathbb{Q}$-filtrable varieties satisfying the strong T-equivariant Kronecker duality indeed resembles that of equivariantly formal spaces [16]. To push the analogy even further, here is a version of the Morse inequalities for these varieties. For the analogous result in equivariant cohomology, see [6, Theorem 4.2].

Theorem 4.5. *Let X be a T-quasiprojective T-linear variety of pure dimension n with isolated fixed points. If X is complete, $\mathbb{Q}$-filtrable, and all equivariant multiplicities are nonzero, then the following inequalities hold for the Chow homology Betti numbers:*

$$b_q + b_{q-1} + \ldots + b_0 \leq b_{n-q} + b_{n-q+1} + \ldots + b_n,$$

for $0 \leq q \leq n$, and

$$2b_1 + 4b_2 + \ldots + 2nb_n \geq n\chi(X),$$

where $\chi(X) = b_0 + b_1 + \ldots + b_n$ is the Euler characteristic, i.e., the number of algebraic rational cells of X. In fact, we get $\chi(X) = |X^T|$. Furthermore, X satisfies Poincaré duality if and only if

$$2b_1 + 4b_2 + \ldots + 2nb_n = n\chi(X).$$

Proof. The proof is an easy adaptation of [6, Theorem 4.2], with a few changes. First note that, as X is T-linear, it is also $\mathbb{G}_m$-linear, where $\mathbb{G}_m$ acts on X via the generic one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ chosen to obtain the $\mathbb{Q}$-filtration. Secondly, since X has an ample T-linearized invertible sheaf, and this sheaf is clearly $\mathbb{G}_m$-linearized, then Theorem 3.4 implies that X satisfies the strong $\mathbb{G}_m$-equivariant Kronecker duality. Thus, by Theorem 3.5, we have $\mathrm{op}A^*_{\mathbb{G}_m}(X) \simeq \mathrm{op}A^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$ and $A_*^{\mathbb{G}_m}(X) \simeq A_*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[t]$, as graded vector spaces, where t is an indeterminate of degree 1. On the other hand, since the $e_x[X]$ are nonzero, the same holds for $e'_x[X]$, the $\mathbb{G}_m$-equivariant multiplicity at x, by [4, Lemma 4.5]. It follows that the map

$$\mathcal{P}_{\mathbb{G}_m} : opA^q_{\mathbb{G}_m}(X) \to A^{\mathbb{G}_m}_{n-q}(X), \quad z \mapsto z \cap [X],$$

is injective for all $q \in \mathbb{Z}$. In view of these results, Brion's argument from [6, Proof of Theorem 4.2] applies verbatim, yielding the result. □

Example 4.6. Let M be a quasismooth monoid, and consider the associated projective embedding $X = \mathbb{P}_\epsilon(M)$. Suppose that X has a unique closed $G\times G$-orbit (i.e., X is **simple**). By the calculations of [26] we get $b_k = b_{n-k}$. Since all the $T\times T$-fixed points in X are attractive, then by Theorem 4.3, X satisfies Poincaré duality for Chow cohomology. Over the complex numbers this reflects the fact that X is rationally smooth. Once more, we point out that the cycle map was not needed in our arguments.

Corollary 4.7. *Let X be a projective G-spherical variety. Let* $\mathrm{Pic}(X)$ *(respectively* $\mathrm{Cl}(X)$*) denote the Picard group (respectively Class group) of X with rational coefficients. If X is $\mathbb{Q}$-filtrable and satisfies Poincaré duality, then* $\mathrm{Pic}(X)\simeq \mathrm{Cl}(X)$.

Proof. Because X is normal, the natural map $\mathrm{Pic}(X)\to \mathrm{Cl}(X)$ is injective. But both, source and target, are finite dimensional vector spaces, so in order to obtain the result it suffices to show that they have the same dimension. By [10], we have $\mathrm{Pic}(X)\simeq \mathrm{op}A^1(X)\simeq \mathrm{Hom}(A_1(X),\mathbb{Q})$. The dimension of the latter vector space is b_1, which equals b_{n-1}, by Poincaré duality. Since $A_{n-1}(X)\simeq \mathrm{Cl}(X)$, the proof is complete. □

Remark 4.8. If X is a complex variety with rational singularities, then we have $Pic(X)\simeq \mathrm{op}A^1(X)$, by [21, Prop. 12.1.4]. So, the conditions of Corollary 4.7 could be slightly relaxed in that case.

Corollary 4.7 admits a combinatorial interpretation in the case of simple group embeddings. Let $X=\mathbb{P}_\epsilon(M)$ be a projective group embedding. Recall that $\Lambda\setminus\{0\}$ indexes the $G\times G$-orbits of X (see Subsection 2.7 for notation). If X is simple, then the unique closed orbit of X is a projective homogeneous variety $G/P_J\times G/P_J^-$, where $J\subset\mathfrak{S}$, P_J is a standard parabolic subgroup, and P_J^- is its opposite (see e.g. [24]). Remarkably, Λ is completely determined by J and the Dynkin diagram of G [25, Section 7.3]. For instance, we have $\Lambda_2\simeq\mathfrak{S}\setminus J$. On the other hand, notice that the number of $G\times G$-stable divisors of X is $|\Lambda_{d-1}|$, where $d=\dim T$.

Next we give a qualitative relation between Λ_{d-1} and J.

Corollary 4.9. *Let M be a quasismooth monoid. If $X = \mathbb{P}_\epsilon(M)$ is simple, then $|\Lambda_{d-1}| = |\mathfrak{S} \setminus J|$.*

Proof. By Theorem 2.17 and Corollary 4.7 we have $\mathrm{Pic}(X) \simeq \mathrm{Cl}(X)$. Since $\mathrm{Cl}(X)$ is freely generated by the $G \times G$-stable divisors of X (since $Cl(G)_{\mathbb{Q}} = 0$), we get $\dim_{\mathbb{Q}} \mathrm{Cl}(X) = |\Lambda_{d-1}|$. Finally, by a result of Brion (see e.g. [23]) the Picard group of X is freely generated by those $B \times B$-stable irreducible divisors which do not contain $G/P_J \times G/P_J^-$. But these correspond to $\mathfrak{S} \setminus J$, by [25, Theorem 5.1]. □

For a complete list of all J's that yield quasismooth monoids M, see [27]. Corollary 4.9 states that Poincaré duality is reflected on the poset structure of the $G \times G$-orbits.

Final remarks

Let X be a complete equidimensional T-variety with isolated fixed points. If all equivariant multiplicities are nonzero (e.g., all fixed points are attractive), then due to Theorem 4.1 the equivariant Poincaré duality map is injective. Thus we get $\mathrm{op}A_T^*(X) \subseteq A_*^T(X)$. An interesting open problem is to describe $\mathrm{op}A_T^*(X)$ as a subgroup of $A_*^T(X)$ in terms of T-invariant cycles. Notice that $\mathrm{op}A_T^*(X)$ carries an additional ring structure. A related task is to assess the effect of this "abstract" product on the associated (geometric) cycles. Solutions to these problems will yield a geometric interpretation of operational Chow groups, at least in the cases of Example 4.2 and those where Poincaré duality holds (Theorem 4.3). Applications to equivariant operational K-theory ([1], [15]) are also envisioned. Notice that all the analysis can be carried out intrinsically using the tools developed in Section 4 and the rich structure of equivariant Chow groups (there is no need for comparing with equivariant cohomology). This will be pursued elsewhere.

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Resumen

En este artículo estudiamos el homomorfismo de dualidad de Poincaré, el cual relaciona cohomología de Chow equivariante y grupos de Chow equivariante en aquellos casos donde un toro algebraico actúa sobre una variedad singular compacta y con puntos fijos aislados. Nuestros resultados proporcionan criterios bajo los cuales el homomorfismo de dualidad de Poincaré es un isomorfismo. Para ello, usamos el teorema de localización en cohomología de Chow equivariante y la noción de célula algebraica racional. Aplicamos nuestros resultados a las variedades esféricas compactas y sus generalizaciones.

Palabras clave: Grupos de Chow, acciones tóricas, descomposiciones celulares, dualidad de Poincaré, variedades esféricas.

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