Duality on 5-dimensional S^1-Seifert bundles

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Abstract

We describe a correspondence between two different links associated to the same K3 orbifold. This duality is produced when two elements, one inside and the other on the boundary of the Kähler cone, are identified. We call this correspondence ∂-*duality.* We also discuss the consequences of ∂-duality at the level of metrics.

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1. Introduction

Orbifolds are geometric objects that can be realized as quotients of Riemannian manifolds by isometries. However, in doing so some finite proper subgroup of the isometries may fix some points. Sometimes they come about as space forms, others in more subtle situations, for instance, as compactifications of moduli spaces. Usually one would like to construct fiber bundles over orbifolds in such a manner that the total space desingularizes geometric data arising from the orbifold. Obviously, additional conditions on both the singularities and the geometric structure on the orbifold should bring additional benefits. In recent years, an specific type of fibration, namely *Riemannian orbifold submersions*, has been intensively studied to establish the existence of Einstein or η-Einstein metrics on manifolds that do not admit special holonomy groups. More precisely, in [4, 6, 7, 8], attention is given to S^1-principal orbibundles over orbifold Kähler surfaces. These bundles can be viewed as a type of real resolution of the singularities of the corresponding orbifold. Under certain conditions, this sort of resolution ends up being the link of the associated Milnor fiber. On the other hand, there is an algebraic geometric procedure to resolve the singularities, namely, minimal models associated to an algebraic variety. The affinity between the resolution and the Milnor fiber, through the link, is well understood in complex dimension 1, where the complex algebraic curve is studied via its 2-dimensional affine cone. This interplay has produced one of the most important relations in singularity theory for surfaces. The extension of this picture to complex dimension 2 brought some remarkable results from Orlik and Wagreich [26]. However, there is no precise description of the relationship of these two types of resolutions at this level.

In this article, we describe a correspondence between two different links associated to the same K3 orbifold. This duality is produced when two elements, one in the interior and the other on the boundary of the Kähler cone, are identified. We call this correspondence *∂-duality.* We also discuss the consequences of ∂-duality at the level of metrics. It will be interesting to determine whether this duality can be interpreted in terms of the Milnor fiber and the minimal model of a K3 surface. This possible connection is encrypted in a variation of the Dynkin diagram of the resolution and its associated equivariant plumbing manifold. This questions –and others– are part of joint work in progress with R. Gonzales.

Here is a concise outline of the paper. In Section 2 some preliminaries on Riemannian orbifolds are presented, avoiding singularities of codimension 1 (due to a significant result of Kollar [18]). We also explain the Hopf map in the context of orbifold Riemannian submersions (a snapshot that will be recurrent on this article). In Section 3, we give a brief introduction to projective Kähler orbifolds, we also exhibit some canonical examples and review some important recent results of Boyer and Galicki relating Kähler geometry and Sasakian geometry. In Section 4, we introduce K3 surfaces and some of their properties, and discuss S^1-orbibundles over K3 surfaces. We conclude the paper in Section 5 explaining ∂-duality and discussing some consequences for the Riemannian metrics associated to the dual pairs.

2. Orbifolds and orbifold Riemannian submersions.

The notion of an orbifold was introduced by Satake in [29] under the name V-manifold. The symbol 'V' in that context indicated the cone-like singularity he was dealing with. In the late seventies Thurston rediscovered the concept of V-manifolds under the name orbifold in his study of the geometry of 3-manifolds [34].

In the sequel $\mathbb{K}$ denotes either $\mathbb{C}$ or $\mathbb{R}$. Let $O(m) = O(m, \mathbb{K})$ be the orthogonal group, and let B_r be the open ball of radius r in $\mathbb{K}^m$ centered at the origin. If G is a subgroup of $O(m)$, then G acts by isometries on B_r; let B_r/G be the associated quotient space.

A compact metric space M is said to be an **orbifold** if every point $p \in M$ has an open neighborhood U_p which is homeomorphic to $B_{r(p)}/G_p$ for some $r(p) > 0$ and some finite subgroup $G_p \subset O(m)$; the groups G_p are called **local uniformizing groups.** Set $\tilde{U}_p = B_{r(p)}$ and let

$$\rho_p : \tilde{U}_p \to \tilde{U}_p/G_p = U_p$$

be the natural projection. A point q of $\tilde{U}_p$ whose isotropy subgroup $\Gamma_q \subset G_p$ is non-trivial is called a **singular** point of $\tilde{U}_p$. The set of all singular points of $\tilde{U}_p$ is denoted by Σ_p. Then the map

$$\rho_p : \tilde{U}_p \backslash \Sigma_p \to U_p \backslash \rho_p(\Sigma_p)$$

is a covering projection. The **singular set** or **orbifold singular locus** of M is defined to be

$$\Sigma(M) = \bigcup_{p \in M} \{\rho_p(\Sigma_p)\}.$$

Note that Σ_p is the union of a finite number of linear subspaces of $\tilde{U}_p$. In this paper the on going assumption is that these subspaces all have codimension at least 2.

We will say that M is a **smooth (or complex) orbifold** if $M \backslash \Sigma(M)$ has the structure of a smooth (or complex) manifold and the maps ρ_p from $\tilde{U}_p \backslash \Sigma_p$ to $U_p \backslash \rho_p(\Sigma_p)$ are local diffeomorphisms (or local biholomorphisms). The orbifold is a **smooth (or complex) manifold** if $\Sigma(M)$ is empty or, equivalently, if $G_p = \{e\}$ for every $p \in M$.

A **Riemannian metric** g on an orbifold M is a Riemannian metric g_p on every $\tilde{U}_p$ that is invariant under the action of G_p and such that each ρ_p is a local isometry from $\tilde{U}_p \backslash \Sigma_p$ to $U_p \backslash \rho_p(\Sigma_p)$. Similarly, one defines a **Hermitian metric** h for complex orbifolds, but this time one requires the maps ρ_p to be local Hermitian isometries.

Remark 2.1. In general one talks about tensors on orbifolds, defining them on the complement of the singular set of the orbifold X (assumed to be of codimension at least 2): a **tensor** θ on an orbifold X is a tensor θ_{ns} on $X\backslash\Sigma(X)$ such that for every chart $\rho_p : \tilde{U}_p \to \tilde{U}_p/G_p = U_p$ the pullback ${\rho_p}^*(\theta_{ns})$ extends to a tensor on $\tilde{U}_p$. Therefore the notions of curvature, Kähler metrics, and Kähler-Einstein metrics on orbifolds make sense (see [7] for details).

The following proposition is a slight variation of the partition of unity argument (see [24] for details).

Proposition 2.2. *Every orbifold admits a Riemannian metric, and every complex orbifold admits a Hermitian metric.* □

Remark 2.3. The definition of smooth (or complex) orbifold given here avoids several technical subtleties that will not affect the subsequent arguments; the skeptical reader is encouraged to seek out other sources (*e.g.* [25]).

Recall that a **submersion** in the smooth setting is a (smooth) map $\pi : M \to B$ of closed Riemannian manifolds (M, g) and (B, g_B) with maximal rank. It follows that for $z \in M$ the tangent space T_zM splits as $V_z \bigoplus H_z$, where

$$V_z = \ker(\pi_{*z}) \quad \text{and} \quad H_z = V_z^{\perp}$$

are the **vertical** and **horizontal** spaces, respectively. If, additionally, π_* is an isometry from H_z to $T_{\pi(z)}B$, one says that π is a **Riemannian submersion**.

Now we briefly explain how to extend this notion to the singular setting. First, consider the extension of charts given previously and consider the (usual) action of G_p on $\tilde{U} \times F$, where F is a closed smooth G_p-manifold, given by

$$\gamma.(\tilde{u}, x) = (\gamma\tilde{u}, \gamma(\tilde{u})x) \text{ for } \gamma \in G_p. \tag{2.1}$$

This action is referred to as $\tilde{U} \times_{G_p} F$. We use this notation in the next definition.

Definition 2.4. Let X and Y orbifolds and let F be a smooth manifold. One says that $\pi : Y \to X$ is a **fiber orbibundle** with fiber F if one can choose charts $\tilde{U}_x/G_x$ in X and charts $\tilde{U}_x \times_{H_x} F$ over Y such that the following diagram

$$\begin{array}{ccc} \tilde{U}_x \times F & \xrightarrow{\rho_x^Y} & \tilde{U}_x \times_{H_x} F \\ \Big\downarrow \tilde{\pi} & & \Big\downarrow \pi \\ \tilde{U}_x & \xrightarrow{\rho_x} & U_x = \tilde{U}_x/G_x \end{array} \tag{2.2}$$

commutes. Here H_x is a subgroup of G_x, $\tilde{\pi}$ and π are projections onto the first factor, and the projections ρ_x^Y and ρ_x are the associated quotient maps. We say that π is a **Riemannian orbifold submersion** if additionally $\tilde{\pi}$ is a Riemannian submersion.

In general the Riemannian metric on $\tilde{U}_x \times F$ is not a product metric, and the decomposition under discussion works only locally.

Remark 2.5. It must be understood that the orbifold fibration is not a fibration in the usual sense. However one can think of this object as a fibration rationally, that is, such that certain tensor power of the fiber is indeed a conventional fiber.

If the fiber F is a vector space of dimension r and all the uniformizing groups act on F as linear transformations, then the orbibundle is called a **vector orbibundle** of **rank** r. If the rank of the vector orbibundle equals 1, we talk about **line orbibundles**. Similarly, if F is a Lie group G, then the orbibundle is called a **principal orbibundle.** Of particular interest is the case when all the uniformizing groups G_x are subgroups of the Lie group G that act freely on the fiber, in which case the total space ends up being a smooth manifold.

Now, let us briefly recall the Hopf fibration in order to generate some examples of orbifold Riemannian submersions. Consider the sphere S^{2n+1} in $\mathbb{C}^{n+1}$ centered at the origin, and let z be the unit outward normal. Let J be the natural almost complex structure. Then Jz defines an integral distribution on S^{2n+1} with totally geodesic leaves. Identifying the leaves as points, one obtains the complex projective space $\mathbb{CP}^n$. The horizontal distribution can be taken to be the orthogonal complement to Jz in the tangent bundle TS^{2n+1}, and one can turn this into a Riemannian submersion, known as the **Hopf fibration** $h : S^{2n+1} \to \mathbb{CP}^n$ with fibers given by great circles.

We explain this fibration with some detail for $n = 1$. The unit sphere S^k is given the standard metric g_k of constant sectional curvature $+1$. We regard $S^3 \subset \mathbb{C}^2$ and $S^2 \subset \mathbb{C} \oplus \mathbb{R}$. The Hopf fibration $H : S^3 \to S^2$ is defined via the rule

$$H(z_1, z_2) = (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

It is not difficult to show that if two points (z_1, z_2) and (z_3, z_4) in the sphere project to the same point, then there is an $r \in S^1$ such that $(z_1, z_2) = r(z_3, z_4)$. Then the fibers are **Hopf circles**, that is, are orbits of points of S^3 under the S^1 action $(z, w) \mapsto (e^{i\theta}z, e^{i\theta}w)$. The O'Neill formulae [20] show that S^2 is a half-radius sphere of constant sectional curvature equal to 4. Hence $H : (S^3, g_3) \to (S^2, \frac{1}{4}g_2)$ is a Riemannian submersion.

Example 2.6. Let $\mathbb{Z}/n\mathbb{Z}$ be the group of n roots of unity. Consider the actions

$$\begin{array}{rcl} \rho_2 : \mathbb{Z}/n\mathbb{Z} \times S^2 & \longrightarrow & S^2 \\ (\gamma, w, t) & \longmapsto & (\gamma w, t) \end{array}$$

and

$$\begin{array}{rcl} \rho_3 : \mathbb{Z}/n\mathbb{Z} \times S^3 & \longrightarrow & S^3 \\ (\gamma, z_1, z_2) & \longmapsto & (\gamma z_1, z_2). \end{array}$$

These two actions (performing as isometries) turn $S^2/\rho_2(\mathbb{Z}/n\mathbb{Z})$ and $S^3/\rho_3(\mathbb{Z}/n\mathbb{Z})$ into Riemannian orbifolds. Moreover, as the group actions are compatible with the Hopf fibration H, one obtains the following commutative diagram:

$$\begin{array}{ccc} S^3 & \xrightarrow{\pi_3} & S^3/\rho_3(\mathbb{Z}/n\mathbb{Z}) \\ \Big\downarrow{\scriptstyle H} & & \Big\downarrow{\scriptstyle \tilde{H}} \\ S^2 & \xrightarrow{\pi_2} & S^2/\rho_2(\mathbb{Z}/n\mathbb{Z}). \end{array} \tag{2.3}$$

Here the induced Hopf map $\tilde{H}$ is the Riemannian orbifold submersion.

Example 2.7. In a similar fashion, one can define different actions with different orbifold structures. For instance, consider $p, q \in \mathbb{Z}$ such that $\gcd(p, q) = 1$, and let $n = pq$. Let a, b be integers subject to $ap - bq = 1$. Here we do not modify the action ρ_2 of $\mathbb{Z}/n\mathbb{Z}$ on S^2 given in the previous example. However, this time the action ρ_3 of $\mathbb{Z}/n\mathbb{Z}$ on S^3 is taken to be

$$(\gamma, z_1, z_2) \mapsto (\gamma^{ap} z_1, \gamma^{bq} z_2).$$

It is clear that these actions are compatible with the Hopf map, and we obtain a commutative diagram similar to Diagram (2.3). Please do not overlook the following interesting fact: the isotropy groups are different over different components of the singular set. For instance, if one considers the north pole $(0, 1)$ and the south pole $(0, -1)$ of S^2, it follows that the action of $\mathbb{Z}/n\mathbb{Z}$ on $H^{-1}((0, 1)) = (z_1, 0)$ and $H^{-1}((0, -1)) = (0, z_2)$ is not faithful. The isotropy groups are $\mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$. This construction is going to be revisited from another point of view in Section 3.

To conclude this section, we state a theorem due to Molino [23] for foliations with compact leaves. The setup of this theorem is the prototype of Riemannian orbifold submersion that will be used in the sequel.

Theorem 2.8. *Let M be a manifold with a p-dimensional Riemannian foliation $\mathcal{F}$ with compact leaves. Then the space of leaves $M/\mathcal{F}$ admits the structure of an orbifold of codimension p. Moreover, the canonical projection $\pi : M \to M/\mathcal{F}$ is an orbifold Riemannian submersion.* □

3. Projective Kähler orbifolds

It is well known that a complex manifold (M, J) always admits a Hermitian metric h that can be written as

$$h = g - i\omega,$$

where g is a Riemannian metric and ω is a 2-form called the **Kähler form** which is of type $(1,1)$ for the almost complex structure J (as follows from the invariance of h under J). If, in addition, we have $d\omega = 0$, then we say that the manifold is **Kähler** and that g is a **Kähler metric**. Sometimes, by abuse of language, one even says that ω is a Kähler metric. There are several characterizations of Kähler metrics: see [35] for a thorough treatment on the matter. From Remark 2.1 (see also [7]), these notions carry over easily to the level of orbifolds.

Formally, a **positive line orbibundle over a compact orbifold** X is a holomorphic orbibundle that carries a Hermitian metric whose associated curvature form Ω with respect to the Hermitian connection is positive, or in other words, $\frac{i}{2\pi}\Omega$ is a closed Kähler form. One can reinterpret this stiffness of style by saying that certain power $L^{\otimes\nu}$ of this line orbibundle is a positive line bundle on X in the usual sense (ν is just the least common multiple of the orders of the isotropy groups). Of course, this is equivalent to saying that $L^{\otimes\nu}$ admits enough holomorphic sections to provide an embedding of X into some complex projective space. This colloquialism is adopted –or tolerated– due to the following version of Kodaira's embedding theorem as proven by Baily [1].

Theorem 3.1. *Let X be a compact complex orbifold that admits a positive orbibundle. Then X is a projective algebraic variety.* $\square$

Remark 3.2. At the level of cohomology, line orbibundles on X can be considered as rational elements and, as such, line orbibundles lie in $H^2(X, \mathbb{Q})$.

An interesting family of examples of projective Kähler orbifolds is given by weighted projective spaces and weighted complete intersections.

Let us consider an affine variety $V \simeq \mathbb{C}^n$. As a vector space, the grading of $V = \bigoplus_k V^k$ is equivalent to saying that V is endowed with a $\mathbb{C}^*$-action acting on the eigen-spaces (or weight-spaces) V^k with weight k. This is equivalent to a $\mathbb{Z}$-grading of the coordinate ring $\mathbb{C}[V]$ (in this case a $\mathbb{Z}$-grading of the ring of polynomials $\mathbb{C}[x_1, \dots x_n]$).

The weights are taken to be strictly positive. Afterwards, one considers the quotient

$$\mathbb{P}(V) = (V \backslash \{0\})/\mathbb{C}^*.$$

This space, usually denoted $\mathbb{CP}(w_0, \dots, w_n)$, is called the **weighted projective space**. Here we have $n = \dim_{\mathbb{C}} V$, and the w_k are the **weights**. It is always assumed $\gcd(w_0, \dots, w_n) = 1$.

Let x_j, for $j = 1, \dots, n$, be coordinates on V such that x_j has weight w_j. Then $\mathbb{CP}(w_0, \dots, w_n)$ is covered by charts

$$\rho : \{x_j = 1\} \simeq \mathbb{C}^{n-1} \longrightarrow \mathbb{CP}(w_0, \dots, w_n).$$

The w_j-th roots of unity in $\mathbb{C}^*$ act trivially on the x_j coordinate and hence preserve the slice $\mathbb{C}^{n-1}$ displayed above. The map ρ is the quotient by $\mathbb{Z}/w_j\mathbb{Z} \subset \mathbb{C}^*$, explicitly given by $(x_l) \mapsto (\exp(2\pi i w_l/w_j)x_l)$.

The orbifold points of $\mathbb{CP}(w_0, \dots, w_n)$ are determined on each stratum. For instance, each vertex $P_i = [0, \dots, 1, \dots, 0]$ is of type

$$\frac{1}{w_i}(w_1, \dots, \widehat{w}_i, \dots, w_n).$$

The general points along the line P_iP_j are orbifold points of type

$$\frac{1}{\gcd(w_i, w_j)}(w_1, \dots, \widehat{w}_i, \dots, \widehat{w}_j, \dots, w_n),$$

with similar orbifold types for higher dimensional strata. We will always assume that $d_i = \gcd(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_n)$ equals 1 for all $i = 0, \dots, n$. This assumption will exclude the case where the singularities have codimension 1.

As in the smooth case, the **tautological line orbibundle** $\mathcal{O}_{\mathbb{P}(V)}(-1)$ over the weighted projective space is the orbibundle over $\mathbb{CP}(w_0,\ldots,w_n)$ whose fiber over $[v]$ is the union of the orbit $\mathbb{C}^*.v$ and $0 \in V$. The structure of vector space is given as follows: any two elements on the fiber $\mathcal{O}_{[v]}(-1)$ can be written $u_i = t_i.v$ for $t_i \in \mathbb{C}$, $i = 1, 2$, so the linear structure is defined via $au_1+bu_2 = (at_1+bt_2).v$. However, this linear structure is not necessarily the one arising from the vector space structure of V, and hence in general $\mathcal{O}_{[v]}(-1) \subset V$ is not a linear subspace.

By definition, weighted projective spaces are projective and hence admit (orbifold) Kähler metrics. One can argue in a similar way as it is done for the smooth case: since the dual $\mathcal{O}_{\mathbb{P}(V)}(1)$ of the tautological line orbibundle ends up being ample, the Kähler metric, associated to it, is the curvature associated to the hermitian metric on $\mathcal{O}_{\mathbb{P}(V)}(1)$. The interested reader can find many details on this type of metric in [28] for instance (where the authors even allowed the orbifolds to have singularities of codimension 1).

A polynomial $f \in \mathbb{C}[z_0,\ldots z_n]$ is a **weighted homogeneous polynomial of degree** d and **weight** $\mathbf{w} = (w_0,\ldots w_n) \in \mathbb{Z}^{n+1}$ if for any $\lambda \in \mathbb{C}^*$ we have

$$f(\lambda\mathbf{z}) = f(\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n) = \lambda^d f(z_0,z_1,\ldots,z_n) = \lambda^d f(\mathbf{z}).$$

The space of weighted homogeneous polynomials of degree d is a λ^d-eigenspace of the induced $\mathbb{C}^*(\mathbf{w})$-action on $\mathbb{C}[z_0,\ldots z_n]$. A **weighted hypersurface** X_f is the zero locus in $\mathbb{CP}(\mathbf{w})$ of a single weighted homogeneous polynomial f. A weighted variety is called a **weighted complete intersection** if the number of polynomials in the collection equals the codimension of X. We denote by $X_d \subset \mathbb{CP}(\mathbf{w})$ the weighted hypersurface of degree d and by $X_{d_1,\ldots,d_c} \subset \mathbb{CP}(\mathbf{w})$ the weighted complete intersection of multidegree d_i (here c denotes the codimension of the variety). We say that the weighted variety is **quasi-smooth** if its affine cone is smooth outside the origin $\mathbf{0}$. Under the quasi-smoothness hypothesis, it is not difficult to see that the orbifold structure on $\mathbb{CP}(\mathbf{w})$

induces a locally cyclic orbifold structure on $X_{d_1,\ldots,d_c}$. Clearly, the resulting orbifolds are of Kähler type.

For example, let x, y, z, u and w be the homogeneous coordinates on $\mathbb{CP}(1,1,1,2,2)$ of weights $1,1,1,2$ and 2 respectively. Let $f = x^3 + y^3 + z^3 + ux + wy$ and $g = x^4 + y^4 + z^4 + u^2 + w^2$ be polynomials of homogeneous degree 3 and 4 respectively. Then the intersection locus of f and g defines an orbifold $X_{3,4} \subset \mathbb{CP}(1,1,1,2,2)$ with only two 2 cyclic singularities, both modeled on $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$.

Now we discuss one of many incarnations of the Hopf map: locally free actions on odd spheres. First, let us consider the lowest possible dimension, say $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$. This time we reinterpret the Hopf map in terms of the vector field ξ on $\mathbb{C}^2$ given by

$$\xi = i\left(w_0 z_0 \frac{\partial}{\partial z_0} + w_1 z_1 \frac{\partial}{\partial z_1}\right),$$

where w_0, w_1 are non-zero real numbers such that quotient w_1/w_0 is rational. Restricted to S^3 this vector field is everywhere tangent to S^3 and defines a nowhere vanishing vector field on S^3. Hence ξ generates a 1-dimensional foliation $\mathcal{F}_\xi$ on the sphere, and this time the associated action is given by the rule

$$(z_0, z_1) \mapsto (e^{2\pi i w_0 t} z_0, e^{2\pi i w_1 t} z_1).$$

We will assume that w_0, w_1 are coprime positive integers (were it not the case, we reparametrize and use complex conjugation if necessary to achieve this assumption). Hence, the leaves of this foliation are all circles. According to Theorem 2.8, the space of leaves $S^3/\mathcal{F}_\xi$ is an orbifold: the weighted projective space $\mathbb{CP}(w_0, w_1)$. We also obtain an orbifold Riemannian submersion

$$\pi : (S^3, \bar{g}_{\mathbf{w}}) \to (\mathbb{CP}(w_0, w_1), g_{\mathbf{w}}).$$

Here the metric $\bar{g}_{\mathbf{w}}$ on the sphere is of **Sasaki** type (see [9]), thus, the metric determines a contact distribution $\mathcal{D} \subset TS^3$ that admits an integrable CR-structure (this structure is inherited from the standard complex structure on $\mathbb{C}^2$). Furthermore, $\bar{g}_{\mathbf{w}}$ defines a Kähler orbifold metric $g_{\mathbf{w}}$ on the weighed projective space.

Of course, one can generalize this procedure to higher dimensions in exactly the same way, and one obtains an orbifold Riemannian submersion from the sphere to the weighted projective space

$$\pi : (S^{2n+1}, \bar{g}_{\mathbf{w}}) \to (\mathbb{CP}(\mathbf{w}), g_{\mathbf{w}}), \tag{3.1}$$

for $\mathbf{w} = (w_0, \dots w_n)$ positive integers satisfying the expected condition: $\gcd(w_0, \dots w_n) = 1$. In recent years, fibrations of this type have been used to establish, due to the intimate relationship between Kähler structures on weighted projective spaces and Sasakian structures on the corresponding total space, existence of Einstein metrics on exotic spheres (see [7] and the references therein). Let us see how to extend this correspondence to weighted complete intersections.

The notion of links of hypersurface singularities was introduced by Milnor in [22]. In [12], Hamm generalized this idea to complete intersections: they are defined as a p-tuple of linearly independent weighted homogeneous polynomials $\mathbf{f} = (f_1, \dots, f_p) \in (\mathbb{C}[z_0, \dots, z_n])^p$ of degrees $d_1, \dots, d_p$ respectively, and weight vector $\mathbf{w}$. Consider the **weighted affine cone** $C_{\mathbf{f}} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} | \, \mathbf{f}(z_0, \dots, z_n) = 0\}$, which has dimension $n + 1 - p$. Let us assume that the origin in $\mathbb{C}^{n+1}$ is the only singularity, and it is isolated. Then we define the link

$$L_{\mathbf{f}} = C_{\mathbf{f}} \cap S^{2n+1},$$

which is smooth, of real dimension $2(n-p)+1$, and $(n-p-1)$-connected (cf. [12, 19]).

It is clear that the link admits a locally free S^1-action inherited from the weighted circle action on the sphere S^{2n+1}. Furthermore, the Riemannian submersion given in Equation (3.1) endows $L_{\mathbf{f}}$ with a Sasakian structure. Thus, one obtains the commutative diagram

$$\begin{array}{ccc} L_{\mathbf{f}} & \longrightarrow & S^{2n+1} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \mathcal{Z}_{\mathbf{f}} & \longrightarrow & \mathbb{CP}(\mathbf{w}), \end{array} \tag{3.2}$$

where the horizontal arrows are Sasakian and Kählerian embeddings, respectively, while $\pi : L_{\mathbf{f}} \to \mathcal{Z}_{\mathbf{f}}$ is an orbifold Riemannian submersion, and the algebraic variety $\mathcal{Z}_{\mathbf{f}}$ is a weighted complete intersection in $\mathbb{CP}(\mathbf{w})$ (see [5] for a proof of this result).

Of course, the S^1-actions with the qualities described above are the associated actions coming from the proper analytic actions of $\mathbb{C}^*$ on the associated weighted affine cone $C_{\mathbf{f}}$ to $Z_{\mathbf{f}}$. It is well known that these actions are determined by the Picard group of the variety $Z_{\mathbf{f}}$. In this case, the action is induced by the natural action of the transition functions of the line orbibundle on the trivializations (see [26] and [18] for generalizations of this procedure in an algebro-geometric setting). In that direction, Boyer and Galicki showed an interesting result (cf. [9]), of which we present a simplified version good enough for our purposes.

Theorem 3.3. *Let (Z, ω) be a polarized Kähler orbifold with rational Kähler form ω, that is, such that $[\omega] \in H^2(Z, \mathbb{Q})$. Then the associated principal S^1-orbibundle $\pi : M_{[\omega]} \longrightarrow Z$ defined by $[\omega]$ determines a Sasakian structure on the total space $M_{[\omega]}$. The curvature two-form on $M_{[\omega]}$ is given by the pullback $\pi^*\omega$ of the Kähler form defining this fibration. Moreover, if the orbifold is locally cyclic (that is, if it has an orbifold atlas all of whose local uniformizing groups are cyclic groups) then M is a manifold.* $\square$

4. K3 orbifolds and S^1-orbibundles

In this section we briefly describe certain results in [10] on circle orbibundles over K3 surfaces. In particular, we discuss the associated

Sasakian metrics on the total space of this bundles coming from pull-backs of orbifold Ricci-flat metrics on the K3 surface. We also give an explicit description of the Kähler cone for a K3 surface of low rank. This calculation will come handy for the last section.

First, we present some facts about K3 surfaces (see [3] or [30] for proofs of the results stated here). A **K3 surface** X is a compact Kähler surface with only du Val singularities such that $H^1(X, \mathcal{O}_x) = 0$ and whose dualizing sheaf ω_X is trivial in the sense that it satisfies $\omega_X = \mathcal{O}_X$.

If X is a K3 surface and $\rho : \tilde{X} \to X$ is a minimal resolution, then ρ induces an isomorphism between $H^1(X, \mathcal{O})$ and $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$, and also satisfies $\omega_{\tilde{X}} = \rho^*\omega_X = \mathcal{O}_{\tilde{X}}$. So $\tilde{X}$ turns out to be a smooth K3 surface.

Non-singular rational curves on a K3 surface X can be blown down to yield rational double points. On the other hand, under resolution of singularities, a rational singularity determines an exceptional locus consisting of smooth rational curves that intersect transversally. In terms of intersection theory, the arrangement of the curves can be viewed as a configuration of a Dynkin diagram of one of the following types: A_n, D_n, E_6, E_7, E_8.

An important feature of smooth K3 surfaces is that any smooth curve C is rational if and only if it satisfies $C.C = -2$ (this follows from a direct application of the adjunction formula). Moreover, any irreducible curve on a smooth K3 surface has self-intersection 0 (mod 2).

If X is non-singular, the dualizing sheaf becomes the line bundle associated to the canonical divisor K_X. In that case $H^2(X, \mathbb{Z})$ is torsion free of rank $b^2(X) = 22$. By means of the intersection form, $H_2(X, \mathbb{Z})$ is endowed with the structure of a lattice L_Λ which is isomorphic to $-E_8 \oplus -E_8 \oplus H \oplus H \oplus H$. Here and for future reference H is the indefinite rank 2 lattice with intersection $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $-E_8$ is the root-lattice associated to the Dynkin diagram E_8, which is even, unimodular and negative definite. It follows from Poincaré duality that in $H^2(X, \mathbb{Z})$ the cup-product is even, unimodular, and indefinite with signature $(b^2_+, b^2_-) = (3, 19)$.

Consider the exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, {\mathcal{O}_X}^*) = \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \tag{4.1}$$

induced by the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow {\mathcal{O}_X}^* \longrightarrow 0.$$

From Lefschetz $(1,1)$-theorem and $H^1(X, \mathcal{O}_X) = 0$, the map c_1 sends isomorphically $\mathrm{Pic}(X)$ onto the Picard lattice $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. We will denote by ρ the rank of the Picard lattice. From the Hodge index theorem, the signature of $\mathrm{Pic}(X)$ equals $(1, \rho - 1)$.

Example 4.1. A **Kummer surface** is defined as the minimal resolution S of the 16 singularities of type A_1 of Z/ι, the quotient of a complex torus Z of dimension 2 by an involution ι on Z which is induced by multiplication by -1 on $\mathbb{C}^2$. The quotient Z/ι is simply-connected and $H^k(Z/\iota, \mathbb{C})$ is the ι-invariant part of $H^k(Z, \mathbb{C})$. Thus, the second Betti number $b^2(Z/\iota)$ equals 3. The blow-up replaces each singular point with a copy of $\mathbb{CP}^1$ with self-intersection -2. This leaves π_1 and b^2_+ invariant but adds 1 to b^2_- for each of the 16 singular points. Hence S is simply-connected with signature $(3, 19)$. Thus S is a smooth K3 surface. Notice that Kummer surfaces are not necessarily projective, however, they admit Kähler metrics: Siu (cf. [32]) showed that every K3 surface is Kähler.

Example 4.2. Consider the **Fermat quartic**

$$X_4 = \{[z_0, \ldots, z_3] \in \mathbb{CP}^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}.$$

From the adjuntion formula, it follows that X_4 has trivial canonical bundle. From Lefschetz hyperplane theorem, X_4 is connected and simply-connected, hence $H^1(X_4, \mathcal{O}) = 0$. Thus X_4 is a smooth K3 surface. Next, we consider a principal circle bundle L_f over X_4. One can always choose a line bundle on $\mathrm{Pic}(X_4)$ such that L_f is simply-connected and spin. From work of Smale on the classification of simply-connected 5-dimensional spin manifolds (cf. [31]), one concludes $L_f = 21\#(S^2 \times S^3)$.

It follows from Theorem 3.3 that L_f admits a Sasakian metric coming from the corresponding Riemannian submersion. Here the corresponding metric on X_4 is of **Calabi-Yau** type, that is, a Ricci-flat Kähler metric. Sasakian metrics satisfying this property are called **null Sasaki-η-Einstein metrics**, and all of them have scalar curvature equal to -4 (see [8]).

Example 4.3. Weighted K3 surfaces of codimension one and two. A classification of quasi-smooth weighted surface complete intersections of codimension 1 and 2 was given by Reid [27] and Iano-Fletcher [13]. All these surfaces are defined in terms of weighted affine cones that are smooth outside the origin, fact that translates into the quasi-smoothness of the corresponding weighted surface. Thanks to the extension of the adjunction formula to weighted surfaces given in [13], it is straightforward to detect the members of these two families that end up being K3 surfaces with at worst rational doble points (compare Tables 1 and 2). In [10] the results of the previous example are generalized to these two lists where we established the existence of Sasakian metrics of constant scalar curvatures on manifolds diffeomorphic to $\#k(S^2 \times S^3)$, where k is the second Betti number of the link, and k ranges from 3 to 21 inclusive. The projections of these metrics, via the Riemannian submersions, on the corresponding weighted K3 surfaces are orbifold metrics of Calabi-Yau type. Actually, in [10] is given a complete classification of null Sasaki η-Einstein metrics in 5-manifolds. Here we present a simplified version of this theorem.

Theorem 4.4. *Let $\pi : L \longrightarrow X$ be a S^1-orbibudle with L a smooth simply-connected 5-manifold and let X be a Calabi-Yau orbifold. Then L admits a null Sasaki η-Einstein structure if L is diffeomorphic to $\#k(S^2 \times S^3)$ for $3 \leqslant k \leqslant 21$.* □

Next, we will calculate the space of Kähler classes of a particular non-singular K3 surface. It will be useful to review the description of the Kähler cone for a smooth K3 surface.

Recall that in a compact complex manifold (X, J) admitting a Kähler metric g, the Kähler form ω of g defines a de Rham cohomology class $[\omega] \in H^2(X, \mathbb{R})$, called the **Kähler class** of g. Since ω is also a $(1,1)$-form, $[\omega]$ lies in the intersection of $H^{1,1}(X, \mathbb{C})$ with $H^2(X, \mathbb{R})$. The **Kähler cone** of X is the set of Kähler classes of X.

On a smooth K3 surface X, the description of the Kähler cone can be made more precise. Let us consider the set

$$\mathcal{C} = \{x \in \mathrm{Pic}(X) \otimes \mathbb{R} \text{ with } x.x > 0\}.$$

Due to the signature $(1, \rho-1)$ of the Picard lattice, the condition $x.x > 0$ determines two disjoint connected cones $\mathcal{C}^+$ and $\mathcal{C}^-$, and since the Kähler classes form a convex subcone of $\mathcal{C}^+ \cup \mathcal{C}^-$, they all belong to one of them, say $\mathcal{C}^+$. The Kähler cone of a K3 (cf. [3] or [30]) is the convex subcone of $\mathcal{C}^+$ given by

$$\mathcal{K}(X) = \{y \in \mathcal{C}^+ : y.d > 0 \text{ for all } d \in \Delta\},$$

where $\Delta = \{d \in \mathrm{Pic}(X) : d.d = -2 \text{ and } d \text{ effective}\}$.

As an example consider number 2 in Table 1: $X_5 \subset \mathbb{CP}(1,1,1,2)$. This weighted K3 surface has only one cyclic singularity, of type A_1. The surface X_5 can be given by different polynomials, $f_1(x,y,z,w) = x^4+y^4+z^4+w^2$, $f_2(x,y,z,w) = x^2y^2+y^2w+z^4$, $f_3(x,y,z,w) = xyx+z^4$, etcetera. Nevertheless, in [2] it is shown that one can find a polynomial f such that the **orbifold Picard group** $\mathrm{Pic}(X) \otimes \mathbb{Q}$ of any weighted K3 surface of Table 1 has rank one. We will assume this type of polynomial as the one defining X_5. (Notice that the second Betti number of the link, and therefore the number of connected sums of $S^2 \times S^3$, is only determined by the type of singularities and not by how many elements in the orbifold Picard group one started with.) After resolving the singularity one obtains a smooth K3 surface with quadratic form determined by the hyperplane bundle and the exceptional divisor arising from A_1. This quadratic form is represented by the matrix $D = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

Let us determine the set of effective classes of (-2)-curves, that is, the set Δ. It is given by integral solutions of the equation

$$2x^2 + 2xy - 2y^2 = -2,$$

which is equivalent to

$$(x - \rho y)(x - \bar{\rho} y) = -1,$$

with $\rho = \dfrac{-1+\sqrt{5}}{2}$. Hence, one obtains

$$\Delta = \{v \in \mathbb{Z}[\rho]\,|v\bar{v} = -1\}.$$

It is not difficult to see that the solutions are generated by odd powers of $\alpha = \left(\frac{1+\sqrt{5}}{2}\right)$ and $\bar{\alpha} = \left(\frac{1-\sqrt{5}}{2}\right)$. These two numbers satisfy the relation $\alpha^3 = 3\alpha + \bar{\alpha}$. One obtains, by induction, $\alpha^{2k+1} = a\alpha + b\bar{\alpha}$ with a, b positive integers. Since the point $(0, 1)$ satisfies the relation given above, owing to $\bar{\alpha} = 1 - \alpha$, the point $(1, -1)$ is also a solution of this last equation. It is clear then that the Kähler cone of $\widetilde{X}_5$ is given by

$$\mathcal{K}(\widetilde{X}_5) = \{(x, y) : x - 2y > 0 \text{ and } x + 3y > 0\}.$$

5. Duality in connected sums of $S^2 \times S^3$

In this section we explain certain duality between k connected sums of $S^2 \times S^3$ for k an integer ranging from 3 to 21. This correspondence comes into sight if one prescribes the Riemannian structure on these 5-manifolds. They will be given in terms of a map which can be thought of as a transversely birational mapping. The aforementioned duality is a consequence of considering the transverse space, in general a K3 orbifold, as a smooth K3 surface with fixed exceptional lattice.

First, let us recall some notions from algebraic geometry. (See e.g., [17], for details).

Let X be a complex surface with a holomorphic line bundle L. What follows can be presented in more generality, the interested reader should consult [17].

Let us denote by $H^0(X, L)$ the vector space of holomorphic sections of L over X, which is known to be finite dimensional over $\mathbb{C}$, of dimension $\ell + 1$, say. Let us denote by $|L|$ the complete linear system of L. The **base locus** $Bs(|L|) \subset X$ of $|L|$ is the set of points at which all sections of $H^0(X, L)$ vanish. One says that $|L|$ is **free** or **base point-free** (or simply that L is **globally generated**) if its base locus is empty. This is equivalent to obtaining, for each $x \in X$, a section $s \in H^0(X, L)$ subject to $s(x) \neq 0$.

Let us choose a basis $s_0, \ldots, s_\ell$ for $H^0(X, L)$. Then one has a natural map

$$\varphi_{|L|} : X - Bs(|L|) \longrightarrow \mathbb{P}H^0(X, L)$$

with rule $\varphi_{|L|}(x) = [s_0(x), \ldots, s_\ell(x)]$. It is costumary to ignore the base locus and construe $\varphi_{|L|}$ as a rational mapping $\varphi_{|L|} : X \dashrightarrow \mathbb{P}H^0(X, L)$. When L is globally generated one obtains a globally defined morphism

$$\varphi_{|L|} : X \longrightarrow \mathbb{P}H^0(X, L).$$

This map is finite, or equivalently satisfies $L.C > 0$ for any irreducible curve C in X, if and only if L is ample. A line bundle is **big** if the map

$$\varphi_{|mL|} : X \dashrightarrow \mathbb{P}H^0(X, L^{\otimes m})$$

is birational onto its image for some m. In this situation L is not, in general, globally generated. A divisor D is **numerically effective** (or **nef** for short) if it satisfies $D.C \geqslant 0$ for any irreducible curve $C \in X$. The notion of amplitud (or ampleness) can also be given numerically: a line bundle L on a smooth surface is ample if and only if satisfies $c_1(L)^2 > 0$ and $L.D > 0$ for every effective divisor on the surface. This characterization is known as **Nakai's criterion** (cf. [3, Corollary 5.4]).

Now, we explain the duality or correspondence between two supposedly different S^1-Seifert bundles. We will always consider elements $X_{\mathbf{w}}$ of one of the two familes of quasi-smooth K3 surfaces depicted in Tables 1 and 2.

Let $\widetilde{X}$ be the minimal resolution $f : \widetilde{X} \to X_{\mathbf{w}}$. By Theorem 4.5, one has for $(X_{\mathbf{w}}, [\omega])$, as a (polarized) projective orbifold, an associated S^1-orbibundle

$$\pi_1 : (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3) \to X_{\mathbf{w}}$$

defined by $[\omega]$. On the other hand, recall (Section 4) that the resolution $\widetilde{X}$ is a smooth projective $K3$ surface, and then, as a consequence of Kodaira's embedding theorem, admits an integral Kähler class $[\omega_{\widetilde{X}}] \in H^2(\widetilde{X}, \mathbb{Z})$ associated to certain ample line bundle A on $\widetilde{X}$ subject to $c_1(A) = [\omega_{\widetilde{X}}]$. Again, we appeal to Theorem 4.5 to conclude the existence of a 5-manifold diffeomorphic, this time, to $21\#(S^2 \times S^3)$. As we mentioned previously, it is important to bear in mind that both $(b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3)$ and $21\#(S^2 \times S^3)$ have scalar curvature -4.

Now let us denote by L_1 the pullback, via f, of L, where $c_1(L) = [\omega]$ with L a positive orbibundle (or ample in the orbifold sense). Notice that L_1 is a big and nef line bundle (almost by definition) that cannot be ample, otherwise the null locus $\mathrm{Null}(L_1)$ of L_1, that is, the set of divisors D such that $L_1.D = 0$, ends up consisting of the exceptional divisors, contradicting Nakai's criterion for ampleness.

We would rather reinterpret the previous paragraph at the level of Kähler classes. Notice that the pullback $f^*[\omega]$ of $[\omega]$ lies on the boundary $\partial\mathcal{K}(\widetilde{X})$ of the Kähler cone of $\widetilde{X}$. From the discussion given above, it follows that the orbifold Riemannian submersion

$$\pi_1 : (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3) \to (X_{\mathbf{w}}, [\omega])$$

induces a natural map

$$\widetilde{\pi}_1 : (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3) \to (\widetilde{X}\backslash\Delta, f^*[\omega]),$$

where Δ denotes de exceptional divisor coming from the resolution of the orbifold. Of course, at the level of the total spaces one obtains the map

$$\hat{f} : 21\#(S^2 \times S^3) \longrightarrow (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3).$$

As an example, consider number 2 in Table 1: $X_5 \subset \mathbb{P}(1,1,1,2)$ with singularity A_1 and Picard number 1. In [10], it is proven that the corresponding S^1-Seifert bundle is diffeomorphic to $20\#(S^2 \times S^3)$.

In the previous section we computed the Kähler cone of the resolution $\widetilde{X}_5$ of X_5, which is given by the set

$$\mathcal{K}(\widetilde{X}_5) = \{(x, y) : x - 2y > 0 \text{ and } x + 3y > 0\}.$$

Observe that a big and nef class that is not ample lies on the boundary of the Kähler cone, that is, in one of the lines $x-2y = 0$ or $x+3y = 0$. From the previous analysis, it is known that at least there is one element that corresponds to an ample class that is integral, in the orbifold sense, in X_5, and the duality between $21\#(S^2\times S^3)$ and $20\#(S^2\times S^3)$ shows up when an integral class inside $\mathcal{K}(\widetilde{X}_5)$ is related to a class in the boundary, that is, a class lying on either the line $x-2y = 0$ or the line $x+3y = 0$. Actually, it is always possible to find an element that provides this duality. We explain this in the next paragraph.

Even though it may no be true that a big and nef line bundle L comes from the pullback of a Hodge orbifold class, in [33], Tosatti showed that this is always the case for projective smooth K3 surfaces: it is enough to apply the basepoint free theorem (see [14, Theorem 6.1]) together with the fact that one is dealing with a Calabi-Yau manifold to conclude that mL is globally generated for m sufficiently large. Thus, any irreducible $D \in \mathrm{Null}(L)$ has negative self-intersection (this follows from the Hodge index theorem, since we obtain $(mL)^2 > 0$). As we mentioned before, in a smooth K3 surface, self-intersections are even, and since $D^2 \geqslant -2$ (see [3, Chapter VIII, Proposition 3.6]) then $D^2 = -2$. By the adjuntion formula one concludes that D is a smooth rational curve. Then the map $\varphi_{|mL|}$ contracts D to a rational double point while X contracts to a projective K3 orbifold with ample line bundle L_1 (in the orbifold sense) and one can assume that its pullback is mL. So, in general, pulling-back an ample class is not enough, as also "sliding" along the boundary may be necessary. The m under discussion is nothing more than the least common denominator of the orders of the cyclic

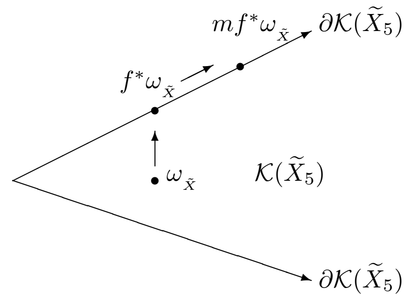

Figure 1: The Kähler cone of $\widetilde{X}_5$: even if $f^*\omega$ does not provide the ∂-duality, a multiple $mf^*\omega_{\tilde X}$ will achieve this goal.

singularities one encounters in the orbifold, so there is a globally defined line bundle on the orbifold. In the example given above $m = 2$ will suffice. For $X_{24,30} \subset \mathbb{CP}(8,9,10,12,15)$, number 84 in Table 2, it is enough to take $m = 180$.

Since a link is the boundary of a Milnor fiber, we will refer to this duality as **∂-duality**. Let us put the discussion given above in theorem form.

Theorem 5.1. *There is ∂-duality, in the sense explained above, between k connected sums of $S^2 \times S^3$ for any $k \in \{3, \dots 20\}$ and $21\#(S^2 \times S^3)$. The following diagram summarizes the relations among these maps*

$$\begin{tikzcd}
 & (b_2(X_{\mathbf{w}}) - 1)\#(S^2 \times S^3) \arrow[dl, "\pi_1"'] \arrow[d, dashed, "\widetilde{\pi}_1"] & 21\#(S^2 \times S^3) \arrow[l, dashed, "\hat{f}"'] \arrow[d, "\pi_0"] \\
(X_{\mathbf{w}}, [\omega]) & (\widetilde{X}\backslash\Delta, f^*[\omega]) \arrow[l] & (\widetilde{X}, [\omega_{\widetilde{X}}]). \arrow[l] \arrow[ll, dashed, bend left, "f"]
\end{tikzcd}$$

Thus, the duality occurs when one considers simultaneously the integral class $[\omega_{\widetilde{X}}]$ *in the Kähler cone* $\mathcal{K}(\widetilde{X})$ *and a class lying on the closure of the Kähler cone that is big but not ample, i.e., a class* $[\omega_\partial] \in \overline{\mathcal{K}(\widetilde{X})} \backslash \mathcal{K}(\widetilde{X})$ *that we must take equal to* $f^*[\omega]$. $\square$

A natural question will be whether one can extend this ∂-duality among k connected sums of $S^2 \times S^3$ with $3 \leqslant k \leqslant 20$. We consider a projective smooth K3 surface X with enough smooth rational curves that intersect each other transversally (this type of K3 surface always can be found, e.g., the Fermat quartic). Let us choose a subset $\mathcal{C} = \{C_1, \ldots, C_s, \ldots, C_l\}$ of rational curves with this sort of intersection. We will perform two blow-downs, the first one contracting all the curves of $\mathcal{C}$, and the other one contracting only $l - s$ curves from $\mathcal{C}$. These contractions manufacture two different K3 orbifolds $X_{\mathbf{w_1}}$ and $X_{\mathbf{w_2}}$, which are projective and hence admit ample line orbibundles L_1 and L_2, respectively. With abuse of notation, we will denote also by L_1 and L_2 the corresponding pullbacks of these two line orbibundles. These two line bundles end up being big and nef on X, so both are globally generated with corresponding maps

$$\varphi_{|L_1|} : X \to \mathbb{P}H^0(X, L_1) \text{ and } \varphi_{|L_2|} : X \to \mathbb{P}H^0(X, L_2).$$

If one pursues ∂-duality on the corresponding links, the null spaces of these line bundles must satisfy $\mathrm{Null}(L_1) = \Delta = \cup_{i=1}^{l} D_i$ and $\mathrm{Null}(L_2) = \Delta \backslash \cup_{i=1}^{s} D_i$. Here each D_i denotes disjoint components of the base locus Δ.

This is not necessarily the case in general. Take for instance $X_7 \subset \mathbb{P}(1,1,2,3)$, number 5 in Table 1, with singularities A_1, A_2. Here the minimal resolution X has null space $\mathrm{Null}(L_1) = D_1 \cup D_2$, where D_1 is just the rational curve created when the singularity of type A_1 is resolved and D_2 consists of the union of two smooth rational curves E_1 and E_2 with $E_1.E_2 = 1$, created when the singularity of type A_2 is resolved. Then the existence of duality between the link $18\#(S^2 \times S^3)$ associated to L_1 and some other connected sum of $S^2 \times S^3$ (with $k \neq 0$) forces the existence of another big and nef line bundle L_2 satisfying $\mathrm{Null}(L_2) = D_1$ (in this case our choice for the ∂-dual is $20\#(S^2 \times S^3)$, the other possibility is taking $\mathrm{Null}(L_2) = D_2$ with ∂-dual $19\#(S^2 \times S^3)$, but the argument works identically). Thus, condition $L_2.D_2 > 0$ is necessary, otherwise the map $\varphi_{|L_2|} : X \to \mathbb{P}H^0(X, L_2)$ will contract D_2 to a point and hence D_2 would belong to the null locus of L_2.

In general, in order to have duality one needs to verify the condition

$$L_2. \cup_{i=1}^{s} D_i > 0. \tag{5.1}$$

With the notation from the last two paragraphs, we have the following result.

Theorem 5.2. *Let $X_{\mathbf{w_1}}$ and $X_{\mathbf{w_2}}$ two K3 orbifolds from either Table 1 or 2. Let $K_1 = (21 - l)\#(S^2 \times S^3)$ and $K_2 = (21 - l + s)\#(S^2 \times S^3)$ be the corresponding links determined by the line orbibundles L_1 and L_2 as explained in the previous paragraphs. Then K_1 is ∂-dual to K_2 if and only if the condition stated on (5.1) is satisfied.* □

Let us interpret this result at the level of metrics. When one considers two classes, first an integral class inside the Kähler cone and then another big and nef class, both in the same smooth K3 surface, the first one has a corresponding Ricci-flat metric on the K3 surface (the celebrated Yau's theorem), while the second one gives rise to a Ricci-flat orbifold metric on the orbifold, obtained from contracting the rational curves belonging to the null space of this big and nef class (see [15]). Moreover, in [16] (see also [33] for a more general statement) it is shown that this metric is smooth Ricci-flat on $X \backslash E$, with E the corresponding set of exceptional divisors, object that can be extended to a closed positive current on the whole K3 surface. The translation of this fact to the corresponding five dimensional Seifert bundle is the existence of one smooth null Sasaki η-Einstein metric on $21\#(S^2 \times S^3)$ (with scalar curvature equal to -4) and a pseudometric on $21\#(S^2 \times S^3)$ that degenerates into l copies of $S^2 \times S^3$ (of course, here l corresponds in a natural way to the number of linearly independent rational curves that determine the null space of the big and nef class). The last metric turns into a smooth null Sasaki η-Einstein metric on $(21-l)\#(S^2 \times S^3)$ also with constant scalar curvature equal -4.

Remark 5.3. From the previous paragraph, it would be tempting to conclude that all manifolds diffeomorphic to $21\#(S^2 \times S^3)$ admit a metric with scalar curvature -4 collapsing on l connected sums of $S^2 \times S^3$ for every $0 \leqslant l \leqslant 18$; however this is not the case. As indicated before, the ∂-duality is determined by the map $\hat{f}$ which exists only in the *transversely birational* sense, that is, only on the space of leaves determined by the action of S^1. Indeed, one cannot use this correspondence to state that the two null Sasakian metrics in display (both with constant scalar curvature -4) can be considered to exist in the same ambient space, that is, in the same differential structure. Actually, the open neighborhood $V_{f^{-1}(x)}$ of $f^{-1}(x)$ is not necessarily diffeomorphic to the open set $\widetilde{U}_x \subset \mathbb{C}^2$ coming from the local orbifold chart $\widetilde{U}_x/\Gamma_x$ (with uniformizing group Γ_x). In fact, $V_{f^{-1}(x)}$ is diffeomorphic to the corresponding Milnor fiber (see [11, page 148]). Whether there exists a natural differential geometric minimal model that contains these two seemingly unrelated Riemannian structures and, therefore, a common place where one can consider both metrics as defining a unique metric on this model amounts to establish a more precise partnership between the minimal resolution of a K3 surface and the link of its affine cone. The existence of a minimal model for S^1-Seifert bundles and its possible applications is part of a joint work in progress with R. Gonzales.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.1	$X_4 \subset \mathbb{P}(1,1,1,1)$		22
No.2	$X_5 \subset \mathbb{P}(1,1,1,2)$	A_1	21
No.3	$X_6 \subset \mathbb{P}(1,1,1,3)$		22
No.4	$X_6 \subset \mathbb{P}(1,1,2,2)$	$3 \times A_1$	19
No.5	$X_7 \subset \mathbb{P}(1,1,2,3)$	A_1, A_2	19
No.6	$X_8 \subset \mathbb{P}(1,1,2,4)$	$2 \times A_1$	20
No.7	$X_8 \subset \mathbb{P}(1,2,2,3)$	$4 \times A_1, A_2$	16
No.8	$X_9 \subset \mathbb{P}(1,1,3,4)$	A_3	19
No.9	$X_9 \subset \mathbb{P}(1,2,3,3)$	$A_1, 3 \times A_2$	15
No.10	$X_{10} \subset \mathbb{P}(1,1,3,5)$	A_2	20
No.11	$X_{10} \subset \mathbb{P}(1,2,2,5)$	$5 \times A_1$	17
No.12	$X_{10} \subset \mathbb{P}(1,2,3,4)$	$2 \times A_1, A_2, A_3$	15
No.13	$X_{11} \subset \mathbb{P}(1,2,3,5)$	A_1, A_2, A_5	15
No.14	$X_{12} \subset \mathbb{P}(1,1,4,6)$	A_1	21
No.15	$X_{12} \subset \mathbb{P}(1,2,3,6)$	$2 \times A_1, 2 \times A_2$	14
No.16	$X_{12} \subset \mathbb{P}(1,2,4,5)$	$3 \times A_1, A_4$	15
No.17	$X_{12} \subset \mathbb{P}(1,3,4,4)$	$3 \times A_3$	13
No.18	$X_{12} \subset \mathbb{P}(2,2,3,5)$	$6 \times A_1, A_4$	12
No.19	$X_{12} \subset \mathbb{P}(2,3,3,4)$	$3 \times A_1, 4 \times A_2$	11
No.20	$X_{13} \subset \mathbb{P}(1,3,4,5)$	A_2, A_3, A_4	13
No.21	$X_{14} \subset \mathbb{P}(1,2,4,7)$	$3 \times A_1, A_3$	16
No.22	$X_{14} \subset \mathbb{P}(2,2,3,7)$	$7 \times A_1, A_2$	13
No.23	$X_{14} \subset \mathbb{P}(2,3,4,5)$	$3 \times A_1, A_2, A_3, A_4$	10
No.24	$X_{15} \subset \mathbb{P}(1,2,5,7)$	A_1, A_6	15
No.25	$X_{15} \subset \mathbb{P}(1,3,4,7)$	A_3, A_6	13
No.26	$X_{15} \subset \mathbb{P}(1,3,5,6)$	$2 \times A_2, A_5$	13
No.27	$X_{15} \subset \mathbb{P}(2,3,5,5)$	$A_1, 3 \times A_4$	9
No.28	$X_{15} \subset \mathbb{P}(3,3,4,5)$	$5 \times A_2, A_3$	9
No.29	$X_{16} \subset \mathbb{P}(1,2,5,8)$	$2 \times A_1, A_4$	16

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.30	$X_{16} \subset \mathbb{P}(1,3,4,8)$	$A_2, 2 \times A_3$	14
No.31	$X_{16} \subset \mathbb{P}(1,4,5,6)$	A_1, A_4, A_5	12
No.32	$X_{16} \subset \mathbb{P}(2,3,4,7)$	$4 \times A_1, A_2, A_6$	10
No.33	$X_{17} \subset \mathbb{P}(2,3,5,7)$	A_1, A_2, A_4, A_6	9
No.34	$X_{18} \subset \mathbb{P}(1,2,6,9)$	$3 \times A_1, A_2$	15
No.35	$X_{18} \subset \mathbb{P}(1,3,5,9)$	$2 \times A_2, A_4$	14
No.36	$X_{18} \subset \mathbb{P}(1,4,6,7)$	A_3, A_1, A_6	12
No.37	$X_{18} \subset \mathbb{P}(2,3,4,9)$	$4 \times A_1, 2 \times A_2, A_3$	11
No.38	$X_{18} \subset \mathbb{P}(2,3,5,8)$	$2 \times A_1, A_4, A_7$	9
No.39	$X_{18} \subset \mathbb{P}(3,4,5,6)$	$3 \times A_2, A_3, A_1, A_4$	8
No.40	$X_{19} \subset \mathbb{P}(3,4,5,7)$	A_2, A_3, A_4, A_6	7
No.41	$X_{20} \subset \mathbb{P}(1,4,5,10)$	$A_1, 2 \times A_4$	13
No.42	$X_{20} \subset \mathbb{P}(2,3,5,10)$	$2 \times A_1, A_2, 2 \times A_4$	10
No.43	$X_{20} \subset \mathbb{P}(2,4,5,9)$	$5 \times A_1, A_8$	9
No.44	$X_{20} \subset \mathbb{P}(2,5,6,7)$	$3 \times A_1, A_5, A_6$	8
No.45	$X_{20} \subset \mathbb{P}(3,4,5,8)$	$A_2, 2 \times A_3, A_7$	7
No.46	$X_{21} \subset \mathbb{P}(1,3,7,10)$	A_9	13
No.47	$X_{21} \subset \mathbb{P}(1,5,7,8)$	A_4, A_7	11
No.48	$X_{21} \subset \mathbb{P}(2,3,7,9)$	$A_1, 2 \times A_2, A_8$	9
No.49	$X_{21} \subset \mathbb{P}(3,5,6,7)$	$3 \times A_2, A_4, A_5$	7
No.50	$X_{22} \subset \mathbb{P}(1,3,7,11)$	A_2, A_6	14
No.51	$X_{22} \subset \mathbb{P}(1,4,6,11)$	A_3, A_1, A_5	13
No.52	$X_{22} \subset \mathbb{P}(2,4,5,11)$	$5 \times A_1, A_3, A_4$	10
No.53	$X_{24} \subset \mathbb{P}(1,3,8,12)$	$2 \times A_2, A_3$	16
No.54	$X_{24} \subset \mathbb{P}(1,6,8,9)$	A_1, A_2, A_8	11
No.55	$X_{24} \subset \mathbb{P}(2,3,7,12)$	$2 \times A_1, 2 \times A_2, A_6$	10
No.56	$X_{24} \subset \mathbb{P}(2,3,8,11)$	$3 \times A_1, A_{10}$	9
No.57	$X_{24} \subset \mathbb{P}(3,4,5,12)$	$2 \times A_2, 2 \times A_3, A_4$	8
No.58	$X_{24} \subset \mathbb{P}(3,4,7,10)$	A_1, A_6, A_9	6

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.59	$X_{24} \subset \mathbb{P}(3,6,7,8)$	$4 \times A_2, A_1, A_6$	7
No.60	$X_{24} \subset \mathbb{P}(4,5,6,9)$	$2 \times A_1, A_4, A_2, A_8$	6
No.61	$X_{25} \subset \mathbb{P}(4,5,7,9)$	A_3, A_6, A_8	5
No.62	$X_{26} \subset \mathbb{P}(1,5,7,13)$	A_4, A_6	12
No.63	$X_{26} \subset \mathbb{P}(2,3,8,13)$	$3 \times A_1, A_2, A_7$	10
No.64	$X_{26} \subset \mathbb{P}(2,5,6,13)$	$4 \times A_1, A_4, A_5$	9
No.65	$X_{27} \subset \mathbb{P}(2,5,9,11)$	A_1, A_4, A_{10}	7
No.66	$X_{27} \subset \mathbb{P}(5,6,7,8)$	A_4, A_5, A_2, A_6	5
No.67	$X_{28} \subset \mathbb{P}(1,4,9,14)$	A_1, A_8	13
No.68	$X_{28} \subset \mathbb{P}(3,4,7,14)$	$A_2, A_1, 2 \times A_6$	7
No.69	$X_{28} \subset \mathbb{P}(4,6,7,11)$	$2 \times A_1, A_5, A_{10}$	5
No.70	$X_{30} \subset \mathbb{P}(1,4,10,15)$	A_3, A_4, A_1	14
No.71	$X_{30} \subset \mathbb{P}(1,6,8,15)$	A_1, A_2, A_7	12
No.72	$X_{30} \subset \mathbb{P}(2,3,10,15)$	$3 \times A_1, 2 \times A_2, A_4$	6
No.73	$X_{30} \subset \mathbb{P}(2,6,7,15)$	$5 \times A_1, A_2, A_6$	9
No.74	$X_{30} \subset \mathbb{P}(3,4,10,13)$	A_3, A_1, A_{12}	5
No.75	$X_{30} \subset \mathbb{P}(4,5,6,15)$	$A_3, 2 \times A_1, 2 \times A_4, A_2$	7
No.76	$X_{30} \subset \mathbb{P}(5,6,8,11)$	A_1, A_7, A_{10}	4
No.77	$X_{32} \subset \mathbb{P}(2,5,9,16)$	$2 \times A_1, A_4, A_8$	8
No.78	$X_{32} \subset \mathbb{P}(4,5,7,16)$	$2 \times A_3, A_4, A_6$	6
No.79	$X_{33} \subset \mathbb{P}(3,5,11,14)$	A_4, A_{13}	5
No.80	$X_{34} \subset \mathbb{P}(3,4,10,17)$	A_2, A_3, A_1, A_9	7
No.81	$X_{34} \subset \mathbb{P}(4,6,7,17)$	$A_3, 2 \times A_1, A_5, A_6$	6
No.82	$X_{36} \subset \mathbb{P}(1,5,12,18)$	A_4, A_5	13
No.83	$X_{36} \subset \mathbb{P}(3,4,11,18)$	$2 \times A_2, A_1, A_{10}$	7
No.84	$X_{36} \subset \mathbb{P}(7,8,9,12)$	A_6, A_7, A_3, A_2	4
No.85	$X_{38} \subset \mathbb{P}(3,5,11,19)$	A_2, A_4, A_{10}	6
No.86	$X_{38} \subset \mathbb{P}(5,6,8,19)$	A_4, A_5, A_1, A_7	5
No.87	$X_{40} \subset \mathbb{P}(5,7,8,20)$	$2 \times A_4, A_6, A_3$	5

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.86	$X_{38} \subset \mathbb{P}(5,6,8,19)$	A_4, A_5, A_1, A_7	5
No.87	$X_{40} \subset \mathbb{P}(5,7,8,20)$	$2 \times A_4, A_6, A_3$	5
No.88	$X_{42} \subset \mathbb{P}(1,6,14,21)$	A_1, A_2, A_6	13
No.89	$X_{42} \subset \mathbb{P}(2,5,14,21)$	$3 \times A_1, A_4, A_6$	9
No.90	$X_{42} \subset \mathbb{P}(3,4,14,21)$	$2 \times A_2, A_3, A_1, A_6$	8
No.91	$X_{44} \subset \mathbb{P}(4,5,13,22)$	A_1, A_4, A_{12}	5
No.92	$X_{48} \subset \mathbb{P}(3,5,16,24)$	$2 \times A_2, A_4, A_7$	7
No.93	$X_{50} \subset \mathbb{P}(7,8,10,25)$	A_6, A_7, A_1, A_4	4
No.94	$X_{54} \subset \mathbb{P}(4,5,18,27)$	A_3, A_1, A_4, A_8	6
No.95	$X_{66} \subset \mathbb{P}(5,6,22,33)$	A_4, A_1, A_2, A_{10}	5

Table 1. Reid's List of 95 Codimension 1 Weighted K3 Surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.1	$X_{2,3} \subset \mathbb{P}(1,1,1,1,1)$		22
No.2	$X_{3,3} \subset \mathbb{P}(1,1,1,1,2)$	A_1	21
No.3	$X_{3,4} \subset \mathbb{P}(1,1,1,2,2)$	$2 \times A_1$	20
No.4	$X_{4,4} \subset \mathbb{P}(1,1,1,2,3)$	A_2	20
No.5	$X_{4,4} \subset \mathbb{P}(1,1,2,2,2)$	$4 \times A_1$	18
No.6	$X_{4,5} \subset \mathbb{P}(1,1,2,2,3)$	$2 \times A_1, A_2$	18
No.7	$X_{4,6} \subset \mathbb{P}(1,1,2,3,3)$	$2 \times A_2$	18
No.8	$X_{4,6} \subset \mathbb{P}(1,2,2,2,3)$	$6 \times A_1$	16
No.9	$X_{5,6} \subset \mathbb{P}(1,1,2,3,4)$	A_1, A_3	18
No.10	$X_{5,6} \subset \mathbb{P}(1,2,2,3,3)$	$3 \times A_1, 2 \times A_2$	15
No.11	$X_{6,6} \subset \mathbb{P}(1,1,2,3,5)$	A_4	18
No.12	$X_{6,6} \subset \mathbb{P}(1,2,2,3,4)$	$4 \times A_1, A_3$	15
No.13	$X_{6,6} \subset \mathbb{P}(1,2,3,3,3)$	$4 \times A_2$	14
No.14	$X_{6,6} \subset \mathbb{P}(2,2,2,3,3)$	$9 \times A_1$	13

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.15	$X_{6,7} \subset \mathbb{P}(1,2,2,3,5)$	$3 \times A_1, A_4$	15
No.16	$X_{6,7} \subset \mathbb{P}(1,2,3,3,4)$	$A_1, 2 \times A_2, A_3$	14
No.17	$X_{6,8} \subset \mathbb{P}(1,1,3,4,5)$	A_4	18
No.18	$X_{6,8} \subset \mathbb{P}(1,2,2,3,5)$	$2 \times A_2, A_4$	14
No.19	$X_{6,8} \subset \mathbb{P}(1,2,3,4,4)$	$2 \times A_1, 2 \times A_3$	14
No.20	$X_{6,8} \subset \mathbb{P}(2,2,3,3,4)$	$6 \times A_1, 2 \times A_2$	12
No.21	$X_{6,9} \subset \mathbb{P}(1,2,3,4,5)$	A_1, A_3, A_4	14
No.22	$X_{7,8} \subset \mathbb{P}(1,2,3,4,5)$	$2 \times A_1, A_2, A_4$	14
No.23	$X_{6,10} \subset \mathbb{P}(1,2,3,5,5)$	$2 \times A_4$	14
No.24	$X_{6,10} \subset \mathbb{P}(2,2,3,4,5)$	$7 \times A_1, A_3$	12
No.25	$X_{8,9} \subset \mathbb{P}(1,2,3,4,7)$	$2 \times A_1, A_6$	14
No.26	$X_{8,9} \subset \mathbb{P}(1,3,4,4,5)$	$2 \times A_3, A_4$	13
No.27	$X_{8,9} \subset \mathbb{P}(2,3,3,4,5)$	$2 \times A_1, 3 \times A_2, A_4$	10
No.28	$X_{8,10} \subset \mathbb{P}(1,2,3,5,7)$	A_2, A_6	14
No.29	$X_{8,10} \subset \mathbb{P}(1,2,4,5,6)$	$3 \times A_1, A_5$	14
No.30	$X_{8,10} \subset \mathbb{P}(1,3,4,5,5)$	$A_2, 2 \times A_4$	12
No.31	$X_{8,10} \subset \mathbb{P}(2,3,4,4,5)$	$4 \times A_1, A_2, 2 \times A_3$	10
No.32	$X_{9,10} \subset \mathbb{P}(1,2,3,5,8)$	A_1, A_7	14
No.33	$X_{9,10} \subset \mathbb{P}(1,3,4,5,6)$	A_2, A_3, A_5	12
No.34	$X_{9,10} \subset \mathbb{P}(2,2,3,5,7)$	$5 \times A_1, A_6$	11
No.35	$X_{9,10} \subset \mathbb{P}(2,3,4,5,5)$	$2 \times A_1, A_3, 2 \times A_4$	9
No.36	$X_{8,12} \subset \mathbb{P}(1,3,4,5,7)$	A_4, A_6	12
No.37	$X_{8,12} \subset \mathbb{P}(2,3,4,5,6)$	$4 \times A_1, 2 \times A_2, A_4$	10
No.38	$X_{9,12} \subset \mathbb{P}(2,3,4,5,7)$	$3 \times A_1, A_4, A_6$	9
No.39	$X_{10,11} \subset \mathbb{P}(2,3,4,5,7)$	$2 \times A_1, A_2, A_3, A_6$	9
No.40	$X_{10,12} \subset \mathbb{P}(1,3,4,5,9)$	A_2, A_8	12
No.41	$X_{10,12} \subset \mathbb{P}(1,3,5,6,7)$	$2 \times A_2, A_6$	12

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.42	$X_{10,12} \subset \mathbb{P}(1,2,5,6,6)$	$A_1, 2 \times A_5$	11
No.43	$X_{10,12} \subset \mathbb{P}(2,3,4,5,8)$	$3 \times A_1, A_3, A_7$	9
No.44	$X_{10,12} \subset \mathbb{P}(2,3,5,5,7)$	$2 \times A_4, A_6$	8
No.45	$X_{10,12} \subset \mathbb{P}(2,4,5,5,6)$	$5 \times A_1, 2 \times A_4$	9
No.46	$X_{10,12} \subset \mathbb{P}(3,3,4,5,7)$	$4 \times A_2, A_6$	8
No.47	$X_{10,12} \subset \mathbb{P}(3,4,4,5,6)$	$2 \times A_2, 3 \times A_3, A_1$	8
No.48	$X_{11,12} \subset \mathbb{P}(1,4,5,6,7)$	A_1, A_4, A_6	11
No.49	$X_{10,14} \subset \mathbb{P}(1,2,5,7,9)$	A_8	14
No.50	$X_{10,14} \subset \mathbb{P}(2,3,5,7,7)$	$A_2, 2 \times A_6$	8
No.51	$X_{10,14} \subset \mathbb{P}(2,4,5,6,7)$	$5 \times A_1, A_3, A_5$	9
No.52	$X_{10,15} \subset \mathbb{P}(2,3,5,7,8)$	A_1, A_6, A_7	8
No.53	$X_{12,13} \subset \mathbb{P}(3,4,5,6,7)$	$2 \times A_2, A_1, A_4, A_6$	7
No.54	$X_{12,14} \subset \mathbb{P}(1,3,4,7,11)$	A_{10}	12
No.55	$X_{12,14} \subset \mathbb{P}(1,4,6,7,8)$	A_1, A_3, A_7	11
No.56	$X_{12,14} \subset \mathbb{P}(2,3,4,7,10)$	$4 \times A_1, A_9$	9
No.57	$X_{12,14} \subset \mathbb{P}(2,3,5,7,9)$	A_2, A_4, A_8	8
No.58	$X_{12,14} \subset \mathbb{P}(3,4,5,7,7)$	$A_4, 2 \times A_6$	6
No.59	$X_{12,14} \subset \mathbb{P}(4,4,5,6,7)$	$3 \times A_3, 2 \times A_1, A_4$	7
No.60	$X_{12,15} \subset \mathbb{P}(1,4,5,6,11)$	A_1, A_{10}	11
No.61	$X_{12,15} \subset \mathbb{P}(3,4,5,6,9)$	$3 \times A_2, A_1, A_8$	7
No.62	$X_{12,15} \subset \mathbb{P}(3,4,5,7,8)$	A_3, A_6, A_7	6
No.63	$X_{12,16} \subset \mathbb{P}(2,5,6,7,8)$	$4 \times A_1, A_4, A_6$	8
No.64	$X_{14,15} \subset \mathbb{P}(2,3,5,7,12)$	A_1, A_2, A_{11}	8
No.65	$X_{14,15} \subset \mathbb{P}(2,5,6,7,9)$	$2 \times A_1, A_5, A_8$	7
No.66	$X_{14,15} \subset \mathbb{P}(3,4,5,7,10)$	A_3, A_4, A_9	6
No.67	$X_{14,15} \subset \mathbb{P}(3,5,6,7,8)$	$2 \times A_2, A_5, A_7$	6
No.68	$X_{14,16} \subset \mathbb{P}(1,5,7,8,9)$	A_4, A_8	10

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

	$X_{\mathbf{w}}$	singularities	$b_2(X_{\mathbf{w}})$
No.69	$X_{14,16} \subset \mathbb{P}(3,4,5,7,11)$	A_2, A_4, A_{10}	6
No.70	$X_{14,16} \subset \mathbb{P}(4,5,6,7,8)$	$A_1, 2\times A_3, A_4, A_5$	6
No.71	$X_{15,16} \subset \mathbb{P}(2,3,5,8,13)$	$2\times A_1, A_{12}$	8
No.72	$X_{15,16} \subset \mathbb{P}(3,4,5,8,11)$	$2\times A_3, A_{10}$	6
No.73	$X_{14,18} \subset \mathbb{P}(2,3,7,9,11)$	$2\times A_2, A_{10}$	8
No.74	$X_{14,18} \subset \mathbb{P}(2,6,7,8,9)$	$5\times A_1, A_2, A_7$	8
No.75	$X_{12,20} \subset \mathbb{P}(4,5,6,7,10)$	$2\times A_1, 2\times A_4, A_6$	6
No.76	$X_{16,18} \subset \mathbb{P}(1,6,8,9,10)$	A_1, A_2, A_9	10
No.77	$X_{16,18} \subset \mathbb{P}(4,6,7,8,9)$	$2\times A_1, 2\times A_3, A_2, A_6$	6
No.78	$X_{18,20} \subset \mathbb{P}(4,5,6,9,14)$	$2\times A_1, A_2, A_{13}$	5
No.79	$X_{18,20} \subset \mathbb{P}(4,5,7,9,13)$	A_6, A_{12}	4
No.80	$X_{18,20} \subset \mathbb{P}(5,6,7,9,11)$	A_2, A_6, A_{10}	4
No.81	$X_{18,22} \subset \mathbb{P}(2,5,9,11,13)$	A_4, A_{12}	6
No.82	$X_{20,21} \subset \mathbb{P}(3,4,7,10,17)$	A_1, A_{16}	5
No.83	$X_{18,30} \subset \mathbb{P}(6,8,9,10,15)$	$2\times A_1, 2\times A_2, A_7, A_4$	5
No.84	$X_{24,30} \subset \mathbb{P}(8,9,10,12,15)$	A_1, A_3, A_8, A_2, A_4	6

Table 2. Fletcher's List of 84 Codimension 2 Weighted K3 surfaces.

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Resumen

Describimos una correspondencia entre dos enlaces asociados a un mismo espacio K3 que soporta a lo más, singularidades cíclicas de tipo *orbifold*. Esta dualidad se hace evidente cuando dos elementos, uno en el interior y el otro en la frontera del cono de Kähler, son identificados. Denominamos a esta correspondencia ∂*-dualidad.* También discutimos las consecuencias de ∂-dualidad al nivel de estructuras riemaniannas.

Palabras Clave: Geometría diferencial, geometría algebraica, espacio de órbitas, superficies K3, submersiones riemannianas.

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