Invariant measures on polynomial quadratic Julia sets with no interior

Alfredo Poirier [1]

November, 2014

Abstract

We characterize invariant measures for quadratic polynomial Julia sets with no interior. We prove that besides the harmonic measure —the only one that is even and invariant—, all others are generated by a suitable odd measure.

MSC(2010): 30D05, 37F05; 37F50, 37L40.

Keywords: Holomorphic dynamics, iteration of polynomials, Mandelbrot set, invariant measures.

[1] *Sección Matemáticas, Departamento de Ciencias, PUCP.*

1. Introduction

For a given a degree $d \geq 2$ polynomial, the **filled Julia set** is the set of points that have bounded orbit under iteration. We denote this set by K. It is well known that this set is a compact invariant subset of $\mathbb{C}$. For this and several other facts related to iteration of rational functions we refer the reader to [3].

In this paper we are concerned with the algebra of continuous functions defined on K. For them, our starting point is the following classical setting.

Let $C(K)$ be the algebra of continuous functions defined on a compact set $K \subset \mathbb{C}$ with values in $\mathbb{C}$. We denote by $Pol(K)$ the linear space of polynomial restrictions to K.

Theorem 1.1 (Lavrientiev, Mergelyan [1],[4])**.** *Let K be a compact set of the plane whose interior is empty. If the complement of K is connected, then $Pol(K)$ is dense in $C(K)$ in the uniform topology.* □

Along this work, $P(z) = z^2 + c$ is a degree two polynomial whose filled Julia set K has no interior. Therefore K is compact with empty interior and connected complement, and Lavrientiev's theorem applies.

2. The harmonic decomposition

Let $P(z) = z^2 + c$ be a degree two polynomial whose filled Julia set K has no interior. As this set is symmetric by the involution $z \mapsto -z$, it is safe to define even and odd objects using a standard procedure.

Given $f \in C(K)$, continuous, its **even** and **odd** parts are defined by the averages

$$\mathcal{E}(f)(z) = \frac{f(z) + f(-z)}{2}, \qquad \mathcal{O}(f)(z) = \frac{f(z) - f(-z)}{2},$$

respectively.

Lemma 2.1. *The odd and even parts of f are continuous functions with norm not bigger than $||f||$. If f is a (restriction of a) polynomial, so are $\mathcal{E}(f)$ and $\mathcal{O}(f)$.*

Proof. Both claims are elementary. □

A continuous function f is **even** if $\mathcal{E}(f) = f$, and **odd** if $\mathcal{O}(f) = f$. Alternatively, as $f = \mathcal{O}(f) + \mathcal{E}(f)$ holds, we have that f to be even is equivalent to $\mathcal{O}(f) = 0$ (that is to $f(z) = f(-z)$), while f to be odd is equivalent to $\mathcal{E}(f) = 0$ (or to $f(z) = -f(-z)$).

Also note that $\mathcal{E}, \mathcal{O} : C(K) \to C(K)$, which recover in turn the symmetric and antisymmetric part, are norm 1 operators. Both $\mathcal{E}$ and $\mathcal{O}$ are projections.

Lemma 2.2 (Reduction lemma)**.** *A continuous function $f \in C(K)$ is even if and only if there exists $g \in C(K)$ such that $f(z) = g(P(z))$. In particular, we have $\mathcal{E}(f \circ P) = f \circ P$ for all $f \in C(K)$. On the other side, if $g(P(z))$ is continuous, then $g(z)$ is continuous. We always have $||f|| = ||g||$. Anyway, f is a polynomial if and only if g is a polynomial.*

Proof. If f is even, as K is closed and $P : K \to K$ is surjective and proper, we have that f factors through P. Conversely, $g(P(z))$ is always even and continuous.

It is clear that f peaks at z_0 if and only if g peaks at $P(z_0)$ and that f is continuous at z_0 if and only if g is continuous at $P(z_0)$.

That g is a polynomial (when f is) was already indicated in Lemma 2.1. □

This lemma gives rise to a unique **even—odd decomposition**

$$f(z) = f_0(z) + g_0(P(z)),$$

where f_0 is odd and g_0 continuous. If we again split g_0 into its odd and even parts as $g_0(z) = f_1(z) + g_1(P(z))$, we get

$$f(z) = f_0(z) + f_1(P(z)) + g_1(P^{\circ 2}(z)).$$

We can continue this process indefinitely.

Proposition 2.3. *Fix $n \geq 0$. For every $f \in C(K)$ there are unique odd continuous functions $f_0, f_1, \ldots f_n$ and $g_n \in C(K)$ subject to*

$$f(z) = f_0(z) + f_1(P(z)) + \cdots + f_n(P^{\circ n}(z)) + g_n(P^{\circ n+1}(z)).$$

Here we have $||f_i|| \leq ||f||$ and $||g_n|| \leq ||f||$.

Proof. Apply induction to the odd—even decomposition of f. □

Corollary 2.4. *In the decomposition above we have*

$$|f_0(z) + f_1(P(z)) + \cdots + f_n(P^{\circ n}(z))| \leq 2||f||.$$

Proof. Indeed, the partial sum is bounded by $|f(z)|+|g_n(P^{\circ n+1}(z))|$. □

The decomposition displayed in Proposition 2.3 is much simpler for polynomials as the process eventually reaches a deadlock.

Proposition 2.5. *Given a polynomial $F \in Pol(K)$, there exist a constant $H(F)$ subject to $|H(F)| \leq ||F||$, and a finite number of odd polynomials, say $F_0, F_1, \ldots F_n$, such that*

$$F(z) = H(F) + F_0(z) + F_1(P(z)) + \cdots + F_n(P^{\circ n}(z)).$$

Those elements are uniquely determined.

Proof. A trivial induction in the degree of F. □

The assignment $H : Pol(K) \to \mathbb{C}$ clearly is linear and annihilates all odd polynomials, hence the symmetry formula $H(\mathcal{E}(f)) = H(f)$. Also, for $F \in Pol(K)$ we have $H(F \circ P) = H(F)$ and $|H(F)| \leq ||F||$.

Theorem 2.6. *Suppose the filled Julia set $K = K(P)$ has no interior. Then there exists a unique norm 1 even invariant measure supported on the Julia set that agrees with H on polynomials. In other words, for all $f \in C(K)$ the measure H satisfies*

- $H(f) = \int f(z)\, dH(z) = \int f(P(z))\, dH(z)$ *(invariance),*
- $H(f) = \int f(z)\, dH(z) = \int \mathcal{E}(f)(z)\, dH(z) = H(\mathcal{E}(f))$ *(symmetry).*

Proof. In fact, when K has no interior, polynomial restrictions to K are dense in $C(K)$. As $H : Pol(K) \to \mathbb{C}$ is continuous, it can be extended uniquely to all of $C(K)$. Since all other properties are satisfied for polynomials, they are satisfied for continuous functions as well. □

As Lyubich proved (cf. [2]), the harmonic measure already satisfies the properties stated in the theorem, so H is actually the **harmonic measure of** K. This corollary is actually true for all polynomial Julia sets. Functions for which the harmonic integral vanish (i.e, f such that $H(f) = 0$) are **harmonic free functions**. For simplicity, we will write H_f for $H(f)$.

Next we retrace our steps with these results in mind. First we apply the odd—even decomposition to the function $f(z) - H_f$ in order to obtain

$$f(z) - H_f = f_0(z) + f_1(P(z)) + \ldots + f_n(P^{\circ n}(z)) + e_n(P^{\circ n+1}(z)).$$

where $f_0, \ldots, f_n$ are odd.

Notice that here we have $0 = H(f_0) = H(f_1) = \ldots = H(f_n)$ because H is even and invariant. We also get $H(e_n) = 0$ by linearity (together with invariance). From our previous work we get further estimates.

Lemma 2.7. *We have $||e_n|| \leq ||f - H_f|| \leq 2||f||$.* □

Corollary 2.8. *For $n < m$ we get*

$$||f_{n+1}(P^{\circ n+1}(z)) + \ldots + f_m(P^{\circ m}(z))|| \leq 2||e_n||.$$

Proof. From Proposition 2.3 with

$$e_n(P^{\circ n+1}(z)) = f_{n+1}(P^{\circ n+1}(z)) + \ldots + f_m(P^{\circ m}(z)) + e_m(P^{\circ m+1}(z))$$

in the role of f we get $||e_m|| \leq ||e_n||$. Then we apply several times Lemma 2.2 and reduce to

$$e_n(z) = f_{n+1}(z) + \ldots + f_m(P^{\circ m-n}(z)) + e_m(P^{\circ m-n+1}(z)).$$

From here we conclude

$$||f_{n+1}(z) + \ldots + f_m(P^{\circ m-n}(z))|| \leq ||e_n|| + ||e_m|| \leq 2||e_n||.$$

□

Lemma 2.9. *If K have no interior, then $||e_n|| \to 0$.*

Proof. Given $\epsilon > 0$, choose a polynomial Q so that $|f(z)-H_f-Q(z)| \leq \epsilon$ on K. Expand Q as $Q(z) = H_Q + \sum_{i=0}^{N} Q_i(P^{\circ i}(z))$. Then, by uniqueness, for $n > N$ we get

$$\begin{aligned} f(z) - H_f - Q(z) &= H_Q + \sum_{i=0}^{N} f_i(P^{\circ i}(z)) - Q_i(P^{\circ i}(z)) \\ &+ \sum_{i=N+1}^{n} f_i(P^{\circ i}(z)) + e_n(P^{\circ n+1}(z)). \end{aligned}$$

Finally, Lemma 2.7 yields $||e_n|| \leq 2||f - H_f - Q|| \leq 2\epsilon$ when applied to $f - H_f - Q$. □

The expansion in the next theorem is **the harmonic decomposition of f.**

Theorem 2.10. *Let K have no interior. Then for $f \in C(K)$ there are odd continuous functions $f_0, f_1, \ldots$ such that*

$$f(z) = H_f + f_0(z) + f_1(P(z)) + f_2(P^{\circ 2}(z)) + \ldots;$$

the convergence here is uniform.

Proof. In fact, for $m \geq n \geq N$ we have

$$||f_{n+1}(P^{\circ n+1}(z)) + \ldots + f_m(P^{\circ m}(z))|| \leq 2||e_n|| \to 0.$$

So, the partial sums form a Cauchy sequence since the discrepancy e_n tends to 0. □

The next result is trivial after inspecting grand orbits. Anyhow, we present an alternative proof.

Lemma 2.11 (Lyubich [2])**.** *If K has no interior, the only invariant continuous functions are the constants.*

Proof. In fact, if $f = f \circ P$, then by matching their harmonic decompositions we get $H_f = H_{f \circ P}$ together with $f_0 = 0, f_1 = f_0, f_2 = f_1, \ldots$. Hence $f(z) = H_f$ is a constant. □

3. The dual decomposition

For the study of measures supported in K we will take the functional analysis approach. Thus, a "measure" on K "is" a linear functional (with values in $\mathbb{C}$) defined on $C(K)$. We denote by $\mathcal{M}(K)$ the space of (complex valued) measures.

Given a measure ν, the **odd** and **even parts** are given by

$$\mathcal{O}(\nu)(f) = \nu(\mathcal{O}(f)) \qquad \mathcal{E}(\nu)(f) = \nu(\mathcal{E}(f)).$$

Evidently, we get $||\mathcal{O}(\nu)||, ||\mathcal{E}(\nu)|| \leq ||\nu||$ (because at the level of functions we have $||\mathcal{O}||, ||\mathcal{E}|| \leq 1$). Also, note the equality $\nu = \mathcal{O}(\nu) + \mathcal{E}(\nu)$. The measure $\mathcal{O}(\nu)$ is odd in the sense that it kills all even functions, while $\mathcal{E}(\nu)$ is even as it kills the odd functions.

Example 3.1. For the delta mass δ_{z_0} based at a point $z_0 \in K$, the even part $\mathcal{E}(\delta_{z_0})$ is given by $\frac{1}{2}\sum_{P(\hat{z})=P(z_0)} \delta_{\hat{z}} = \frac{\delta_{z_0}+\delta_{-z_0}}{2}$. In fact, we get

$$\mathcal{E}(\delta_{z_0})(f) = \mathcal{E}(f)(z_0) = \frac{1}{2}\sum_{P(\hat{z})=P(z_0)} f(\hat{z}) = \left(\frac{1}{2}\sum_{P(\hat{z})=P(z_0)} \delta_{\hat{z}}\right)(f).$$

As a by-product we obtain

$$\mathcal{O}(\delta_{z_0}) = \frac{\delta_{z_0} - \delta_{-z_0}}{2}.$$

It is important to set some notation straight. Instead of the customary $d\nu(z)$ we will use $\nu(z)$ most of the time. In this way, given $f \in C(K)$, we write

$$\nu(f) = \int f\,\nu = \int f(z)\,\nu(z)$$

when needed. We will even use $\nu(z)$, meaning ν, when the context calls for it.

The measure $\nu \circ P$ (or $\nu(P(z))$ in brief) is by convention the even measure that satisfies

$$\int g(P(z))\nu(P(z)) = \int g(z)\nu(z).$$

Example 3.2. The harmonic measure is even as the relation $H(f) = H(\mathcal{E}(f))$ is equivalent to $H(f) = \mathcal{E}(H)(f)$.

Also, for $f \in C(K)$ we set $h_f(z) = f(z)\,H(z)$, where $H(z)$ is the standard harmonic measure as defined in Section 2. Then we have $h_f(P(z)) = f(P(z))\,H(z)$. In fact, as both of the above measures are even, it is enough to check the equality

$$\int g(P(z))\,h_f(P(z)) = \int g(P(z)) f(P(z))\,H(z).$$

For them, however, we readily get

$$\begin{aligned}\int g(P(z))\, h_f(P(z)) &= \int g(z)\, h_f(z)\\ &= \int g(z) f(z)\, H(z)\\ &= \int g(P(z)) f(P(z))\, H(z),\end{aligned}$$

where the fist equality is given by convention, the second by definition of h_f, and the third by the invariance of the harmonic measure. This notable fact is what justifies our convention for the dynamical push-forward of the measure.

Example 3.3. We claim that

$$\mathcal{E}(\delta_{z_0}) = \frac{\delta_{z_0} + \delta_{-z_0}}{2} = \delta_{z_1} \circ P$$

holds (as usual, we have $P(z_0) = z_1$). In fact, let $f(z) = f_0(z) + g(P(z))$ with f_0 odd. Then we have

$$\begin{aligned}\int f_0(z) + g(P(z))\, \delta_{z_1}(P(z)) &= \int g(P(z))\, \delta_{z_1}(P(z))\\ &= \int g(z)\, \delta_{z_1}(z)\\ &= g(z_1).\end{aligned}$$

On the other side, oddness of f_0 implies $\frac{1}{2}\sum_{P(\hat{z})=z_1} f_0(\hat{z}) = 0$, so we get

$$\int f_0(z) + g(P(z))\,(\frac{1}{2}\sum_{P(\hat{z})=z_1} \delta_{\hat{z}}) = \frac{1}{2}\sum_{P(\hat{z})=z_1} (f_0(\hat{z}) + g(P(\hat{z}))) = g(z_1).$$

Thus, the two values coincide, and the measures agree.

An easy induction delivers $\delta_{z_n} \circ P^{\circ n} = \dfrac{1}{2^n} \displaystyle\sum_{P^{\circ n}(\hat{z})=z_n} \delta_{\hat{z}}$ as well, for $z_n = P^{\circ n}(z_0)$.

Lemma 3.4. *The measures τ and $\tau \circ P$ have the same norm.*

Proof. Notice that $\tau(f) = \tau \circ P(f \circ P)$ implies $||\tau|| \leq ||\tau \circ P||$.

Now take $f + g \circ P$ with f odd subject to $||f + g \circ P|| \leq 1$. Then $||g|| = ||g \circ P|| = ||\mathcal{E}(f + g \circ P)|| \leq 1$ forces

$$||\tau \circ P(f + g \circ P)|| = ||\tau \circ P(g \circ P)|| = ||\tau(g)|| \leq ||\tau||\,||g|| \leq ||\tau||.$$

□

Lemma 3.5. *All even measures have the form $\tau \circ P$ for some $\tau \in \mathcal{M}(K)$.*

Proof. Let ν be a measure that kills all odd functions. For the functional

$$\tau(f) = \int f(P(z))\,\nu(z),$$

the convention $\int f(z)\,\tau(z) = \int f(P(z))\,\tau(P(z))$ joining forces with the symbolism $\tau(f) = \int f(z)\,\tau(z)$ leads us to $\nu(z) = \tau(P(z))$. □

In view of Lemma 3.5, we have a natural splitting

$$\nu(z) = \nu_0(z) + \sigma(P(z)),$$

where ν_0 is odd. However, before iterating this odd–even decomposition, practice gained in the manipulation of continuous functions suggests we better subtract the "harmonic" part first. For that, we set

$$H_\nu = \nu(1) = \int \nu(z).$$

Whenever we have $H_\nu = 0$, we say that ν is **harmonic free**.

Proposition 3.6. *Fix $n \geq 0$. There are unique odd measures $\nu_0, \ldots \nu_n$ and a measure τ_n such that*

$$\nu(z) = H_\nu\, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z)) + \tau_n(P^{\circ n+1}(z)).$$

This decomposition is unique provided $H_{\tau_n} = 0$. In this case we have $||\tau_n|| \leq 2||\nu||$ and $||\nu_i|| \leq ||\nu||$.

Proof. Trivial. □

What is not trivial is the following asymptotic decomposition.

Theorem 3.7. *If K has no interior, the partial sums*

$$H_\nu \, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z))$$

converge $*$-weak to ν.

Proof. Key here is to understand how

$$\nu(z) = H_\nu \, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z)) + \tau_n(P^{\circ n+1}(z))$$

acts on the function

$$f(z) = H_f + f_0(z) + f_1(P(z)) + \ldots + f_n(P^{\circ n}(z)) + e_n(P^{\circ n+1}(z)).$$

To begin with, by definition H_f is the way how $dH(z)$ acts on f. Therefore $H_\nu \, dH(z)$ paired against f gives $H_f H_\nu$.

Next, $\nu_i(P^{\circ i}(z))$ acts on $g(P^{\circ m}(z))$, with $m > i$, as $\nu_i(z)$ acts on $G(P^{\circ m-i}(z))$, hence kills them all since ν_i is odd and the said functions are even. This applies to H_f, $f_m(P^{\circ m}(z))$, for $m > i$, and to $e_n(P^{\circ n+1}(z))$. When $m < i$ then $\nu_i(P^{\circ i}(z))$ acts on $f_m(P^{\circ m}(z))$ in the same way as $\nu_i(P^{\circ i-m}(z))$ acts on $f_m(z)$, thus annihilating them. We also have $\displaystyle\int f_i(P^{\circ i}(z))\nu_i(P^{\circ i}(z)) = \int f_i(z)\nu_i(z)$ by reduction, the surviving term at this stage.

Finally, it should be clear by now that $\tau_n(P^{\circ n+1}(z))$ annihilates all the f_i. Also, evaluating at the constant function 1 we get

$$\nu(1) = H(\nu)H(1) + \nu_0(1) + \ldots + \nu_n(1) + \tau_n(1).$$

Since we have relations $\nu_i(1) = 0$ and $H_\nu H(1) = \nu(1)H(1) = \nu(1)$, we conclude the equality $\tau_n(1) = 0$. Therefore, $\tau_n(P^{\circ n+1}(z))$ acts merely on $e_n(P^{\circ n+1}(z))$.

Collecting our findings we obtain

$$\int f(z)\nu(z) = H_f H_\nu + \sum_{i=0}^{n} \int f_i(z)\nu_i(z) + \int e_n(z)\tau_n(z).$$

By the above formula, the action on f of $\nu(z) - \tau_n(P^{\circ n+1}(z))$ (i.e, of $H_\nu\, dH(z) + \nu_0(z) + \cdots + \nu_n(P^{\circ n}(z)))$ is $H_f H_\nu + \sum_{i=0}^{n} \int f_i(z)\nu_i(z)$; which in turn equals $\int f(z)\nu(z) - \int e_n(z)\tau_n(z)$. However,

$$\left| \int e_n(z)\tau_n(z) \right| \leq ||e_n||\, ||\tau_n|| \leq 2||\nu||\, ||e_n||$$

converges to 0, so we are done. □

Example 3.8. We try the decomposition of a delta mass. Let $z_0 \in K$. Then we have

$$\delta_{z_0}(z) = dH(z) + \Delta_0(z) + \ldots + \Delta_n(P^{\circ n}(z)) + \ldots,$$

since the harmonic part is $\delta_{z_0}(1) = 1$.

Now take $f \in C(K)$ odd (so that $f(z) + f(-z) = 0$ for all $z \in K$). Then the formula

$$\int f(z)\, \Delta_0(z) = \int f(z)\, \delta_{z_0}(z) = f(z_0) = \frac{f(z_0)}{2} - \frac{f(-z_0)}{2},$$

shows that the odd part of δ_{z_0} is $\dfrac{\delta_{z_0} - \delta_{-z_0}}{2}$ (compare also Example 3.3).

In general, (we use here the convention $z_n = P^{\circ n}(z_0)))$ for f odd we get

$$\int f(P^{\circ n}(z))\, \Delta_n(P^{\circ n}(z)) = \int f(P^{\circ n}(z))\, \delta_{z_0}(z) = f(z_n),$$

and we conclude that Δ_n is the odd part of δ_{z_n}, that is $\dfrac{\delta_{z_n} - \delta_{-z_n}}{2}$. In short, we have

$$\delta_{z_0} = 1 + \sum_{i=0}^{\infty} \mathcal{O}(\delta_{z_i}) \circ P^{\circ i}.$$

A measure ν is **invariant** when for all $f \in C(K)$ we have

$$\int f(z)\,\nu(z) = \int f(P(z))\,\nu(z).$$

The harmonic measure $dH(z)$ and the delta masses δ_{z_f} located at fixed points z_f are prototypical examples of invariant measures. This is in sharp contrast with the function case where we only have one invariant object. Other examples of invariant measures are averages along periodic orbits.

The following is a characterization of invariant measures using the canonical decomposition.

Theorem 3.9. *Suppose K has no interior. If $\nu(z) = \alpha + \nu_0(z) + \nu_1(P(z)) + \ldots$ is an invariant measure, then $\nu_0 = \nu_1 = \nu_2 = \ldots$.*

Proof. For any odd test function f we get thanks to invariance

$$\begin{aligned}\int f(z)\nu_n(z) &= \int f(P^{\circ n}(z))\nu_n(P^{\circ n}(z))\\ &= \int f(P^{\circ n}(z))\nu(z)\\ &= \int f(z)\nu(z)\\ &= \int f(z)\nu_0(z).\end{aligned}$$

Therefore ν_n and ν_0 are the same functional. □

As an extra remark, we should indicate that not all odd functions give rise to invariant measures. For instance, we will see briefly that the odd part of a delta mass seldom determines an invariant measure.

Corollary 3.10. *If K has no interior, then the space of even invariant measures supported in K is one-dimensional.* □

Theorem 3.11. *Suppose K has no interior. Let ν_0 be an odd measure. Then the partial sums $\mu_n = \nu_0 + \nu_0 \circ P + \ldots + \nu_n \circ P^{\circ n}$ converge (∗-weak) to an invariant measure if and only if there is a constant M so that $||\mu_n|| \leq M$.*

Proof. If the sequence μ_n converges ∗-weak, then their norms certainly form a bounded sequence.

On the other side, if $||\mu_n||$ is bounded, it carries ∗-weakly convergent subsequences. Therefore it is enough to prove that for all $f \in C(K)$ the limit of $\mu_n(f)$ exists. Given $\epsilon > 0$, let N be such that for $n \geq N$ we have

$$f(z) = H_f + f_0(z) + \ldots + f_n(P^{\circ n}) + e_n(P^{\circ n+1}(z)),$$

with $||e_n|| \leq \epsilon$. When we take $m > n \geq N$, we get

$$|\mu_m(f) - \mu_n(f)| = |(\mu_n - \mu_m)(e_N \circ P^{\circ N+1})| \leq 2M||e_N|| \leq 2M\epsilon.$$

□

Example 3.12. Let $z_0 \in K$ be a non-periodic point outside the orbit of the critical point (any point with a countable number of exceptions would do). We use Theorem 3.11 to prove that the odd part of the delta mass δ_{z_0} does not generate an invariant measure.

If $\pm z_{-1}$ are the two preimages of z_0, the measures $\delta_{\pm z_{-1}}(P^{\circ i}(z))$ have total mass 1 and support $(P^{\circ i})^{-1}(\pm z_{-1})$, mutually disjoint sets. The bottom line is that

$$\sum_{i=0}^{n-1} (\frac{\delta_{z_{-1}} - \delta_{-z_{-1}}}{2}) \circ P^{\circ i}$$

has norm n.

4. Iteration and reduction

In this section we study the iteration process as an operator acting both on continuous functions and on measures. For better understanding, we introduce in parallel the process of reduction.

The **iteration operator it** is defined in continuous functions as $\mathbf{it}(f)(z) = f(P(z))$ and in measures as $\mathbf{it}(\nu)(z) = \nu(P(z))$. The **reduction operator red** is defined as follows. If $\varphi(z) = \mathcal{O}(\varphi)(z) + \psi(P(z))$, then we set $\mathbf{red}(\varphi)(z) = \psi(z)$, both for functions and measures.

When K has no interior and

$$\varphi(z) = H_\varphi + \varphi_0(z) + \sum_{i=1}^{\infty} \varphi_i(P^{\circ i}(z)),$$

with φ_i odd, holds, then we write

$$\mathbf{red}(\varphi)(z) = H_\varphi + \sum_{i=1}^{\infty} \varphi_i(P^{\circ i-1}(z)).$$

Proposition 4.1. *The adjoint operator of* $\mathbf{red} : C(K) \to C(K)$ *is given by* $\mathbf{it} : \mathcal{M}(K) \to \mathcal{M}(K)$*, while the adjoint of* $\mathbf{it} : C(K) \to C(K)$ *is* $\mathbf{red} : \mathcal{M}(K) \to \mathcal{M}(K)$*. Both are norm* 1 *operators.*

Proof. For $f \in C(K)$ let $f(z) = \mathcal{O}(f)(z) + g(P(z))$ and for $\nu \in \mathcal{M}(K)$ let $\nu(z) = \mathcal{O}(\nu)(z) + \tau(P(z))$. Then we have

$$\begin{aligned} \int f(z)\,\mathbf{red}^*(\nu)(z) &= \int \mathbf{red}(f)(z)\,\nu(z) = \int g(z)\,\nu(z) \\ &= \int g(P(z))\,\nu(P(z)) \\ &= \int \mathcal{O}(f)(z) + g(P(z))\,\nu(P(z)) \\ &= \int f(z)\,\mathbf{it}(\nu)(z). \end{aligned}$$

Therefore, we obtain $\mathbf{it}(\nu) = \mathbf{red}^*(\nu)$.

The other identity is tackled in a similar way.

About the norm, this should be obvious by now. □

Next we comment briefly about the operators $I - \lambda\,\mathbf{it}$ and $I - \lambda\,\mathbf{red}$, with $\lambda \in \mathbb{C}$.

Lemma 4.2. *Both in $C(K)$ and in $\mathcal{M}(K)$ the operators $I - \lambda\,\mathbf{it}$ and $I - \lambda\,\mathbf{red}$ are invertible for $|\lambda| < 1$.*

Proof. In fact, both **it** and **red** have norm 1. □

Lemma 4.3. *Both in $C(K)$ and in $\mathcal{M}(K)$, for $|\lambda| > 1$, the operators $I - \lambda\,\mathbf{it}$ are closed, injective but not surjective, while the $I - \lambda\,\mathbf{red}$ are closed, surjective but not injective.*

Proof. We first attack the surjectivity of $I - \lambda\,\mathbf{red}$. Given ψ in the appropriate space, we define $\varphi(z) = -\sum_{i=0}^{\infty} \psi(P^{\circ i+1}(z))/\lambda^{i+1}$. From

$$\begin{aligned}
\lambda\,\mathbf{red}(\varphi)(z) &= -\lambda \sum_{i=0}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i+1} \\
&= -\sum_{i=0}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i} \\
&= -\psi(z) - \sum_{i=1}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i} \\
&= -\psi(z) + \varphi(z),
\end{aligned}$$

we get $\{I - \lambda\,\mathbf{red}\}(\varphi) = \psi$, and the operator is surjective. Evidently, a surjective operator has closed range. Also, for any odd ψ, the element $\sum_{i=0}^{\infty} \psi(P^{\circ i}(z))/\lambda^{i}$ is well defined (since $|\lambda| > 1$) and belongs to the kernel of $I - \lambda\,\mathbf{red}$.

The properties for the operator $I - \lambda\,\mathbf{it}$ follow by duality. □

When $|\lambda| = 1$, the study of those operators is not simple. We will be concerned specially with the case $\lambda = 1$, since they help characterize invariant measures.

Proposition 4.4. *For $\nu \in \mathcal{M}(K)$ the following properties are equivalent.*
- *The measure ν is invariant;*
- *the condition $\{I - \mathbf{it}\}(\nu) = \mathcal{O}(\nu)$ holds;*
- *the measure $\{I - \mathbf{it}\}(\nu)$ is odd;*
- *the measure ν belongs to the kernel of $I - \mathbf{red}$.*

Proof. Everything is trivial. □

Proposition 4.5. *If K has empty interior, the kernel of $I - \mathbf{it}$ is one dimensional: it consists of the constants or of the multiples of the harmonic measure, depending in the case. These operators are not closed.*

Proof. It is clear that the constants (or constant multiples of H) are the only members of the kernel of $I - \mathbf{it}$.

To prove that this operator acting on continuous functions is not closed, we note that the space of all functions annihilated by the harmonic measure is a codimension one space in where $I-\mathbf{it}$ acts injectively. Therefore it is enough to construct a sequence of harmonic free functions φ_n of norm greater or equal to 1 such that $||\{I - \mathbf{it}\}(\varphi_n)||$ converges to 0. With that in mind, let z_f be a non-critical fixed point of P. Let $F : K \to [-1, 1]$ be any continuous function such that $F(z_f) = 1$ and $F(-z_f) = -1$. Write $F_0 = \mathcal{O}(F)$. Notice that $F(z_f) = -F(-z_f)$ implies $F_0(z_f) = F(z_f) = 1$. Therefore we get $1 \leq ||F_0|| \leq ||F|| = 1$. Now for $\varphi_n(z) = \dfrac{1}{n}\displaystyle\sum_{i=0}^{n-1} F_0(P^{\circ i}(z))$ we have $\varphi_n(z_f) = 1$, and so $||\varphi_n|| \geq 1$. However by construction the function $\{I - \mathbf{it}\}(\varphi_n)(z) = \dfrac{F_0(z) - F_0(P^{\circ n}(z))}{n}$ has norm at most $2/n$.

For measures we proceed similarly: for z_0 a point that is not eventually periodic (compare Example 3.12), we take the odd measure $\nu_0 = \mathcal{O}(\delta_{z_0})$ and define $\varphi_n(z) = (1/n)\sum_{i=0}^{n-1} \nu_0(P^{\circ i}(z))$. A trivial calculation gives then $||\varphi_n|| = 1$ and $||\{I - \mathbf{it}\}(\varphi_n)|| = 2/n$. □

Corollary 4.6. *If K has empty interior, the image of $Id - \mathbf{red}$ is dense in the space of harmonic free objects. This operator is not closed.*

Proof. This follows from Proposition 4.5 by duality. □

References

[1] T. Gamelin; *Uniform Algebras*, American Mathematical Society, 2005.

[2] M. Lyubich; *The measure of maximal entropy of a rational endomorphism of a Riemann sphere,* Funktsional. Anal. i Prilozhen. 16 (1982), no. 4, 78-79. (Russian).

[3] J. Milnor; *Dynamics in One Complex Variable, Introductory Lectures,* Vieweg, 1999.

[4] W. Rudin; *Real and Complex Analysis*, McGraw Hill, 1974.

Resumen

En este artículo caracterizamos medidas invariantes sobre conjuntos de Julia sin interior asociados con polinomios cuadráticos. Probamos que más allá de la medida armónica —la única par e invariante—, el resto son generadas por su parte impar.

Palabras clave: Dinámica holomorfa, iteración de polinomios, conjunto de Maldelbrot, medidas invariantes.

Alfredo Poirier
Sección Matemáticas
Departamento de Ciencias
Pontificia Universidad Católica del Perú
Av. Universitaria 1801, San Miguel, Lima 32, Perú
apoirie@pucp.edu.pe