Cremona Symmetry in Gromov-Witten Theory

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Abstract

We establish the existence of a symmetry within the Gromov-Witten theory of $\mathbb{CP}^n$ and its blowup along points. The nature of this symmetry is encoded in the Cremona transform and its resolution, which lives on the toric variety of the permutohedron. This symmetry expresses some difficult to compute invariants in terms of others less difficult to compute. We focus on enumerative implications; in particular this technique yields a one line proof of the uniqueness of the rational normal curve. Our method involves a study of the toric geometry of the permutohedron, and degeneration of Gromov-Witten invariants.

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1. Introduction

1.1 Overview

How many rational curves of degree d pass through r general points in $\mathbb{CP}^n$? Call this number $N_{d,r}^n$. Determining $N_{d,r}^n$ is at the heart of classical enumerative geometry and is captured by the stationary genus-0 Gromov-Witten theory of $\mathbb{P}^n$.

For a smooth projective variety X, a collection of cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X, \mathbb{Z})$, and an effective curve class $\beta \in H_2(X, \mathbb{Z})$, the genus-$g$, class β, Gromov-Witten invariant of X with insertions $\{\gamma_i\}$ is denoted

$$\langle \gamma_1, \ldots, \gamma_r \rangle_{g,\beta}^X.$$

These invariants contain enumerative information, but only in some cases do they precisely correspond to the number of genus-g curves in X of class β intersecting the Poincaré dual of each γ_i. Such invariants are simply called **enumerative**.

The term **stationary** refers to Gromov-Witten invariants with only point insertions. Since $\mathbb{P}^n$ is convex, its (genus-0) Gromov-Witten theory is enumerative. Hence, computing such Gromov-Witten invariants yields exact enumerative information,

$$\langle pt^r \rangle_{0,d}^{\mathbb{P}^n} = N_{d,r.}^n$$

For this reason, and others, the genus-0 Gromov-Witten theory of $\mathbb{P}^n$ is of basic interest in enumerative geometry and Gromov-Witten theory, and is well studied; indeed, it has been completely determined using several different methods. These include two of the most important tools in Gromov-Witten theory: localization and Virasoro constraints.

Computing Gromov-Witten invariants via localization consists of a two step process. Graber-Pandharipande [10] showed that the (all genus) Gromov-Witten theory of any nonsingular complex projective variety with a strong torus action may be reduced to Hodge integrals

via virtual localization. In turn, Faber-Pandharipande [5] provides an algorithm computing such Hodge integrals.

Alternatively, one may compute using Virasoro constraints. The Virasoro constraints are an infinite set of differential equations which conjecturally are satisfied by the Gromov-Witten generating series of a given target space. The Gromov-Witten invariants of the target are completely determined by these constraints. The Virasoro conjecture has been proved for $\mathbb{P}^n$; the Virasoro conjecture for target curves, including $\mathbb{P}^1$, was proved by Okounkov-Pandharipande [20], and the general case was proved by Givental [8].

Here we introduce a new method for the computation of certain genus-0 Gromov-Witten invariants of $\mathbb{P}^n$ and its blow-up along points. This method exploits a new symmetry of the invariants, which arises from the geometry of the Cremona transform and its resolution, which lives on the toric variety associated to the permutohedron, Π_n. This symmetry expresses some difficult to compute invariants in terms of others that are less difficult to compute. In this way, where it applies, it is often very computationally effective, yielding a computational tool for enumerative geometry. Additionally, this symmetry has not been observed using other techniques, including those mentioned above, and as such yields new insight into the structure of the genus-0 Gromov-Witten theories of $\mathbb{P}^n$ and its blow-up along points.

The permutohedral variety is of independent interest and admits a modular description. The permutohedral variety $X_{\Pi_{(n-1)}}$ was first constructed as an iterated toric blow-up by Kapranov in [13, Section 4.3]. Losev and Manin in [18] then proved $X_{\Pi_{(n-1)}}$ is isomorphic to the moduli space $\overline{M}_{0,2|n}$ of chains of $\mathbb{P}^1$'s with marked points $x_0 \neq x_\infty$ and $y_1, \ldots, y_n$, where the points y_i may collide but not with x_j; here we use the notation of [1, 19]. In addition, the permutohedral variety is isomorphic to the Hassett space $\overline{M}_{0,(1,1,1/n,\ldots,1/n)}$ of weighted pointed curves, here of genus zero, with two points of weight one and n points of weight $1/n$ [19]. The permutohedral variety (by definition) admits an S_n action by permuting the n points generating the permutohedron. But the

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permutohedron is also symmetric about the origin, and hence admits an S_2 action. (This symmetry resolves the Cremona transform on projective space, as explained below.) The cohomology of the permutohedral variety is thus a representation of $S_2 \times S_n$. Bergstöm and Minabe study the cohomology of the permutohedral variety, compute the character of this representation, and provide an excellent introduction to the subject in [1]. Thus, our description of the action and geometry of Cremona symmetry on the cohomology of the permutohedral variety may also be interpreted in terms of the cohomology of the associated Losev-Manin and Hassett spaces.

This work builds upon the work of many others. The Cremona transform was first studied in the context of Gromov-Witten theory on $\mathbb{P}^2$ by Crauder-Miranda [4] and Göttsche-Pandharipande [9]. This technique on $\mathbb{P}^2$ was used with success by Bryan-Leung [3]. The $\mathbb{P}^3$ case was proved by Bryan and the first author [2], used by the first author and Liu-Mariño [14], and was inspired by the beautiful work of Gathmann [7].

Cremona symmetry is an example of **toric symmetry** as a computational tool in Gromov-Witten theory. Toric symmetry of $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ have been studied in [15] and [16], respectively. Cremona symmetry of $\mathbb{P}^n$ is the first higher-dimensional example of toric symmetry in Gromov-Witten theory to be studied.

1.2   Summary

In what remains of the introduction we establish the notation necessary to state the main results, proceed to do so and discuss applications of the results to enumerative geometry. We then move towards a proof of our results, beginning with a description of the Cremona transform on $\mathbb{P}^n$, which leads to a discussion of the blow-up of $\mathbb{P}^n$ along points, which we call X, the permutohedron Π_n, and its toric variety X_{Π_n}.

We then study the geometry and intersection theory of the permutohedral variety, X_{Π_n}, and its blow-up along points, $\widehat{X}$. We find a symmetry of the polytope of X_{Π_n} which yields a nontrivial action on the cohomology of $\widehat{X}$. Since Gromov-Witten invariants are functorial under

isomorphism, this action on cohomology ascends to the Gromov-Witten theory of $\widehat{X}$, producing a symmetry of the Gromov-Witten invariants of $\widehat{X}$. This will establish our main result (Theorem 1.1).

However, we are interested in the invariants of $\mathbb{P}^n$ and X, its blow-up along points, as opposed to the invariants of $\widehat{X}$, which is a much more complicated iterated blow-up. So, we must prove that invariants of $\widehat{X}$ descend to X. We use degeneration to do so, in Corollary 1.2.

The cut-down moduli stack introduced by Bryan-Leung [3] allows one to trade point insertions for conditions of passing through blow-up points (Proposition 4.1). Thus the stationary genus-0 invariants of $\mathbb{P}^n$ are equal to certain invariants of X; Theorem 1.1 provides a symmetry of the invariants of X, thus exposing a symmetry of the invariants of $\mathbb{P}^n$ itself.

1.3   Preliminaries

Let X be a nonsingular complex projective variety and $\beta \in H_2(X;\mathbb{Z})$ be a curve class in X. The moduli stack $\overline{M}_{0,r}(X,\beta)$ parametrizes isomorphism classes of stable maps

$$f : C \longrightarrow X$$

from possibly nodal rational curves $(C, p_1, \ldots, p_r)$ with r marked points to X representing β. This moduli space comes equipped with a virtual fundamental class $[\overline{M}_{0,r}(X,\beta)]^{\mathrm{vir}}$ of dimension

$$\mathrm{vdim}\left(\overline{M}_{0,r}(X,\beta)\right) = (\dim X - 3) - K_X \cdot \beta + r,$$

where K_X is the canonical divisor class on X.

The genus-0, primary, stationary Gromov-Witten invariants of X are defined by integration over this virtual fundamental class as

$$\langle \mathrm{pt}^r \rangle_\beta^X = \int_{[\overline{M}_{0,r}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{r} \mathrm{ev}_i^*(\mathrm{pt}),$$

where $\mathrm{ev}_i : \overline{M}_{0,r}(X,\beta) \to X$ is the ith evaluation map given by

$$[f : (C, p_1, \ldots, p_r) \to X] \longmapsto f(p_i)$$

and $\mathrm{pt} \in H^0(X; \mathbb{Q})$ denotes the class Poincaré dual to a point in X.

For foundational results in Gromov-Witten theory, we refer the reader to the excellent book [12].

Let $\pi : \tilde{X} \to X$ be the blow-up of X along a subvariety $Z \subset X$. An effective curve class $\tilde{\beta}$ in $H_2(\tilde{X}; \mathbb{Z})$ is **nonexceptional** with respect to X if

$$\langle \mathrm{pt}^r \rangle_{\tilde{\beta}}^{\tilde{X}} = \langle \mathrm{pt}^r \rangle_{\pi_* \tilde{\beta}}^{X}.$$

In this case, we also refer to $\beta = \pi_* \tilde{\beta} \in H_2(X; \mathbb{Z})$ as **nonexceptional**.

1.4   Main Results

Theorem 1.1. *Let X be the blow-up of $\mathbb{P}^n$ along $m > n + 1$ general points $(p_1, \ldots, p_m) \in \mathbb{P}^n$. Let $h \in H_2(X; \mathbb{Z})$ denote the pullback of the class of a general line in $\mathbb{P}^n$, and let $e_i \in H_2(X; \mathbb{Z})$ denote the class of a line in the exceptional divisor over p_i.*

Let $\beta = dh - \sum_{i=1}^m a_i e_i \in H_2(X; \mathbb{Z})$, for $d \in \mathbb{Z}_{\geqslant 0}$ and $a_i \in \mathbb{Z}$. Suppose β is a nonexceptional class in X with respect to $\hat{X}$ (constructed in Section 2). Then

$$\boxed{\langle \mathrm{pt}^k \rangle_{\beta}^{X} = \langle \mathrm{pt}^k \rangle_{\beta'}^{X},}$$

where $\beta' = d'h - \sum a_i' e_i$, and $d', a_i' \in \mathbb{Z}$ are given by

$$d' = nd - (n-1) \sum_{i=1}^{n+1} a_i$$

$$a_i' = \begin{cases} d - \displaystyle\sum_{\substack{j \neq i \\ j \leqslant n+1}} a_j & 1 \leqslant i \leqslant n+1 \\ a_i & i > n+1. \end{cases}$$

Corollary 1.2. *The class $\beta = nh - e_1 - \cdots - e_{n+3} \in H_2(X; \mathbb{Z})$ is nonexceptional. Therefore*

$$\langle \ \rangle_{nh - e_1 - \cdots - e_{n+3}}^{X} = \langle \ \rangle_{h - e_{n+2} - e_{n+3}}^{X} = 1.$$

Remark 1.3. The invariant $\langle \ \rangle^X_{nh-e_1-\cdots-e_{n+3}}$ counts the number of rational curves in $\mathbb{P}^n$ of degree n and passing through $n+3$ general points. The rational normal curve has these properties. The central equality is, of course, an application of Theorem 1.1. The last equality corresponds to the fact that there is a unique line through two general points in $\mathbb{P}^n$.

So, Corollary 1.2 is a Gromov-Witten theoretic proof of the uniqueness of the rational normal curve through $n+3$ points. This is of course very classical; a good reference is a combination of two theorems in Harris [11, Theorems 1.18, 18.9].

2. The Cremona Transform

The Cremona transform is the rational map

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

defined by

$$(x_0 : \cdots : x_n) \mapsto (x_1 \cdots x_n : x_0 x_2 \cdots x_n : \cdots : \prod_{j \neq i} x_j : \cdots : x_0 \cdots x_{n-1}).$$

Note that on the open set $U = \{(x_0 : x_1 : \cdots : x_n) \in \mathbb{P}^n \mid x_i \neq 0\}$, the Cremona transform has the familiar form

$$(x_0 : x_1 : \cdots : x_n) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \cdots : \frac{1}{x_n} \right).$$

Note that the Cremona transform is undefined precisely on the set

$$\{(x_0 : \cdots : x_n) \mid x_j = x_k = 0 \text{ for some } j \neq k\},$$

which is the union of all torus-invariant subvarieties of $\mathbb{P}^n$ of codimension at least 2 under the standard torus $\mathbb{T} = (\mathbb{C}^\times)^n$ action on $\mathbb{P}^n$ given by

$$(\lambda_1, \ldots, \lambda_n) \cdot (x_0 : \cdots : x_n) = (x_0 : \lambda_1 x_1 : \cdots : \lambda_n x_n).$$

So, we resolve the Cremona transform by iteratively blowing up along these $\mathbb{T}$-invariant subvarieties.

Let $X = X_0 = X_0(\mathfrak{m})$ denote the blow-up of $\mathbb{P}^n$ at $\mathfrak{m}$ distinct points, where $\mathfrak{m} \geqslant n + 1$. When $\mathfrak{m} > n + 1$, note X_0 is not a toric variety. However we perform an iterated blow-up of $\mathbb{P}^n$, beginning with X_0, of a very toric flavor.

Let $\{p_0, \ldots, p_n\}$ denote the torus fixed points of $\mathbb{P}^n$ under the standard action of $\mathbb{T} = (\mathbb{C}^\times)^n$. The $\mathbb{T}$-invariant subvarieties of $\mathbb{P}^n$ are indexed by subsets of $\{0, \ldots, n\}$, and we perform an iterated blow-up of X_0 along (the proper transforms of) these subvarieties.

First, let $\mathcal{Z}_0 = \{p_0, \ldots, p_\mathfrak{m}\}$ and, again, denote by

$$X_0 = \mathrm{Bl}_{\mathcal{Z}_0} \mathbb{P}^n$$

the blow-up of $\mathbb{P}^n$ along $\mathcal{Z}_0$.

We now iterate, and blow-up $X_j = X_j(\mathfrak{m})$ along the proper transform of $\mathbb{T}$-invariant subvarieties of dimension $j+1$ to obtain $X_{j+1} = X_{j+1}(\mathfrak{m})$. For $0 \leqslant j < n - 2$, let $Z_{i_0, \ldots, i_{j+1}} \subset X_j$ denote the proper transform of the $(j + 1)$-dimensional $\mathbb{T}$-invariant subvariety of $\mathbb{P}^n$ through $\{p_{i_0}, \ldots, p_{i_{j+1}}\} \subseteq \{p_0, \ldots, p_n\}$. Now set

$$\mathcal{Z}_{j+1} = \bigcup_{\substack{\alpha \subset \{0, \ldots, n\} \\ |\alpha| = j+2}} Z_\alpha \subset X_j.$$

To complete the construction, let $X_{j+1} = \mathrm{Bl}_{\mathcal{Z}_{j+1}} X_j$ denote the blow-up of X_j along $\mathcal{Z}_{j+1}$. So we have the iterated sequence of blow-ups

$$\hat{X} = X_{n-2} \xrightarrow{\pi_{n-2}} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 \xrightarrow{\pi_0} \mathbb{P}^n.$$

For example X_1 is the blow-up of X_0 along

$$\mathcal{Z}_1 = Z_{0,1} \cup Z_{0,2} \cup \cdots \cup Z_{n-1,n},$$

where $Z_{i,j}$ is the proper transform in X_0 of the torus invariant line l_{ij} in $\mathbb{P}^n$ through p_i and p_j.

In order to understand the geometry of this construction, it is useful to consider the toric case $\mathfrak{m} = n + 1$.

3. Toric Construction

For the basics of toric geometry, including all of the toric constructions here, we refer the reader to Fulton's classic text [6].

3.1 The base space $\mathbb{P}^n$

We use the standard fan $\Sigma = \Sigma_{\mathbb{P}^n}$ for $\mathbb{P}^n$, with primitive generators $\rho_0 = (-1, \ldots, -1)$, $\rho_1 = (1, 0, \ldots, 0), \ldots, \rho_n = (0, \ldots, 1)$; note that ρ_i is the ith standard basis vector of $\mathbb{R}^n$ for all $1 \leqslant i \leqslant n$. The cones of Σ are generated by the proper subsets of $\{\rho_0, \ldots, \rho_n\}$.

With this assignment, the $\mathbb{T}$-fixed points correspond to cones as follows:

$$p_0 \longleftrightarrow \langle \rho_1, \ldots, \rho_n \rangle$$
$$p_1 \longleftrightarrow \langle \rho_0, \rho_2, \ldots, \rho_n \rangle$$
$$\vdots$$
$$p_n \longleftrightarrow \langle \rho_0, \ldots, \rho_{n-1} \rangle.$$

3.2 The blow-up at points, X_0

Blowing up the point p_0 corresponds to subdividing Σ along the cone $\langle \rho_1, \ldots, \rho_n \rangle$, creating a new primitive generator

$$\rho_{1,\ldots,n} = \rho_1 + \cdots + \rho_n = (1, \ldots, 1).$$

To obtain Σ_{X_0}, we subdivide $\Sigma_{\mathbb{P}^n}$ along all cones corresponding to $\mathbb{T}$-fixed points.

The cohomology of X_0 is then generated by $\{D_\alpha\}$, where ρ_α is a primitive generator in Σ_{X_0}. The primitive generators introduced as a result of subdivision correspond to exceptional divisors. In particular, we have the following isomorphism of presentations of the cohomology

ring of X_0:

$$D_i \longleftrightarrow -H + \sum_{j \neq i} E_i$$

$$D_{0,1,\ldots,\hat{i},\ldots,n} \longleftrightarrow -E_i.$$

Here H and $\{E_i\}$ form the geometric basis for cohomology, namely H is the hyperplane class and E_i is the exceptional divisor over p_i. Note that

$$K_{X_0} = \sum_{\rho_\alpha \in \Sigma_{X_0}} D_\alpha = -(n+1)H + (n-1)\sum_{i=0}^{n} E_i.$$

3.3 The blow-up at lines, X_1

Let $0 \leqslant i < j \leqslant n$. To blow-up the line Z_{ij} in X_0, which is the proper transform of the line l_{ij} in $\mathbb{P}^n$ through p_i and p_j, we subdivide the cone indexed by the complement of $\{i, j\}$ in $\{0, \ldots, n\}$. Thus the 1-skeleton of Σ_{X_1} is

$$\Sigma_{X_1}^1 = \{\rho_i, \rho_{0,\ldots,\hat{i},\ldots,n}, \rho_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n} : 0 \leqslant i < j \leqslant n\}.$$

The cones of higher dimension are given by subdivision.

The isomorphism between toric and geometric bases of $H^*(X_1, \mathbb{Z})$ is given by

$$D_i \longleftrightarrow -H + \sum_{j \neq i} E_j + \sum_{i \notin \{j,k\}} E_{jk}$$

$$D_{0,1,\ldots,\hat{i},\ldots,n} \longleftrightarrow -E_i$$
$$D_{0,1,\ldots,\hat{i},\ldots,\hat{j},\ldots,n} \longleftrightarrow -E_{ij}.$$

Here H and E_i are as above, and E_{ij} is the exceptional divisor above Z_{ij}.

3.4 The full permutohedral case, $\widehat{X}(n+1)$

Let $[n] = \{0, \ldots, n\}$. In general, we have

$$D_i \longleftrightarrow -H + \sum_{\substack{i \notin \alpha \\ \alpha \subsetneqq [n]}} E_\alpha$$

$$D_\alpha \longleftrightarrow -E_\alpha, \text{ for all } \alpha \subsetneqq [n], |\alpha| > 1.$$

Expressing the canonical bundle in both bases, we compute

$$K_{\hat{X}} = \sum_{\alpha \subsetneqq [n]} D_\alpha$$

$$= -(n+1)H + \sum_{\alpha \subsetneqq [n]} (n - |\alpha|)E_\alpha.$$

The cohomology ring $H^*(\hat{X}, \mathbb{Z}) = A^*(\hat{X})$ is given by

$$A^*(\hat{X}) \cong \mathbb{Z}[\{D_\alpha\}]/I,$$

where I is the ideal generated by all

(i) $D_{\alpha_1} \cdot D_{\alpha_2} \cdot \cdots \cdot D_{\alpha_k}$ for $\rho_{\alpha_1}, \ldots, \rho_{\alpha_k}$ not in a cone of Σ; and

(ii) $\sum_\alpha (e_i \cdot \rho_\alpha) D_\alpha$ for e_i the ith standard basis vector in $\mathbb{R}^n$.

3.5 The general case, $\widehat{X}(m)$

The cohomology of $\hat{X}(m)$ (or $X_j(m)$) is now easy to express in terms of that of $\hat{X}(n+1)$ (or $X_j(n+1)$). Indeed, cohomology is generated in codimension two, and we have

$$H^2(\hat{X}(m)) = H^2(\hat{X}(n+1)) + \sum_{i > n+1} \mathbb{Z} \cdot [E_i].$$

Further, as E_i is far from a general line and the invariant subvarieties, we have

$$E_i \cdot H = E_i \cdot E_\alpha = 0,$$

for all $i > n+1$, $i \neq \alpha$.

3.6   Toric Symmetry

The fan Σ_{Π_n} of the permutohedral variety $\widehat{X}(n+1) = X_{\Pi_n}$ is thus symmetric about the origin. Reflecting through the origin sends ρ_α to its complement

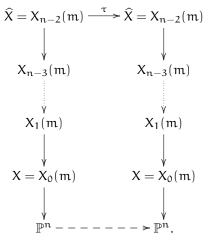$$\rho_\alpha \mapsto \rho_{[n]\setminus\alpha}.$$

Any symmetry of the fan of a toric variety induces an isomorphism on the variety itself; call this map $\tau : \widehat{X} \to \widehat{X}$. We immediately see that the action of τ on cohomology is given by

$$\tau^* D_\alpha = D_{[n]\setminus\alpha}.$$

It is elementary to verify that τ is a resolution of the Cremona transform.

4. Proof of the Main Theorem

In the previous two sections we established the existence of the diagram

$$
\begin{array}{ccc}
\widehat{X} = X_{n-2}(m) & \xrightarrow{\ \tau\ } & \widehat{X} = X_{n-2}(m) \\
\downarrow & & \downarrow \\
X_{n-3}(m) & & X_{n-3}(m) \\
\vdots & & \vdots \\
X_1(m) & & X_1(m) \\
\downarrow & & \downarrow \\
X = X_0(m) & & X = X_0(m) \\
\downarrow & & \downarrow \\
\mathbb{P}^n & \dashrightarrow & \mathbb{P}^n.
\end{array}
$$

Note that Gromov-Witten invariants are functorial under isomorphism, i.e. for any $\hat{\beta} \in H_2(\widehat{X})$ we get

$$\langle \tau^* pt^k \rangle^{\widehat{X}}_{\hat{\beta}} = \langle pt^k \rangle^{\widehat{X}}_{\tau_* \hat{\beta}}.$$

But $\tau^*pt = pt$ and thus

$$\langle pt^k \rangle^{\hat{X}}_{\hat{\beta}} = \langle pt^k \rangle^{\hat{X}}_{\tau_*\hat{\beta}}.$$

Furthermore, $\hat{\beta}$ must be of the form

$$\hat{\beta} = d\hat{h} - \sum_{i=1}^{m} a_i \hat{e}_i.$$

We compute

$$\tau_* \hat{\beta} = \hat{\beta}',$$

where

$$\hat{\beta}' = d'\hat{h} - \sum_{i=1}^{m} a_i' \hat{e}_i$$

is given by

$$d' = nd - (n-1)\sum_{i=1}^{n+1} a_i$$

$$a_i' = \begin{cases} d - \sum_{\substack{j \neq i \\ j \leqslant n+1}} a_j & 1 \leqslant i \leqslant n+1 \\ a_i & i > n+1. \end{cases}$$

This symmetry then descends to X for nonexceptional classes. $\qquad \square$

Moreover, this symmetry of X further descends to $\mathbb{P}^n$ itself via the following proposition.

Proposition 4.1 (Bryan-Leung [3]). *Let* $\beta = dh - \sum a_i e_i \in H_2(X; \mathbb{Z})$. *Suppose that* $a_i = 0$ *for some* i. *Then*

$$\langle pt^r \rangle^X_\beta = \langle pt^{r-1} \rangle^X_{\beta - e_i}.$$

$\qquad \square$

In particular, in the case $a_i = 0$ for all i, we have

$$\langle pt^{n+3} \rangle^{\mathbb{P}^n}_{nh} = \langle \; \rangle^X_{nh - e_1 - \cdots e_{n+3}}.$$

5. Nonexceptional classes

In order for Theorem 1.1 to be of use, and in order to establish Corollary 1.2, we must identify nonexceptional classes. We do so using degeneration of relative Gromov-Witten invariants.

5.1   Relative GW theory

Let $Y \subset X$ be a nonsingular divisor. The Gromov-Witten theory of X relative to Y is defined in the algebraic setting in [17]. Let $\beta \in H_2(X, \mathbb{Z})$ be a curve class such that $\beta \cdot [Y] \geqslant 0$, and let $\vec{\mu}$ be a partition of this nonnegative number. The moduli stack $\overline{M}_{g,n}(X/Y, \beta, \mu)$ parameterizes stable relative maps with relative multiplicities determined by $\vec{\mu}$. Note that the target of a relative stable map may be a k-step degeneration of X along Y; again see [17].

Let δ_i be classes in $H^*(Y, \mathbb{Q})$. A cohomology-weighted partition μ is an unordered set of pairs

$$\{(\mu_1, \delta_{i_1}), \dots, (\mu_s, \delta_{i_s})\},$$

where μ_i is a part of $\vec{\mu}$.

Now, let γ_j be classes in $H^*(X, \mathbb{Q})$. The genus-g, class β Gromov-Witten invariant with insertions $\gamma_1, \dots, \gamma_r$ relative to the cohomology weighted partition μ is defined via integration against the virtual class

$$\langle \gamma_1, \dots, \gamma_r \mid \mu \rangle_{g,\beta}^{X/Y} = \frac{1}{|\text{Aut}(\mu)|} \int_{[\overline{M}_{g,r}(X/Y, \beta, \mu)]^{vir}} \prod_{j=1}^r \text{ev}_j^*(\gamma_j) \cup \prod_{i=1}^s \text{ev}_i^* \delta_i.$$

5.2   Degeneration

Let $\mathcal{X} \to \mathbb{A}^1$ be a family of projective schemes such that the fibers $\mathcal{X}_t$ are smooth for $t \neq 0$ and the special fiber $\mathcal{X}_0$ has two irreducible components

$$\mathcal{X}_0 = X \coprod_Z W$$

intersecting transversally along a connected smooth divisor $Z \subset \mathcal{X}_0$. Then, for a virtual dimension zero class $\beta \in H_2(\mathcal{X}_t, \mathbb{Z})$, the degeneration formula of [17] yields

$$\langle \ \rangle_\beta^{\mathcal{X}_t} = \sum_{\mu, \beta = \beta_1 + \beta_2} \langle \ | \ \mu \rangle_{\beta_1}^{X/Z} \cdot C(\mu) \cdot \langle \ | \ \mu^\vee \rangle_{\beta_2}^{W/Z}, \qquad (5.1)$$

where μ remains a cohomology weighted partition, $\mu^\vee$ is its dual, and $C(\mu)$ is a combinatorial factor.

If β does not split, and we only may consider $\beta = \beta_1$, we are forced into the empty partition $\mu = \emptyset$, in which case $C(\mu) = 1$, our relative $\langle \ | \ \emptyset \rangle_\beta^{X/Z}$ invariant is in fact an absolute invariant, and we have

$$\langle \ \rangle_\beta^{\mathcal{X}_t} = \langle \ \rangle_\beta^X.$$

We now use this technique to equate invariance of X_j and its blow-up X_{j+1} by using deformation to the normal cone to set up our degeneration.

5.3   Proof of Corollary 1.2

For $0 \leqslant j < n - 2$ let $Z_\alpha = Z_{i_0, \ldots, i_{j+1}}$ be the proper transform of the $j + 1$-dimensional subvariety of $\mathbb{P}^n$, as defined in Section 2. Then Z_α is isomorphic to the blow-up of $\mathbb{P}^{j+1}$ at all the $0, 1, \ldots, j - 1$-dimensional $\mathbb{T}$-equivariant subvarieties. Also as above, we continue to let $E_\alpha = E_{i_0, \ldots, i_{j+1}}$ be the exceptional divisor of the blow-up at Z_α. Then

$$E_\alpha \cong \mathbb{P}(L^{(n-j-1)\oplus}) \cong Z_\alpha \times \mathbb{P}^{n-j-2},$$

where L is the restriction of the line bundle

$$-H + \sum_{\substack{\emptyset \neq \zeta \subset [n] \\ |\zeta| < j+2}} E_\zeta.$$

Recall that $e_\alpha = e_{i_0, \ldots, i_{j+1}}$ is the class of a line in the fibers of E_α. Set $Z_{j+1} = \cup_\alpha Z_\alpha \subset X_j$ where the union is over $\alpha \subset [n]$ and $|\alpha| = j + 2$, here X_{j+1} is defined by $X_{j+1} = Bl_{Z_{j+1}} X_j$. Then $A_1(X_{j+1}) =$

$H_2(X_{j+1}, \mathbb{Z})$ is generated by the collection $\{e_\alpha\}$. Our convention is that $e_\emptyset = h$ represents the proper transform of the class of a line in $\mathbb{P}^n$. We also use the same notation to denote the proper transform of all these classes at the subsequent stages of blow-ups.

Now let $\alpha = \{i_0, \dots, i_{j+1}\}$ be fixed. For any $\gamma \subset \alpha$ with $|\gamma| \leqslant j$, we denote by $_\alpha e_\gamma$ the generators of $A_1(Z_\alpha)$. Also let $\pi_\alpha : E_\alpha \to Z_\alpha$ be the projection and $S_\alpha \subset E_\alpha$ be the divisor given by $\mathbb{P}(L^{(n-j-2)\oplus})$. We have $\mathcal{O}_{E_\alpha}(1) \sim \pi_\alpha^* L + S_\alpha$.

Lemma 5.1. *We have the following relations in* $A_1(X_{j+1})$*:*

$$_\alpha e_\emptyset = h - \sum_\delta e_\delta + (j+1)e_\alpha,$$

where δ runs over all $\delta \subset \alpha$ and $|\delta| = j+1$. Also, if $j \geqslant 1$, then for any $k \in \alpha$ we have

$$_\alpha e_k = e_k - \sum_\epsilon e_\epsilon + je_\alpha,$$

where ϵ runs over all $\epsilon \subset \alpha$, $|\epsilon| = j+1$, and $k \in \epsilon$. More generally, for any $\gamma \subset \alpha$ with $|\gamma| < j+1$, we have

$$_\alpha e_\gamma = e_\gamma - \sum_\epsilon e_\epsilon + (j+1-|\gamma|)e_\alpha,$$

where ϵ runs over all $\epsilon \subset \alpha$ and $|\epsilon| = j+1$ and $\gamma \subset \epsilon$.

Proof. Let $_\alpha e_\emptyset = \sum_{|\zeta| \leqslant j+1} x_\zeta e_\zeta + xe_\alpha$ and $_\alpha e_k = \sum_{|\zeta| \leqslant j+1} y_\zeta e_\zeta + ye_\alpha$. We intersect both sides of these equations by the divisor E_α. We can see easily that $E_\alpha \cdot e_\zeta = 0$ for any ζ with $|\zeta| \leqslant j$, and

$$E_\alpha \cdot {}_\alpha e_\emptyset = \mathcal{O}_{E_\alpha}(-1) \cdot {}_\alpha e_\emptyset = -\pi_\alpha^* L \cdot {}_\alpha e_\emptyset = 1 - (j+2) = -j-1,$$

where $j + 2 = \sum_{\eta \subset \alpha, |\eta| = j+1} E_\eta \cdot {}_\alpha e_\emptyset$. Similarly, $E_\alpha \cdot e_k = 1 - (j+1)$, where $j + 1 = \sum_{\eta \subset \alpha, |\eta| = j+1} E_\eta \cdot {}_\alpha e_k$. Also note that $\mathcal{O}_{E_\alpha}(-1) \cdot e_\alpha = -1$, from which it follows that $x = j$ and $y = j - 1$. The other unknowns are found by intersecting both sides of the relations above by the divisors H and E_ζ's. $\square$

Now consider the degeneration of X_j into $X_{j+1} \coprod_{Z_\alpha} W_\alpha$, where

$$W_\alpha = \mathbb{P}(\mathcal{O}_{Z_\alpha} \oplus L^{(n-j-1)\oplus}).$$

For a virtual dimension zero class $\beta \in A_2(X_j)$, the gluing formula Equation (5.1) yields

$$\langle \, \rangle_\beta^{X_j} = \sum_{\mu, \beta = \beta_1 + \beta_2} \langle \, | \, \mu \rangle_{\beta_1}^{X_{j+1}/Z_\alpha} \cdot C(\mu) \cdot \langle \, | \, \mu^\vee \rangle_{\beta_2}^{W_\alpha/Z_\alpha}. \qquad (5.2)$$

Let $\tilde{Z}_\alpha \subset W_\alpha$ be the copy of Z_α "at infinity" given by $\mathbb{P}(\mathcal{O}_{Z_\alpha}) \subset W_\alpha$. Denote by $_\alpha \tilde{e}_\gamma$ the corresponding class in $\tilde{Z}_\alpha$.

Lemma 5.2. *The following relations hold in* $A_1(W_\alpha)$. *We have*

$$_\alpha \tilde{e}_\emptyset = {}_\alpha e_\emptyset - (j+1)e_\alpha.$$

If $j \geqslant 1$ *then*

$$_\alpha \tilde{e}_k = {}_\alpha e_k - j e_\alpha.$$

More generally, for any $\gamma \subset \alpha$ *with* $|\gamma| < j+1$ *we have*

$$_\alpha \tilde{e}_\gamma = {}_\alpha e_\gamma - (j+1-|\gamma|)e_\alpha.$$

In particular, e_α *together with the elements of* $\{_\alpha \tilde{e}_\gamma\}_{\gamma \subset \alpha, |\gamma| < j+1}$ *gives a positive basis for* $A_1(W_\alpha)$.

Proof. These follow from $\mathcal{O}_{W_\alpha}(1) \cdot {}_\alpha \tilde{e}_\emptyset = \mathcal{O}_Z(1) \cdot {}_\alpha \tilde{e}_k = \mathcal{O}_Z(1) \cdot {}_\alpha \tilde{e}_\gamma = 0$ and $\mathcal{O}_{W_\alpha}(1)|_{Z_\alpha} \sim \mathcal{O}_{Z_\alpha}(1)$. $\qquad \square$

Now in the degeneration formula, if the class β splits as $\beta_1 + \beta_2$, where $\beta_2 = \sum_\alpha (b_\alpha e_\alpha + \sum_\gamma {}_\alpha a_\gamma {}_\alpha \tilde{e}_\gamma)$ for nonnegative integers b_α and $_\alpha a_\gamma$, then by the lemmas above β_1 is forced to be

$$\beta_1 = \beta - \sum_{|\alpha|=j+2} \left(b_\alpha e_\alpha + \sum_{\gamma \subset \alpha, |\gamma| \leqslant j} {}_\alpha a_\gamma \left(e_\gamma - \sum_{\gamma \subset \epsilon \subset \alpha, |\epsilon|=j+1} e_\epsilon \right) \right).$$
$$(5.3)$$

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Now if the coefficients of h and e_ks in $\beta = dh - \sum_{i=1}^{m} a_i e_i$ are such that the class β does not afford any nontrivial splitting into effective curve classes, then in the degeneration formula (5.2) there will be no contributions from GW theory of the W_αs. For example, we can apply this in the following situation.

Proposition 5.3. *If β is an extremal ray in the cone of effective curves then in (5.3) we must have $\beta_1 = \beta$.*

Proof. The class $\sum_\epsilon e_\epsilon - e_\gamma$ is not effective by Lemma 5.1, and hence β_1 cannot be effective unless all the coefficients in the right hand side of (5.3) are zero. $\qquad\square$

Proof of Corollary 1.2. It follows from Proposition 5.3, because $\beta = nh - e_1 - \cdots - e_{n+3}$ is an extremal ray. $\qquad\square$

Remark 5.4. In general, one would like to equate the invariants of a space and its blow-up for any curve class which is far away from the exceptional divisors. However no such general theorem exists; in general it is very difficult to determine the behavior of Gromov-Witten invariants under rational maps. This holds true even in case the rational map is a blow-up. However, note that the proof of Corollary 1.2 applies to much more than the rational normal curve alone. So one may use degeneration for cases of interest in the absence of a more universal theorem.

Acknowledgments

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Resumen: En este trabajo establecemos la existencia de una simetría en el marco de la teoría de Gromov-Witten para $\mathbb{CP}^n$ y su explosión a lo largo de puntos. La naturaleza de esta simetría queda codificada en la transformación de Cremona y su resolución en una variedad tórica del permutoedro. Esta simetría expresa algunos invariantes difíciles de calcular junto con otros que no lo son tanto. Nos centramos en implicaciones enumerativas; en particular esta técnica ofrece una prueba en una línea de la unicidad de la curva racional normal. Nuestro método involucra un estudio de la geometría tórica del permutoedro, así como el de la degeneración de los invariantes de Gromov-Witten.

Palabras clave: teoría de Gromov-Witten, geometría enumerativa, invariantes estacionarios, transformación de Cremona, espacio proyectivo, permutoedro, variadad tórica permutoedral, espacio de Losev-Manin.

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