On some generalizations of Tate Cohomology: an overview

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Abstract

This paper is an overview of the developments and generalizations of Tate Cohomology. The number of such generalizations is high and the literature on many of them is vast. Hence, we do not pretend to give a complete account of all the branches that have developed from the original ideas of Tate. This is rather an overview of how the ideas developed.

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Introduction

Tate cohomology was defined by John Tate in [49]. His declared intention was to give a compact account of the reciprocity law for Class Field Theory. The latter is, in short, a family of isomorphisms

$$\hat{\mathrm{H}}^p(\pi, \mathbb{Z}) \xrightarrow{\cup c_{E/F}} \hat{\mathrm{H}}^{p+2}(\pi, A), \tag{$*$}$$

defined for every finite Galois extension E/F of local or global fields with finite Galois group π. In $(*)$, A is the multiplicative group $E^\times$ of E in the local case and the *idèle* class group in the global case. In both situations, an action of π on A is naturally defined, and hence also the classical cohomology groups. Also, $\hat{\mathrm{H}}$ denotes the Tate cohomology, while $\cup c_{E/F}$ denotes the so called cup-product with a canonical generator $c_{E/F}$ of $\mathrm{H}^2(\pi, A)$. According to Tate, the family of automorphisms defined in $(*)$ is ultimately a way to reduce to a minimum the authentic arithmetic results in Class Field Theory and to show how all the others are a consequence of a more general algebraic theory. We start with a summary of Tate's original point of view in Section 1.

Next we give an outline of the rest of the paper.

As Weibel tells in his brief history of homological algebra [53], the book [13], published by Cartan and Eilenberg in 1956, collected and put order in several definitions, constructions, and results at that time available on the subject. Among them was also Tate cohomology for finite groups. In Section 2, we then follow Chapter XII of [13], where most features of Tate cohomology for finite groups are present. Given a finite group π, a *complete resolution of* $\mathbb{Z}$ is defined as a, possibly unbounded, exact complex of projective π-modules $\boldsymbol{T}$ such that the image of the differential $d_0^{\boldsymbol{T}}$ is exactly $\mathbb{Z}$ (see Definition 2.2). For every other π-module M, the *n-th Tate cohomology groups of* M, denoted by $\hat{\mathrm{H}}^\bullet(\pi, M)$, are the cohomology of the complex $\mathrm{Hom}_\pi(\boldsymbol{T}, M)$. The definition of these groups $\hat{\mathrm{H}}^\bullet(\pi, -)$ is functorial in M. Except for the degrees 0 and 1, they are the regular cohomology in positive degree and the regular (shifted by 1) homology in negative degree, given as follows:

$$\hat{\mathrm{H}}^n(\pi, A) = \begin{cases} \mathrm{H}^n(\pi, A) & \text{for } n > 0, \\ \mathrm{H}_{-n-1}(\pi, A) & \text{for } n < -1. \end{cases}$$

Moreover, for every short exact sequence of π-modules, we have a long and unbounded exact sequence in cohomology. Since Tate cohomology is both effaceable and coeffaceable, dimension shifting techniques are applicable. For instance, cup product can be transported from cohomology to any degree:

$$\cup : \hat{\mathrm{H}}^p(\pi, M) \otimes \hat{\mathrm{H}}^q(\pi, N) \longrightarrow \hat{\mathrm{H}}^{p+q}(\pi, M \otimes N).$$

The theory developed until then heavily relied on the existence of a complete resolution in the category of π-modules for a finite group π, that is, in the category of modules over the group ring $\mathbb{Z}[\pi]$. The subsequent generalizations of Tate cohomology have taken several directions. The first one, proposed by Farrell in [21], enlarges the class of groups to those with finite virtual cohomological dimension. We give an overview of his idea in Section 3.1. After Farrell, as we see in Section 3.2, Ikenaga [29] pushed it to the groups with finite generalized cohomological dimension, provided that a complete resolution exists, and Buchweitz [42] brought it to Gorenstein rings, for any ring R, not necessarily a group ring.

For these kind of viewpoints, two ingredients are necessary: the existence of at least a complete resolution, in the sense defined previously, and a way to ensure that the cohomology defined from it does not depend on the particular choice. Cornick and Kropholler in [17] resolved the latter issue by slightly modifying the definition itself: the exact complex $\boldsymbol{T}$ must also satisfy the condition that $\mathrm{Hom}(\boldsymbol{T}, -)$ is an exact functor. This leads eventually to the use of totally acyclic complexes of projective modules. We discuss this axiom and some methods to build complete resolutions in Section 3.6.

In Section 3.3 we show how, slightly before Cornick and Kropholler's work, Mislin used satellites to define a generalization for Tate cohomology over any group. Despite the fact that no complete resolution is involved, and inspired by Gedrich and Gruenberg, Mislin justifies the name using the concept of $\mathcal{P}$-completion. In this context, Tate cohomology is the unique, up to equivalence, $\mathcal{P}$-completion of the classical group cohomology $\mathrm{H}^\bullet(G,-)$.

Section 3.4 is devoted to the developments of the idea of Tate in the context of strongly Gorenstein rings, as Buchweitz called them in [42]. These are left-right Noetherian rings with finite injective dimension as modules over themselves. Buchweitz redacted a long preprint in which he established the equivalence of

- right bounded complexes with bounded and finitely generated cohomology, modulo finite complexes of finitely generated projective modules;
- complete resolutions, in the sense of acyclic projective complexes up to homotopy;
- maximal Cohen-Macaulay modules up to projective modules.

The definition of the bifunctor $\widehat{\mathrm{Ext}}_R^n(-,-)$ is then given through the complete resolution. Finally, Buchweitz was able to give the following definition:

$$\hat{\mathrm{H}}^n(G,-) = \widehat{\mathrm{Ext}}_{R[G]}(R,-).$$

This is yet another generalization of Tate cohomology, since here G is acting on R-modules and not just on abelian groups. This setting is extended to modules of finite Gorenstein dimension in [2] and finally to complexes of finite Gorenstein dimension by Veliche [51].

In the same years, around 1986, in a private letter, Vogel developed another method to generalize Tate cohomology. His ideas have been written and published by Goichot [24] and they substantially converge to the framework of stable cohomology. We present this cohomology theory in Section 3.5, following the work done by Avramov and Veliche in [3].

Although from different points of view the works we have mentioned so far revolve around the same objects, a different perspective was taken by Jørgensen in [32]. Denote by $e_* : \mathcal{K}_{\text{tac}}(\mathcal{P}(R)) \to \mathcal{K}(\mathcal{P}(R))$ the embedding of the homotopic category of totally acyclic complexes of projective R-modules inside the homotopic category of projective resolutions of R-modules. If e_* has a right adjoint $e^!$, we called the image $e^!(\boldsymbol{P})$ of a complex in $\mathcal{K}(\mathcal{P}(R))$ a *totally acyclic projective approximation of* $\boldsymbol{P}$. Relying on the fact that, when a module M has a complete resolution $\boldsymbol{T} \to \boldsymbol{P}$ and $e^!$ exists, we have $\boldsymbol{T} \simeq e^!(\boldsymbol{P})$, Jørgensen disregards the problem of the existence of a complete resolution and directly defines

$$\widehat{\text{Ext}}^n(M,N) = \mathrm{H}^n(\mathrm{Hom}_R(e^!(\boldsymbol{P}), N)),$$

where $\boldsymbol{P}$ is any projective resolution of M. The motivations for this approach are both technical a philosophical: a totally acyclic projective approximation exists for modules over a wide class of rings, and, moreover, according to Jørgensen, this generalization is more faithful to the original idea of Tate. We explain his approach, as well as a comparison with the other generalizations seen so far, in Section 3.6.

Compared to the classical group cohomology, but still in the context of groups, two additional phenomena arise and deserve attention when considering Tate cohomology: periodicity of cohomology and non-zero products in negative degrees. The former is already in plain sight since the beginning (see $(*)$ or directly Section 2.3), but it expands further in the generalizations, especially in the ramifications of Tate cohomology in topology. The latter was studied by Benson and Carlson in [5]. We briefly report on these facts in Sections 3.7 and 3.8.

Profinite groups require a special treatment. Indeed, although finite groups are trivially profinite, the category of discrete G-modules does not have enough projectives if G is an infinite profinite group. In Section 3.9 we described various ideas to overcome this issue and build a Tate cohomology theory nonetheless.

We conclude the paper with a generalization of Tate cohomology to algebras in Section 4, from some very early results of Nakayama published in [37] through a huge gap until recent times with papers like [6] or [7]. The latter in particular extends to Calabi-Yau categories some of the ideas of Benson and Carlson.

Notation and Conventions

Along the whole paper, we will use the following conventions. The letters G and H will denote groups; in honour of [13], π will be used to refer to finite groups.

In general, rings will be denoted by the letters R or K, while algebras with capital greek letters like Λ. Modules over rings will be denoted with capital regular letters like M and N, or A and B. The category of left and right modules over R will be denoted by $\mathcal{M}(R)$ and $(R)\mathcal{M}$ respectively. Likewise, complexes of left modules over a ring R will be denoted by $\mathcal{C}(R)$, and each complex in bold, like $(\boldsymbol{A}, d^{\boldsymbol{A}})$ and $(\boldsymbol{B}, d^{\boldsymbol{B}})$; note that we will often just write $\boldsymbol{A}$ and $\boldsymbol{B}$. This choice allows us to denote by A^* and $\boldsymbol{A}^*$ respectively the duals $\mathrm{Hom}_R(A, R)$ and $\mathrm{Hom}_{\mathcal{C}(R)}(\boldsymbol{A}, R)$ without confusion. In the latter, R denotes also the complex concentrated in degree 0 given by R, even if not in bold. For complexes, the degree will be written both as A_n and A^n, but the direction of the differential will always be consistent with the traditional conventions: $d_n^A : A_n \to A_{n-1}$ and $d_A^n : A^n \to A^{n+1}$.

Throughout the paper, when the base ring R is the group ring $\mathbb{Z}[G]$ of a group G, we will use the notation $\mathcal{M}(G)$, $\mathcal{C}(G)$, and $\mathcal{D}(G)$ as short forms for the categories $\mathcal{M}(\mathbb{Z}[G])$, $\mathcal{C}(\mathbb{Z}[G])$, and $\mathcal{D}(\mathbb{Z}[G])$ respectively. Likewise, we will write $\mathrm{Hom}_G(-,-)$ instead of $\mathrm{Hom}_{\mathbb{Z}[G]}(-,-)$, $- \otimes_G -$ for $- \otimes_{\mathbb{Z}[G]} -$, and Ext_G instead of $\mathrm{Ext}_{\mathbb{Z}[G]}$. Therefore, we will make no distinction between G-modules, that is, abelian groups on which G acts as groups, and $\mathbb{Z}[G]$-modules. Given a group G, the *fixed points functor* will be denoted by $(-)^G : \mathcal{M}(G) \to \mathcal{A}b$ and it is explicitly defined by

$$M^G = \{m \in M \mid gm = g \text{ for every } g \in G\}.$$

It is left exact and we will denote by $\mathrm{H}^n(G, M)$ its right derived functors. These are also called the *group cohomology functors of G*. Equivalently, since the equality $M^G = \mathrm{Hom}_G(\mathbb{Z}, M)$ holds, $\mathrm{H}^n(G, M)$ will also be the n-th cohomology group of the right derived functor $R\mathrm{Hom}_{\mathcal{C}(G)}(\mathbb{Z}, -)$ applied to M, the latter seen as a complex concentrated in degree 0. The same conventions apply also to the functor $(-)_G : \mathcal{M}(G) \to \mathcal{A}b$, explicitly defined by the quotient

$$M_G = \frac{M}{\{gm - g \mid m \in M, g \in G\}}.$$

In this case, we have the equality $M_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$, hence M_G is right exact and $\mathrm{H}_n(G, M)$ denotes both the left derived functor of $(-)_G$ and the n-th cohomology group of the left derived functor $\mathbb{Z} \otimes^L_{\mathcal{C}(G)} -$ applied to M. In the context of group rings, the ideal I generated by all the elements of the form $1 - g$, with $g \in G$, is called *augmentation ideal*. With this notation, M_G can also be defined as M/IM.

Standard references for all these constructions and notations are for example [13], [25], [10], [52], and [33]. Specifically for complexes over rings, Section 1 of [51] is an excellent, compact, and clear summary.

1. First appearance of Tate cohomology

We start with an outline of where and how Tate introduced the core of the cohomology theory that now bears his name. Since it was still in its infancy, we will be vague about the details, especially on the field theoretic side. What we omit about the latter can be found in Section 2.4. In short, the aim of Class Field Theory is to describe the abelian extensions of a field K, that is, the Galois extensions L of K such that $\mathrm{Gal}(L/K)$ is abelian. In principle, it would be even more interesting to drop the latter assumption, but results are less satisfactory. Let A then denote the multiplicative group $K^\times$ of K in the local case or the idèle class group in the global case. Let G denote a finite group of automorphisms of K. Following the remark that all the main results of Class Field Theory come from a few arithmetical facts about A expressed in cohomological terms, Tate showed how these can be summarized in the vanishing of two cohomological groups relative to an auxiliary group $\bar{A}$. This is the *splitting module associated to a non trivial element α of* $\mathrm{H}^2(G,A)$. That is, a G-module such that A is a G-submodule of $\bar{A}$ and α belongs to the kernel of the natural map $\mathrm{H}^2(G,A) \to \mathrm{H}^2(G,\bar{A})$ given by the inclusion $A \hookrightarrow \bar{A}$. Here is Tate's result.

Theorem 1.1. *Let G be a finite group of order n and A a G-module. Let α be a non trivial element of $\mathrm{H}^2(G,A)$ and $\bar{A}$ a splitting G-module for α. Then the following two conditions are equivalent.*

- *For every subgroup H of G, we have $\mathrm{H}^1(H,A) = 0$ and $\mathrm{H}^2(H,A)$ is cyclic of order $|H|$ generated by the restriction of α to H.*
- *For every subgroup H of G, we have $\mathrm{H}^1(H,\bar{A}) = 0$ and $\mathrm{H}^2(H,\bar{A}) = 0$.*

The proof given in [49] revolves around the following two short exact sequences:

$$0 \longrightarrow A \longrightarrow \bar{A} \longrightarrow I \longrightarrow 0,$$

$$0 \longrightarrow I \longrightarrow \mathbb{Z}[H] \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

The first one comes for the definition of $\bar{A}$. The second one from the fact that the augmentation ideal I is also the kernel of the *degree* map

$$\begin{array}{rccc} \deg: & \mathbb{Z}[H] & \longrightarrow & \mathbb{Z} \\ & \displaystyle\sum_{h\in H} n_h h & \longmapsto & \displaystyle\sum_{h\in H} n_h. \end{array}$$

See Section 2.1 for more details. Using a standard and key tool of group cohomology (see [10] Chapter III §6), from these two short exact sequences Tate obtained two long exact sequences that he wrote as

$$\mathrm{H}^0(H,I) \longrightarrow \mathrm{H}^1(H,A) \longrightarrow \mathrm{H}^1(H,\bar{A}) \longrightarrow \mathrm{H}^1(H,I) \longrightarrow \dots$$

$$\mathrm{H}^0(H,\mathbb{Z}[H]) \longrightarrow \mathrm{H}^0(H,\mathbb{Z}) \longrightarrow \mathrm{H}^1(H,I) \longrightarrow \mathrm{H}^1(H,\mathbb{Z}[H]) \longrightarrow \dots.$$

The key point here is that in this diagram Tate denoted by $\mathrm{H}^0(H,-)$ not the standard cohomology but what he called *'reduced' 0-dimensional group*. This, given an H-module X and denoting by N the element

$\sum_{h \in H} h \in \mathbb{Z}[H]$, is the quotient X^H/NX, now commonly denoted by $\hat{\mathrm{H}}^0(H, X)$ and called the *0-th Tate cohomology group of H with coefficients in X*. We review the basic facts about it in Section 2.2.

The second of Tate's theorems states the following.

Theorem 1.2 (Theorem 2 in [49], or Tate's Theorem)**.** *Let G and A be as in Theorem 1.1 and satisfying any of the two additional equivalent axioms in it. Then the cup product by α yields an isomorphism from $\mathrm{H}^{n-2}(G, \mathbb{Z})$ to $\mathrm{H}^n(G, A)$ for every $n > 2$.*

Tate concluded his paper announcing that negative cohomology groups can be introduced and that Theorem 1.2 can be proved for all integers n, adding a remark when $n = 0$. In this case, indeed, we obtain the isomorphism

$$G^{\mathrm{ab}} = G/[G, G] \simeq \mathrm{H}^{-2}(G, \mathbb{Z}) \xrightarrow{\sim} \hat{\mathrm{H}}^0(G, A),$$

where $\hat{\mathrm{H}}^0(G, A)$ is called the *idèle class norm residues* group. The inverse of the isomorphism in the center coincides with the so called *reciprocity law* or *norm residue symbol*.

Tate did not publish the more general results he claimed, but in a Séminaire Bourbaki in 1953, [46], Serre developed explicitly these statements by dimension shifting (see Section 2.3).

2. Basic definitions and properties

Most of the work of Tate on this subject is unpublished, or has not been published by Tate. Hence, we have to rely on what other mathematicians have written. It should be noted, though, that many of the ideas described in this section can be directly traced back to him.

A first formal framework for Tate cohomology is developed in Chapter 12 of *Homological Algebra*, published by Cartan and Eilenberg in 1956 ([13]). Since many of the generalizations that stem from Tate's ideas rely on a big part of the theoretical structure exposed in [13], we will give an account of it already at this point, keeping its notation when possible.

2.1 Augmented rings and algebras

A *left augmented ring* is a triple (R, ε, Q) where R is a ring and $\varepsilon : R \to Q$ is an epimorphism of left R-modules. The module Q is called *augmentation module*, ε is the *augmentation epimorphism*, or just the *augmentation*. The kernel I of ε is called the *augmentation ideal*.

Definition 2.1. For a right R-module M and a left R-module N,

- the *homology of the augmented ring R with coefficients in M* is

$$\mathrm{Tor}_n^R(M, Q) = \mathrm{H}_n(M \otimes_R^L Q);$$

- the *cohomology of the augmented ring R with coefficients in N* is

$$\mathrm{Ext}_R^n(Q, N) = \mathrm{H}^n(R\mathrm{Hom}_{\mathcal{C}(R)}(Q, N)).$$

The (co)homology groups can be computed using a projective resolution of either of Q or M, or an injective resolution of N. The particularity of this setting is that, being R projective, a resolution of the kind

$$\ldots \longrightarrow P_1 \longrightarrow R \longrightarrow Q \longrightarrow 0 \tag{2.1}$$

can always be used for Q.

Given a ring K, a special kind of augmented rings is given by the *supplemented algebras*, that is, K-algebras Λ together with a morphism of K-algebras $\varepsilon : \Lambda \to K$.

Since ε has to be surjective, the triple $(\Lambda, \varepsilon, K)$ is an augmented ring. Moreover, if $\mu : K \to \Lambda$ is the ring homomorphism giving the algebra structure, note that $\varepsilon \circ \mu = \mathbb{1}_K$ necessarily holds.

A supplemented algebra is given for example when we have a ring K and a group G, with the group ring $K[G]$. In this context, usually only multiplicative augmentations are considered and such augmentations only depend on the choice of a map $G \to K$. Without loss of generality, we take the *unit augmentation* defined by $\varepsilon : g \mapsto 1_K$. The triple $(K[G], \varepsilon, K)$ is then a supplemented algebra.

When K is the ring of integers, we recover in this way the group (co)homology of G in the sense of right and left derived functors respectively of $(-)^G$ and $(-)_G$. On the other hand, taking for example K as a field, we are in the context of G acting on K-vector spaces, as in Section 3.7, and so on.

2.2 Tate Cohomology for finite groups

Let us fix $\mathbb{Z}$ as base ring and analyze the case of a finite group π. The discussion of [13] revolves around the properties of the *norm* N *of* π, that is, the element $\sum_{x\in\pi} x$ in $\mathbb{Z}[\pi]$ or, in the same notation, the corresponding π-module homomorphism given by the multiplication by N.

Recall that the augmentation ideal I is generated by the elements $x-1$, for $x \in \pi$. Moreover, we have $N(x-1_\pi) = 0_{\mathbb{Z}[\pi]}$ and $xN = N$ for every $x \in \pi$. Therefore, given a π-module A, we obtain

$$IA \subset \ker(N), \qquad NA \subset A^\pi.$$

This implies that N induces a homomorphism $\tilde{N} : A_\pi \to A^\pi$. Starting from this remark, Cartan and Eilenberg show that the family of functors

$$\begin{array}{lcll} \hat{\mathrm{H}}^n(\pi, A) & = & \mathrm{H}^n(\pi, A), & \text{for } n > 0, \\ \hat{\mathrm{H}}^0(\pi, A) & = & A^\pi/NA, & \\ \hat{\mathrm{H}}^{-1}(\pi, A) & = & \ker(N)/IA, & \\ \hat{\mathrm{H}}^n(\pi, A) & = & \mathrm{H}_{-n-1}(\pi, A), & \text{for } n < -1, \end{array}$$

gives a *connected sequence of functors*, in the sense that, in addition to being functors, for each short exact sequence of π-modules we have the usual long exact sequence in cohomology. This is, in a very concrete way, the *Tate cohomology of the (finite) group* π and it is basically the same construction exposed by Serre in 1953 ([46]).

2.3 Complete resolutions and main features

The real interesting developments in the exposition of [13] are the first definition of a complete resolution and its use to compute (or define) the Tate cohomology of a finite group π in one step. This definition has been since then generalized. We will review it in more detail in Section 3.6.

Definition 2.2. Let π be a finite group. A *complete resolution of* $\mathbb{Z}$ is a pair $(\boldsymbol{X}, e)$ where $\boldsymbol{X}$ is an exact complex of projective π-modules and e is an element of $(X_{-1})^\pi$ that generates the image of the differential $d_0 : X_0 \to X_{-1}$.

With this definition, we recover the Tate cohomology of a π-module A as if it were the classical group cohomology:

$$\hat{\mathrm{H}}^n(\pi, A) = \mathrm{H}^n(\mathrm{Hom}_\pi(\boldsymbol{X}, A));$$

we refer to Section XII.3 of [13] for details. Many generalizations of this cohomology pass through the existence of a complete resolution. In the case of finite groups, the existence of such a resolution mainly relies on the fact that, given a finitely generated free $\mathbb{Z}[\pi]$-module X, also $\mathrm{Hom}(X, \mathbb{Z})$ is finitely generated and free, with the action of π defined by $(gf)(x) = f(g^{-1}x)$. More precisely, we have the isomorphim $\mathrm{Hom}(\mathbb{Z}[\pi], \mathbb{Z}) \simeq \mathbb{Z}[\pi]$. Moreover, if $\boldsymbol{X}$ is an exact complex of finitely generated free $\mathbb{Z}[\pi]$-modules, also $\mathrm{Hom}(\boldsymbol{X}, \mathbb{Z})$ is exact. Let us then consider the projective resolution $\boldsymbol{P} \xrightarrow{\varepsilon} \mathbb{Z}$ given by

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow \mathbb{Z}[\pi] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

as in (2.1), where all the P_i are free and finitely generated. From the previous remarks, if we write $\boldsymbol{P}^* = \mathrm{Hom}(\boldsymbol{P}, \mathbb{Z})$, the sequence $\mathbb{Z} \to \boldsymbol{P}^*$ is also a free right resolution. We then consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] & \mathbb{Z}[\pi] \arrow[d, "\varepsilon"] & & \\ & & \mathbb{Z} \arrow[d, "\varepsilon^*"] & & \\ & & \mathbb{Z}[\pi] \arrow[r] & P_1^* \arrow[r] & \cdots \end{tikzcd} \tag{2.2}$$

The composition $\varepsilon^* \circ \varepsilon$ can be computed explicitly and gives the multiplication by N. Contracting 2.2, that is, taking $X_n = P_n$ for $n \geq 0$ and renumbering $X_n = P^*_{-n+1}$ for $n < 0$, we obtain a complete resolution $(\boldsymbol{X}, \varepsilon^*(1))$.

The details of this construction can be found again in [13]. Another technique to recursively construct a complete resolution is described in [48].

In order to give a more explicit example, we consider the case of a finite cyclic group π. Denote the order of π by k and fix a generator σ. In this setting, the group ring $\mathbb{Z}[\pi]$ is isomorphic to $\mathbb{Z}[x]/(x^k - 1)$. The norm element becomes $\sum_{i=0}^{k-1} \sigma^i$ and the augmentation ideal is generated by $\sigma - 1$. It is then straightforward to verify that we have the following very simple complete resolution:

$$\ldots \xrightarrow{N} \mathbb{Z}[\pi] \xrightarrow{\sigma-1} \mathbb{Z}[\pi] \xrightarrow{N} \mathbb{Z}[\pi] \xrightarrow{\sigma-1} \mathbb{Z}[\pi] \longrightarrow \ldots.$$

This also means that, for any G-module A, we have

$$\hat{\mathrm{H}}(\pi, A) = \begin{cases} A^\pi / NA & \text{for } n \text{ even}, \\ \ker(N)/(\sigma - 1)A & \text{for } n \text{ odd}. \end{cases} \tag{2.3}$$

The existence of a complete resolution in other contexts is, in general, not true. We will come back to this in more detail in Section 3.6.

Apart from the elegance of a unified construction, defining Tate cohomology through the existence of a complete resolution gives practical tools to connect homology and cohomology. Among others, dimension shifting and cup products are probably the most important.

Recall that a G-module is *induced* if, for some abelian group A, it is isomorphic to $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ considered with the diagonal action on the left $g(h \otimes a) = gh \otimes ga$. Dually, a G-module is *coinduced* if it is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ for some abelian group A with action $(gf) : h \mapsto gf(g^{-1}h)$. Only in this section, we denote by A_* the kernel of the right projection $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A \to A$ and by A^* the cokernel of the constant embedding $A \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$. Furthermore, let A be, in addition, a π-module for a finite group π. Nevertheless, we can still apply the aforementioned definitions to the underlying abelian group. Since it can be proved that the Tate cohomology of any induced or coinduced π-module is trivial, using the long exact sequence in cohomology we have the isomorphisms

$$\hat{\mathrm{H}}^{n-1}(\pi, A) \simeq \hat{\mathrm{H}}^{n}(\pi, A_*) \qquad \text{and} \qquad \hat{\mathrm{H}}^{n+1}(\pi, A) \simeq \hat{\mathrm{H}}^{n}(\pi, A^*),$$

for every n. Roughly speaking, this means that if, *for some convenient fixed* n, some property is true for every π-module, then it is also true for every n. This is what informally is called *dimension shifting*.

Coming to the second of the tools we have mentioned, the *cup product* is a family of morphisms. They are normally defined for the classical homology and cohomology; in addition, they can also be extended to Tate cohomology, using indeed dimension shifting. In its latter form, they can be axiomatically defined as follows.

Proposition 2.3. *For every finite group π and every pair of π-modules A and B, there exists a unique family of homomorphisms*

$$\cup_{pq} : \hat{\mathrm{H}}^{p}(\pi, A) \otimes \hat{\mathrm{H}}^{q}(\pi, B) \to \hat{\mathrm{H}}^{p+q}(\pi, A \otimes B)$$

such that the map $\cup_{00} : \hat{\mathrm{H}}^{0}(\pi, A) \otimes \hat{\mathrm{H}}^{0}(\pi, B) \to \hat{\mathrm{H}}^{0}(\pi, A \otimes B)$ is the one induced by the tensor product itself and, respectively,

- *for each short exact sequence $0 \to A' \to A \to A'' \to 0$ such that $0 \to A' \otimes B \to A \otimes B \to A'' \otimes B \to 0$ is still exact, for every p and q, and for every $y \in \hat{\mathrm{H}}^{q}(\pi, B)$,*
- *for every short exact sequence $0 \to B' \to B \to B'' \to 0$ such that $0 \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$ is still exact, for every p and q, and for every $x \in \hat{\mathrm{H}}^{p}(\pi, A)$,*

we have commutative diagrams

$$\begin{tikzcd}
\hat{\mathrm{H}}^{p}(\pi, A'') \arrow[r, "\delta"] \arrow[d, "\cup_{pq} y"'] & \hat{\mathrm{H}}^{p+1}(\pi, A') \arrow[d, "\cup_{p+1,q} y"] \\
\hat{\mathrm{H}}^{p+q}(\pi, A'' \otimes B) \arrow[r, "\delta"'] & \hat{\mathrm{H}}^{p+q+1}(\pi, A' \otimes B),
\end{tikzcd}$$

$$\begin{tikzcd}
\hat{\mathrm{H}}^{q}(\pi, B'') \arrow[r, "\delta"] \arrow[d, "x\cup_{pq}"'] & \hat{\mathrm{H}}^{q+1}(\pi, B') \arrow[d, "x\cup_{p,q+1}"] \\
\hat{\mathrm{H}}^{p+q}(\pi, A \otimes B'') \arrow[r, "\delta"'] & \hat{\mathrm{H}}^{p+q+1}(\pi, A \otimes B').
\end{tikzcd}$$

The family is often collectively denoted by $\cup$.

2.4 Application to Class Field Theory

We want to give here a brief account of the so called Class Field Theory. The literature on the topic is extensive, therefore our intention is only to show how Tate cohomology is linked to field theory, in order to complete what we have already cited from the paper by Tate [49].

A *discrete valuation v over a field K* is a group homomorphism $v : (K^{\times}, *) \to (\mathbb{Z}, +)$ such that $v(x+y) \geq \inf(v(x), v(y))$. The function v is formally extended to a map $K \to \mathbb{Z} \cup \{\infty\}$ by the assignment $v(0) = \infty$. Each discrete valuation defines a *valuation ring* $R_v = \{x \in K \mid v(x) \geq 0\}$ and a *valuation ideal* $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$. The ring R is local and $\mathfrak{m}_v$ is its maximal ideal. A *discrete valuation field* is a pair (K, v) where K is a field and v is a discrete valuation over it. Attached to it, we have the *residue class field k_v of* (K, v), that is, the residue class field $R_v/\mathfrak{m}_v$ of R_v.

Given a prime p, we denote by $v_p : \mathbb{Z} \to \mathbb{N}$ the function that associates to each integer n the exponent of the maximum power of p dividing n. By extending each v_p so that $v_p(a/b) = v_p(a) - v_p(b)$, we obtain easy examples of valuations over the field $\mathbb{Q}$. Another example is provided by the field of formal power series $k[[T]]$ over a field k of finite characteristic.

A *local field* is a discrete valuation field (K, v) that is complete with respect to the topology defined by the norm $|x-y| = e^{-v(x-y)}$ and such that its residue class field is finite. If we consider the completions $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to each valuation v_p, it turns out that the examples briefly described above are essentially the only ones, see Proposition 5.2 in [39].

A similar axiomatic description is available also to define *global fields*. In this case, it can be proven that they again are of two kinds:

- number fields, that is, finite field extensions of $\mathbb{Q}$;
- function fields, that is, finite field extensions of $k(x)$, for some finite field k.

As we have already mentioned, one of the main aims of Class Field Theory is to describe all the abelian extensions of a field K: an extension F/K is said to be *abelian* if it is finite, Galois, and if the Galois group $\mathrm{Gal}(F/K)$ is abelian. If we denote by G_K the *absolute Galois group of* K, that is, the Galois group $\mathrm{Gal}(K^s/K)$ of the separable closure K^s of K over K itself, this aim can be achieved by determining $G_K/[G_K, G_K]$, that is, the maximal abelian quotient of G_K.

In this context, the interesting similarity among the two classes of fields described so far is that they both give rise to a class formation.

Definition 2.4.
- A *formation* is a topological group G together with a topological G-module A with a continuous action.
- A *layer* is a pair E and F of subgroups of G such that $[E : F]$ is finite. Moreover, a layer E/F is said to be *normal* if F is normal in E.
- A formation is a *class formation* if, for every normal layer E/F, we have $\mathrm{H}^1(E/F, A^F) = 0$ and $\mathrm{H}^2(E/F, A^F)$ is cyclic of order $[E : F]$.

Given a normal layer E/F, note that, since it is finite, we can consider the Tate cohomology groups $\hat{\mathrm{H}}^n(E/F, A^F)$ with any integral index n. The typical examples of formations are given by the absolute Galois group of a field K endowed with the Krull topology and taking A to be the groups $(K, +)$ or $(K^\times, *)$. Here, taking into account the topology is necessary: otherwise, the usual Galois correspondence between normal subgroups and normal extensions would fail; the right objects in this case are normal closed subgroups. The action of G is continuous if the stabilizer of each element is open.

Theorem 2.5. *Let K be a local field. Then the pair $(G_K, K^\times)$ is class formation.*

This result mainly follows from the so called Hilbert's Theorem 90 (see Proposition 2 in §X.1 of [45]) and the computation of the Brauer group of a local field (see §XIII.3 in [45]). We refer for example to §XII.4 of [45] for complete details.

Combining now Theorems 1.1 and 2.5, we obtain a proof of the Tate's Theorem 1.2 already stated in Section 1.

In the so called global case, the situation is more involved because the classifying object is much less simple than $K^\times$. Let K be a global field. An *idèle of* K is a family $\{x_v\}_v$ in $\prod_v K_v^\times$, such that $x_v \in U_v$, for almost all valuations v, where U_v denotes the multiplicative group of the units in the valuation ring R_v of K_v.

Theorem 2.6. *Let K be a global field. The pair (G_K, I_K) is a class formation.*

3. Tate cohomology for bigger classes of groups and rings

The classical Tate cohomology is defined for a finite group π using $\mathbb{Z}$ as base ring and π-modules as coefficients. Generalizations have been pursued in several directions. A natural one, after the discussion in Section 2.3, is to search for complete resolutions in bigger categories, as for [21], [29], and [42]. Others, like Vogel (reported in [24]) and [35] follow other paths.

3.1 Farrell-Tate cohomology

The *cohomological dimension of a group* G is the smallest integer n such that for every G-module M the groups $\mathrm{H}^m(G, M)$ are all trivial if $m > n$. It is denoted by $\mathrm{cd}(G)$. If such an integer does not exist, we say that G has infinite cohomological dimension, and we write $\mathrm{cd}(G) = \infty$.

We say that a group G has *virtually a property* P if it contains a subgroup of finite index for which property P holds (see [10] Chapter VIII.11).

A group G is then *virtually of finite cohomological dimension* if it has a subgroup of finite index with finite cohomological dimension. In [44], Serre proved that, in this case, this property is also verified for every other subgroup of finite index and, furthermore, their cohomological dimension is the same. The latter depends then only on G and is denoted by $\mathrm{vcd}(G)$. It is called the *virtual cohomological dimension of* G. For example, finite groups have finite virtual cohomological dimension, since the trivial subgroup 0 has finite index and finite cohomology dimension.

Farrell, in [21], extended Tate cohomology from finite groups to this class. His idea is to modify the definition of complete resolution. A key part in the construction exposed in [13] is that we can splice the two projective (free) complexes $\boldsymbol{P} = \ldots \to P_1 \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ and $\mathrm{Hom}(\boldsymbol{P}, \mathbb{Z})$. This is done via two identifications: $\mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z}) \simeq \mathbb{Z}[G]$ and $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}$. To see why this approach cannot be immediately extended, observe that the composition $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}[G]$ associates to 1 the element $\displaystyle\sum_{g \in G} g$. If G is finite, this is the norm element, but otherwise, this is not defined in $\mathbb{Z}[G]$. The idea of Farrell is still to use two complexes, but splicing them in degree $\mathrm{vcd}(G)$ rather than in degree 0.

Definition 3.1. Let G be a group with finite virtual cohomological dimension. A *complete resolution* for G is a pair $(\boldsymbol{X}, \boldsymbol{P})$, where $(\boldsymbol{X}, d^{\boldsymbol{X}})$ is an exact complex of projective G-modules and $(\boldsymbol{P}, d^{\boldsymbol{P}})$ is a projective resolution of $\mathbb{Z}$, also of G-modules, for which we have both $X_n = P_n$ and $d_n^X = d_n^P$ for n sufficiently large.

Proposition 3.2 ([21], Proposition 6). *If G has finite virtual cohomological dimension, there exists a complete resolution for G.*

A sketch of the construction is as follows. Let H be a torsion-free normal subgroup of finite index of G and let π denote the quotient G/H. Let $(\boldsymbol{Y}, d^{\boldsymbol{Y}})$ be a complete resolution for π as in (2.2) so that

$$\begin{tikzcd}
\cdots \arrow[r] & Y_1 \arrow[r, "d_1^Y"] & Y_0 \arrow[rr, "d_0^Y"] \arrow[dr, "\varepsilon"'] & & Y_{-1} \arrow[r, "d_{-1}^Y"] & Y_{-2} \arrow[r, "d_{-2}^Y"] & \cdots \\
& & & \mathbb{Z}. \arrow[ur, hook, "\mu"'] & & &
\end{tikzcd}$$

Let also $(\boldsymbol{Q}, d^{\boldsymbol{Q}})$ be a G-projective resolution of $\mathbb{Z}$. If $s = \mathrm{vcd}(G)$, then it can be shown that $K = \mathrm{Im}(d_s^Q)$ is an H-projective module (see Lemma 5 in [21]). Finally, define the complex $\boldsymbol{X}$ as $X_n = Y_{n-s} \otimes K$ and the complex $\boldsymbol{P}$ such that $P_n = Q_n$ for $n < s$, and $P_n = X_n$ for $n \geq s$, setting the s-th differential to be $d_s^X(x \otimes k) = \varepsilon(x) \cdot k$. We refer to §1 of [21] for the facts needed to prove that $(\boldsymbol{X}, \boldsymbol{P})$ is actually a complete resolution for G.

The *Farrell-Tate cohomology of a G-module A* is now defined as

$$\hat{\mathrm{H}}^n(G, M) = \mathrm{H}^n(\mathrm{Hom}_G(\boldsymbol{X}, M))$$

for a fixed complete resolution $(\boldsymbol{X}, \boldsymbol{P})$ of $\mathbb{Z}$. It is independent from it up to a canonical isomorphism.

By construction, it is immediate to see that, for $n > \mathrm{vcd}(G)$, the functors $\hat{\mathrm{H}}^n(G, -)$ and $\mathrm{H}^n(G, -)$ are isomorphic as connected sequences of functors.

Farrell-Tate cohomology retains many of the features of the original Tate cohomology. For example, it is also both effaceable and coeffaceable, so dimension shifting is still possible and morphisms like $\mathrm{res}_H^G : \hat{\mathrm{H}}^n(G, -) \to \hat{\mathrm{H}}^n(H, \rho_H^G)$ exist for any n as extensions of the usual res_H^G for $n > \mathrm{vcd}(G)$, where ρ_H^G denotes the forgetful functor from $\mathcal{C}(G)$

to $\mathcal{C}(H)$. From this, it can be shown that this cohomology is still torsion: if H is torsion-free, the index $[G:H]$ annihilates $\hat{\mathrm{H}}^n(G,-)$.

The motivation of Farrell in this work is to measure the obstruction to the Bieri-Eckmann duality for groups with finite virtual cohomological dimension. Following [8] we introduce the following definition.

Definition 3.3. Let G be a group, C a right G-module, and n a positive integer. Then G is a *duality group of dimension n with respect to C* if there exists an element $\lambda \in \mathrm{H}_n(G,C)$ such that

$$\lambda\cap : \mathrm{H}^k(G,M) \xrightarrow{\sim} \mathrm{H}_{k-n}(G,C\otimes M)$$

is an isomorphism for every left G-module M and every k. The module C is called the *dualizing module of G*. The symbol $\cap$ denotes the *cap product*; for its definition, symmetric to Definition 2.3, we refer to Chapter XI [13].

The dualizing module and the dimension depend only on G. Actually, we have the identity $n = \mathrm{cd}(G)$ (see Proposition 2 in [8]). This duality is a generalization of the Poincaré duality and we refer to said paper and to [9] for any further topological remark.

At this point, following the spirit of the virtual properties, one may wonder if a *virtual duality group*, that is, a group having a finite index subgroup satisfying (3.3), has a Bieri-Eckmann duality. The answer is in general no and, for a left G-module M, the obstruction is represented by $\hat{\mathrm{H}}(G,M)$; see Theorem 2 and Remark 4 in [21].

After the work of Farrell, there have been various attempts of generalizing Tate cohomology to bigger classes of groups or base rings. In this way, the connections between Tate cohomology and various types of dimensions will become apparent.

3.2 Generalized (co)homological dimension

The mere existence of a complete resolution is not enough to generate a Tate-like cohomology: tools to shift degree along their terms are necessary at least to define the morphisms $f^n : \hat{\mathrm{H}}^n(-,M) \to \hat{\mathrm{H}}^n(-,N)$ from a homomorphism $f : M \to N$. That is, to make the $\hat{\mathrm{H}}^n$'s functors. This was the intention of Ikenaga in giving the following notion.

Definition 3.4. Consider a group G. The *generalized cohomological dimension of* G is the integer

$$\underline{\mathrm{cd}}(G) = \sup(k \in \mathbb{N} \mid \mathrm{Ext}^k_G(M, F) \neq 0, M\ \mathbb{Z}\text{-free}, F\ G\text{-free}).$$

By taking $M = \mathbb{Z}$, we recover the usual cohomological dimension. Some properties of the generalized cohomological dimension and its dual, the *generalized homological dimension* $\underline{\mathrm{hd}}(-)$, are investigated, for example, in [29] and [30]. We remark the following two facts.

1. If $\mathrm{vcd}(G) = n$, then $\underline{\mathrm{cd}}(G) = n$.
2. If G is finite, then $\underline{\mathrm{cd}}(G) = 0$. The converse is also true, see [19]).

For our discussion, the important result is the following one.

Proposition 3.5 ([29] Proposition 13)**.** *If a group G has finite generalized cohomological dimension, then any two complete resolutions of G (in the sense of Definition 3.1) are homotopically equivalent.*

Hence, the next concept is well defined.

Definition 3.6. Let G be a group with finite generalized cohomological dimension. If G possesses a complete resolution, we can extend the Farrell-Tate cohomology as

$$\hat{\mathrm{H}}^n(G, M) = \mathrm{H}^n(\mathrm{Hom}_G(\boldsymbol{X}, M)),$$

where $\boldsymbol{X}$ is the acyclic $\mathbb{Z}G$-projective part of a complete resolution of G.

Also in this context, and given a subgroup H, restriction and corestriction can be defined. Shapiro's lemma continues to hold, and hence $\hat{\mathrm{H}}(G, -)$ is also effaceable and coeffaceable. Using dimension shifting, the $\cup$-product can again be extended to any degree.

3.3 Satellites and completions

After Tohoku [25], usually homology and cohomology are considered as derived functors and computed through resolutions. A slightly different point of view is possible though. The main reference is again [13], Chapter 3. We are now recalling the basic definitions and then we are coming back to our main topic.

Definition 3.7. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between two abelian categories. If $\mathcal{A}$ has enough projectives (resp. injectives), the *left satellites functors* S_nF (resp. the *right satellites* S^nF) are recursively defined as follows on an object $A \in \mathcal{A}$: consider a short exact sequence $0 \to M \xrightarrow{\varphi} P \to A \to 0$ with P projective (resp. $0 \to A \to Q \xrightarrow{\psi} N \to 0$ with Q injective), then

$$\begin{array}{ll} S_0F = F & (\text{resp. } S^0F = F), \\ S_1F = \ker(F(\varphi)) & (\text{resp. } S^1F = \operatorname{coker}(F(\psi))), \\ S_nF = S_1S_{n-1}F & (\text{resp. } S^nF = S^1S^{n-1}F). \end{array}$$

These families of functors, when they exist, are well defined. If they both exist, they may be regrouped as $\{S^nF\}_{n\in\mathbb{Z}}$, where $S^nF = S_{-n}F$ for negative n. If the functor F is *half exact*, that is, if it preserves the exactness in the center of any exact sequence $0 \to A' \to A \to A'' \to 0$, then we have an associated long exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & S^{n-1}F(A'') \arrow[d, "\delta^{n-1}"] & & \\ & S^nF(A') \arrow[r] & S^nF(A) \arrow[r] & S^nF(A'') \arrow[d, "\delta^n"] & \\ & & & S^{n+1}F(A') \arrow[r] & \cdots \end{tikzcd}$$

Note that, when it makes sense, this sequence exists for any n.

Both the left and right derived functors and left and right satellites are examples of *connected sequences of functors*, that is, families of functors $T^\bullet$ together with *connecting homomorphisms* $\delta^\bullet$ associated to each short exact sequence. If F is left (resp. right) exact, its left (resp. right) derived functors coincide with its left (resp. right) satellites.

More or less at the same time as Ikenaga, the notion of completion of a cohomological functor was introduced by Gedrich and Gruenberg in [22].

Definition 3.8. (1) Given a pair of integers r and s, with $r \leq s$ and possibly $r = -\infty, s = +\infty$, an (r,s)-*cohomological functor in an abelian category* $\mathcal{A}$ is a family $\{T^n\}_{n=r}^{s}$ of additive functors $T^n : \mathcal{A} \to \mathcal{A}b$ such that for any short exact sequence we have the usual long exact sequence in cohomology (for the existing indexes).

(2) A *complete cohomological functor* is a $(-\infty,+\infty)$-cohomological functor.

(3) Given an $(r,+\infty)$-cohomological functor $U^\bullet$, a *completion of* $U^\bullet$ is a complete cohomological functor $T^\bullet$ such that U^k is naturally isomorphic to T^k for every $k > n$, for some $n \geq r$.

Given a $(r,+\infty)$-cohomological functor, a category of completions can be defined and a *terminal completion* is a terminal object in this category. Finally, by definition, such a completion is unique up to a unique (collection of) isomorphism(s).

In their aforementioned paper, they proved the following proposition.

Proposition 3.9 ([22] Proposition 1.2)**.** *For a complete cohomological functor $T^\bullet$, the following are equivalent:*

(1) $T^n(P) = 0$ for every projective object P of $\mathcal{A}$ and every n;

(2) for every integer k, $T^\bullet$ is a terminal completion of $T^{\geq k}$;

(3) for every integer k, we have $T^i \simeq S^{i-k}T^k$ for every $i \leq k$, where S^n is the n-th right satellite.

This result can be specialized to the category of modules over a ring.

Theorem 3.10 ([22] Theorem 1.3). *Let M be an R-module. The $(0,\infty)$-cohomological functor $\operatorname{Ext}^{\bullet}_R(M,-)$ has a terminal completion if and only if there exists an integer m, in general depending on M, such that $\operatorname{Ext}^n_R(M,P)=0$ for every projective R-module and every $n \geq m$.*

In the case of a group G of finite virtual cohomological dimension d, the value m can always be chosen as d. Finally, taking $M=\mathbb{Z}$, we see that the Farrell-Tate cohomology is *the* terminal completion of the usual cohomology.

On the other hand, in general, a terminal completion needs not to exist. In [35], Mislin then weakens the definitions and calls *$\mathcal{P}$-complete* a complete cohomological functor $T^{\bullet}$ such that $T^n(P)=0$ for every n and for every projective object P. The definition of *$\mathcal{P}$-completion* follows. In this case, we do have an existence theorem.

Theorem 3.11 ([35] Theorem 2.2). *All complete cohomological functors admit a unique, up to equivalence, $\mathcal{P}$-completion.*

Specializing the proof to G-modules, Mislin is able to present the following definition.

Definition 3.12. Let G be any group and consider the complete cohomological functor $\mathrm{H}^n : \mathcal{M}(G) \to \mathcal{A}b$ defined as $\mathrm{H}^n(G,M)$ for $n \geq 0$ and 0 for $n<0$. We denote by $\hat{\mathrm{H}}^{\bullet}$ its $\mathcal{P}$-completion

$$\hat{\mathrm{H}}^n(G,M) = \varinjlim_{k\geq 0} S^{-k}\mathrm{H}^{k+n}(G,M).$$

Being the Farrell-Tate cohomology *also* the terminal completion of the usual cohomology, this definition coincides with it when G has finite virtual cohomological dimension.

In a similar fashion, with respect to satellites, Benson and Carlson gave in [5] another definition of Tate cohomology for an arbitrary group using the loop-space functors of Heller [26]. In Theorem 4.1, Mislin proves that the two are equivalent. We will come back to this and other interesting results found in that paper in Section 3.7.

A compact but careful review of the results illustrated until this point is also in §5 of [34]. In that paper, Kropholler used the generalized version of Tate cohomology to prove the following result.

Theorem 3.13 ([34] Theorem 3.1.3 and 3.1.4)**.** *Let G be a soluble group or a characteristic zero linear group. If $\mathbb{Z}$ has a projective resolution of finitely projective modules, then G has finite virtual cohomological dimension.*

3.4 Tate cohomology in the Gorenstein context

Around 1986, Buchweitz started writing a 150 pages long paper [42] with the title *Maximal Cohen-Macaulay modules and Tate Cohomology over Gorenstein Rings*. Even if it has never been published in a journal, it is the reference for many developments in each of the three topics mentioned so far. The general program, as stated by Buchweitz, is the equivalence of the following three data:

- right bounded complexes with bounded and finitely generated cohomology, modulo finite complexes of finitely generated projective modules;
- complete resolutions, defined as acyclic projective complexes up to homotopy;
- maximal Cohen-Macaulay modules (see Definition 3.15) up to projective modules.

Buchweitz proves these equivalences in the context of modules over a left-right Noetherian ring that has finite injective dimension as a module over itself. He calls them *strongly Gorenstein rings*. The name choice is to distinguish them from common Gorenstein rings, that are required to be, in addition, commutative. On the other hand, Buchweitz's notation allows us to consider also group rings. The restriction to this context will anyway not be needed until the definition of a maximal Cohen-Macaulay module. Hence, until then, R can be any associative ring with unit.

It must be remarked that even a summary of all the results explained by Buchweitz is above the possibilities of this paper. We will then only sketch the framework in which he worked, keeping his choice of using the language of derived categories.

Recall that, given a ring R, the derived category $\mathcal{D}(R)$ is the localization of the homotopy category $\mathcal{K}(R)$ with respect to quasi-isomorphisms. That means that the objects of $\mathcal{D}(R)$ are complexes and the morphisms between two complexes $\boldsymbol{A}$ and $\boldsymbol{B}$ can be represented as pair of arrows $\boldsymbol{A} \xleftarrow{s} \boldsymbol{S} \xrightarrow{t} \boldsymbol{B}$ where both s and t are morphisms of complexes up to homotopy and s is a quasi-isomorphism. References about triangulated and derived categories are [23] and [33]. A classical and fundamental result is that $\mathcal{D}(R)$ is equivalent to the category $\mathcal{K}^-(\mathcal{P}(R))$, that is, the homotopy category of complexes of projective modules bounded above. Roughly speaking, the equivalence is given by 'taking projective resolutions'.

A complex of R-modules is *perfect* if it is isomorphic, in $\mathcal{D}(R)$, to a finite complex of finitely generated projective modules. The perfect complexes form a triangulated full subcategory of $\mathcal{D}^b(R)$, the *derived category of bounded complexes*, denoted by $\mathcal{D}^b_{\text{perf}}(R)$. We can then give the following definition.

Definition 3.14. The *stabilized derived category of* R is the triangulated quotient

$$\underline{\mathcal{D}^b(R)} = \mathcal{D}^b(R)/\mathcal{D}^b_{\text{perf}}(R).$$

The objects of $\underline{\mathcal{D}^b(R)}$ are the same as $\mathcal{D}^b(R)$, while for the morphisms we have the following property:

$$\mathrm{Hom}_{\underline{\mathcal{D}^b(R)}}(\boldsymbol{A}, \boldsymbol{B}[n]) \simeq \mathrm{Hom}_{\mathcal{D}^b(R)}(\boldsymbol{A}, \boldsymbol{B}[n]), \qquad \text{for } n \text{ big enough},$$

where n depends on $\boldsymbol{A}$ and $\boldsymbol{B}$. Hence the word 'stabilized': what counts is, somehow, the 'tail' of $\boldsymbol{B}$. This is the first of the data listed by Buchweitz.

The second one is the full subcategory of $\mathcal{K}(\mathcal{P}(R))$ given by the acyclic complexes of finitely generated projective R-modules. Buchweitz denotes it by $\underline{\mathrm{APC}(R)}$.

Finally, for the third one, we need the following.

Definition 3.15. A *maximal Cohen-Macaulay* module over a strongly Gorenstein ring R is a left module M such that $\mathrm{Ext}^i_R(M, R) = 0$ for $i \neq 0$. In short they are denoted by MCM. They constitute a full subcategory of $\mathcal{C}(R)$ denoted by $\mathrm{MCM}(R)$.

The interesting properties of Gorenstein rings and MCM modules are numerous, we refer for example to [4] or [18].

We need one last construction, that we will use also later, to fit all these object in a unique elegant diagram. Following for example [26] or [1], we proceed as follows.

Given the category $\mathcal{C}(R)$ for some ring R, or more general an abelian category $\mathcal{A}$, the *stabilized category of finitely generated R-modules* is the category $\underline{\mathcal{C}(R)}$ such that $\mathcal{O}bj(\underline{\mathcal{C}(R)}) = \mathcal{O}bj(\mathcal{C}(R))$ while

$$\mathrm{Hom}_{\underline{\mathcal{C}(R)}}(M, N) = \frac{\mathrm{Hom}_R(M, N)}{\mathcal{P}(M, N)},$$

where the denominator is the group of morphisms $f : M \to N$ that factors through a projective module. We will denote $\mathrm{Hom}_{\underline{\mathcal{C}(R)}}(M, N)$ by $\underline{\mathrm{Hom}}_R(M, N)$.

By a universal property of the stabilization, we obtain a canonical decomposition that gives the diagram

$$\begin{tikzcd}
\mathcal{C}(R) \arrow[r] \arrow[d] & \mathcal{D}^b(R) \arrow[r] & \underline{\mathcal{D}^b(R)} \\
\underline{\mathcal{C}(R)} \arrow[urr, dashed, "\iota_R"'] & &
\end{tikzcd}.$$

Recall that, given a complex of left R-modules $\boldsymbol{A}$, the *n-th syzygy of $\boldsymbol{A}$* is $\Omega_n(\boldsymbol{A}) = \mathrm{coker}(d_A^{-n})$.

Theorem 3.16 ([42] Theorem 4.4.1)**.** *Let R be a strongly Gorenstein ring. The 0-th syzygy functor induces an equivalence of categories via* $\Omega_0 : \underline{\mathrm{APC}(R)} \longrightarrow \underline{\mathrm{MCM}(R)}$*, the functor ι_R induces an equivalence* *of categories* $\underline{\mathrm{MCM}(R)} \longrightarrow \underline{\mathcal{D}^b(R)}$*, and the two structures induced on* $\underline{\mathrm{MCM}(R)}$ *agree.*

All these maps fit into the following commutative diagram (up to isomorphisms of functors):

$$
\begin{tikzcd}
 & 0 \arrow[r] & P(R) \arrow[r] \arrow[dl, equal] & \mathrm{MCM}(R) \arrow[rr] \arrow[dl] & & \underline{\mathrm{MCM}(R)} \arrow[r] \arrow[dl] \arrow[dd, "\iota_R"] & 0 \\
0 \arrow[r] & P(R) \arrow[r] \arrow[dr] & \mathcal{C}(R) \arrow[r] \arrow[dr] & \underline{\mathcal{C}(R)} \arrow[r] \arrow[ddrr, phantom, ""] \arrow[drr, "\iota_R"] & 0 & & \underline{\mathrm{APC}(R)} \arrow[ul, "\Omega_0"'] \arrow[dl, "\sigma_R"] \\
 & 0 \arrow[r] & \mathcal{D}^b_{perf}(R) \arrow[r] & \mathcal{D}^b(R) \arrow[rr] & & \underline{\mathcal{D}^b(R)} \arrow[r] & 0.
\end{tikzcd}
\tag{3.1}
$$

For the explicit description of the morphism $\underline{\mathrm{APC}(R)} \xrightarrow{\sigma_R} \underline{\mathcal{D}^b(R)}$, we refer to [42]. Buchweitz denotes its quasi-inverse by $\mathbf{CM}$ and the quasi-inverse of ι_R by $\mathbf{M}$. These functors give a second diagram ([42] Theorem 5.6.7), again commutative up to isomorphisms of functors:

$$
\begin{tikzcd}
 & \mathcal{D}^b(R) \arrow[dl, "\mathbf{CM}"'] \arrow[d, "\mathbf{M}"] \arrow[dr] & \\
\underline{\mathrm{APC}(R)} \arrow[r, "\Omega_0"] & \underline{\mathrm{MCM}(R)} \arrow[r, "\iota_R"] & \underline{\mathcal{D}^b(R)}.
\end{tikzcd}
$$

Given an R-module M, its image $\mathbf{M}(M)$ is called the *maximal Cohen-Macaulay approximation of* M.

Now that we have defined and established the equivalence among our three categories, we can finally state the generalization of Tate cohomology proposed by Buchweitz.

Definition 3.17. Let M and N be two modules (but they can even be complexes with bounded cohomology) over a strongly Gorenstein ring R. The *n-th Tate cohomology group of M with value in N* is

$$\underline{\mathrm{Ext}}^n_R(M,N) = \underline{\mathrm{Hom}}_{\mathcal{D}^b(S)}(M, N[n]).$$

Equivalently, we have

$$\mathrm{H}^n(\mathrm{Hom}_R(\mathbf{CM}(M), N)) = \underline{\mathrm{Ext}}_R^n(M, N) = \underline{\mathrm{Hom}}_R(\Omega_R^n(\mathbf{M}(M)), N),$$

where Ω_R^n is the loop-space functor attached to the stabilized category $\underline{\mathcal{C}(R)}$.

This new Tate cohomology enjoys similar properties of the classical one: it is functorial in both components, it is effaceable and coeffaceable (and hence it allows dimension shifting), and it has a product. Moreover, for n big enough, we have $\underline{\mathrm{Ext}}_R^n(M, N) = \mathrm{Ext}_R^n(M, N)$ and $\underline{\mathrm{Ext}}_R^{-n}(M, N)$ is isomorphic to $\mathrm{Tor}_{i-1}^R(N, M^*)$.

The examples that can be analyzed with this machinery are vast, but the essential exposition we could give here prevents us from saying more. As expected though, the classical Tate cohomology is recovered through $\hat{\mathrm{H}}^n(G, M) = \underline{\mathrm{Ext}}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M)$.

Given the existence of an MCM approximation for every module over a strongly Gorenstein ring, this kind of modules may also be used instead of the projectives modules to build resolutions and, hence, a cohomology theory. This has been done, for example, in [2]. In that paper, the theory is actually developed for *totally reflexive modules* over a left and right Noetherian ring. Given an R-modules M, we denote by M^* the *dual of* M, that is $\mathrm{Hom}_R(M, R)$.

An R-module M is *reflexive* if the natural morphism $M \to M^{**}$ is bijective. A reflexive module is *totally reflexive* if also

$$\mathrm{Ext}_R^n(M, R) = 0 = \mathrm{Ext}_{R^{\mathrm{op}}}^n(M^*, R)$$

holds for every $n > 0$.

On the other hand, on a local Gorenstein ring R a finite module M is totally reflexive if and only if it is MCM (see [11], Theorem 3.3.10.d). To keep the generality of [2], we will consider totally reflexive modules, but said equivalence should be kept in mind. Denote by $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{P}$ the full subcategories of $\mathcal{C}(R)$ respectively constituted by finite R-modules, totally reflexive R-modules, and projective R-modules. Note the inclusion $\mathcal{G} \subset \mathcal{P}$.

Definition 3.18. Let M be an R-module. The *Gorenstein dimension of* M, denoted by $G\dim_R(M)$, is the minimum integer g for which there exists a $\mathcal{G}$-resolution of M of the form

$$0 \longrightarrow G_g \longrightarrow G_{g-1} \longrightarrow \dots \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

If no such g exists, M is said to have infinite Gorenstein dimension.

Denote now also by $\tilde{\mathcal{G}}$ the finite R-modules with finite Gorenstein dimension and by $\tilde{\mathcal{P}}$ the finite R-modules of finite projective dimension. To understand the relevance of these two classes, we remark that they coincide for a Gorenstein ring R if and only if R itself is regular.

We need an extra technical refinement before constructing the so called relative cohomology: an exact complex A is called *proper* if for every $M \in \mathcal{G}$ the complex $\mathrm{Hom}_R(M, A)$ is still exact. A $\mathcal{G}$-resolution $\boldsymbol{G} \to M$ of an R-module M is *proper* if $\dots \to G_1 \to G_0 \to M \to 0$ is proper. Denote by $\bar{\mathcal{G}}$ the full subcategory of $\mathcal{F}$ of finite R-modules having a proper $\mathcal{G}$-resolution. We need this refinement because $\tilde{\mathcal{G}}$ does contain $\bar{\mathcal{G}}$ but possibly strictly.

Definition 3.19. For every pair of R-modules $M \in \bar{\mathcal{G}}$ and N, we define the *relative cohomology of* M *with coefficients in* N as

$$\mathrm{Ext}^n_{\mathcal{R}}(M, N) = \mathrm{H}^n(\mathrm{Hom}^n_R(\boldsymbol{G}, N)),$$

where $\boldsymbol{G}$ is a proper $\mathcal{G}$-resolution of M.

For a discussion of the properties of the relative cohomology and their connection with the Gorenstein projective dimension, we refer to §4 of [2]. Now, if we also consider a $\mathcal{P}$-resolution $\boldsymbol{P}$ of M, the identity $\mathbb{1}_M$ can be lifted to a morphism $\boldsymbol{P} \to \boldsymbol{G}$ to give collection of morphisms

$$\varepsilon^n_{\mathcal{R}}(M, N) : \mathrm{Ext}^n_{\mathcal{R}}(M, N) \longrightarrow \mathrm{Ext}^n_R(M, N).$$

In this context (see Definition 3.27), a complete resolution M is a diagram $\boldsymbol{T} \xrightarrow{\theta} \boldsymbol{P} \xrightarrow{\varphi} M$ where $\boldsymbol{T}$ is a totally acyclic complex, $\boldsymbol{P}$ is in

$\mathcal{P}$, and θ^n is an isomorphism for n big enough. Here *totally acyclic* means that both $\boldsymbol{T}$ and $\boldsymbol{T}^*$ are acyclic. In the same style as the previous paragraphs, we give the following.

Definition 3.20. Given a strongly Gorenstein ring R and R-modules M and N, the n-th *Tate cohomology group of* M *with coefficients in* N is

$$\widehat{\operatorname{Ext}}^n_{\mathcal{R}}(M,N) = \mathrm{H}^n(\operatorname{Hom}^n_R(\boldsymbol{T},N)).$$

Also in this case we obtain a natural map:

$$\varepsilon^n_R(M,N) : \operatorname{Ext}^n_R(M,N) \longrightarrow \widehat{\operatorname{Ext}}^n_R(M,N).$$

This definition of the Tate cohomology enjoys the expected properties.

Theorem 3.21. *For every n, $\widehat{\operatorname{Ext}}^n_R : \tilde{\mathcal{G}}^{\mathrm{op}} \times \mathcal{M}(R) \to \mathcal{A}b$ is a functor. Moreover*

1. *these functors and the morphisms $\varepsilon^n_R(M,N)$ are independent of $\boldsymbol{T}$ and φ;*
2. *the module M has finite Gorenstein dimension $G\dim_R(M) = g$ if and only if $\varepsilon^n_R(M,N)$ is an isomorphism whenever $n > g$;*
3. *the R-module M has finite projective dimension if and only if we have $\widehat{\operatorname{Ext}}^n_R(M,-) = 0$ for every n;*
4. *the R-module N has finite projective dimension if and only we have if $\widehat{\operatorname{Ext}}^n_R(-,N) = 0$ for every n; and*
5. *for every short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\tilde{\mathcal{G}}$ and $0 \to N' \to N \to N'' \to 0$ in $\mathcal{C}(R)$, we have long exact sequences in cohomology.*

The proofs of (1) and (2) are in [2], Theorem 5.2. That of (3) and (4) are combined in [2], Theorem 5.2 and Theorem 5.9. For (5) see [2], Proposition 5.4 and [14], Theorem 5.4 (see also the following remark).

Remark 3.22. **(a)** In [51], Veliche generalized both the definition of Gorenstein dimension and Tate cohomology to complexes bounded above in the controvariant component and below in the covariant component. See also Definition 3.28 further in this paper.

(b) The functor $\widehat{\mathrm{Ext}}_R^n$ is defined as $\mathcal{G}^{\mathrm{op}} \times \mathcal{M}(R) \to \mathcal{A}b$ instead of $\mathcal{M}(R)^{\mathrm{op}} \times \mathcal{M}(R) \to \mathcal{A}b$ only to ensure the existence of a complete resolution. In fact, Christensen and Jorgensen [14] proved that Tate cohomology can be computed, already in the general context of complexes just mentioned, through a complete *injective* resolution of N, in the sense of [41]. Hence, provided the necessary hypotheses to ensure the existence of a complete projective or injective resolutions, the Tate cohomology is *balanced*.

(c) Analogous results are available for Tate homology, see [28].

(d) It is enough to verify Properties 3 and 4 only for a single index n or when $\widehat{\mathrm{Ext}}_R^0(M, M) = 0$. The proof is again in [2], Theorem 5.9.

Through the so called comparison morphisms δ_R^n (see [3], Theorem 7.1), the relative, absolute, and Tate cohomology fit in the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}_{\mathcal{R}}^0(M,N) \xrightarrow{\varepsilon_{\mathcal{R}}^0} \mathrm{Ext}_R^n(M,N) \xrightarrow{\varepsilon_R^0} \dots \\ \dots &\xrightarrow{\varepsilon_R^n} \widehat{\mathrm{Ext}}_R^n(M,N) \xrightarrow{\delta_R^n} \mathrm{Ext}_{\mathcal{R}}^{n+1}(M,N) \xrightarrow{\varepsilon_{\mathcal{R}}^{n+1}} \mathrm{Ext}_R^{n+1}(M,N) \longrightarrow \dots . \end{aligned} \tag{3.2}$$

3.5 Tate-Vogel or stable cohomology

In the 80s, in a private letter, Vogel developed another method to generalize the Tate cohomology. His ideas have been written in a paper by Goichot [24] and taken over several times by other authors. We present here a modern formulation developed by Avramov and Veliche in [3], presented with the language of differentially graded-categories, or DG-categories. The basic example of a DG-category is a DG-algebra, seen as the set of homomorphisms of the category with one object. A reference for this topic is the introductory paper by Toën [50]. Here, we will consider the DG-category of complexes of R-modules, for an associative ring R. This is an *enhanced category* where, given two complexes $(\boldsymbol{A}, d^{\boldsymbol{A}})$ and $(\boldsymbol{B}, d^{\boldsymbol{B}})$, the complex $\mathrm{Hom}_R(\boldsymbol{A}, \boldsymbol{B})$ is defined as

$$\mathrm{Hom}_R(\boldsymbol{A}, \boldsymbol{B})_n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_R(A_i, B_{i+n}) = \mathrm{Hom}_R(\boldsymbol{A}, \boldsymbol{B})^{-n},$$

with differentials

$$d(\alpha_i) = d^{\boldsymbol{B}}\alpha_i - (-1)^i \alpha_{i-1} d^{\boldsymbol{A}}.$$

There is no risk of confusion, since the normal set of morphisms in $\mathcal{C}(R)$ would be denoted by $\mathrm{Hom}_{\mathcal{C}(R)}(-,-)$. In fact, if any of the two complexes is concentrated in one degree, for example $\boldsymbol{B}$, we have the equality $\mathrm{H}^n(\mathrm{Hom}_R(\boldsymbol{A}, B)) = \mathrm{H}^n\mathrm{Hom}_{\mathcal{C}(R)}(\boldsymbol{A}, B)$ for every n, since B is the only non-zero component, no matter the degree. In this context, if $\boldsymbol{P} \to M$ and $\boldsymbol{Q} \to N$ are two projective resolutions of two R-modules M and N respectively, we have that $\mathrm{Hom}_R(\boldsymbol{P}, \boldsymbol{Q})$ and $\mathrm{Hom}_R(\boldsymbol{P}, N)$ are quasi-isomorphic and satisfy

$$\mathrm{H}(\mathrm{Hom}_R(\boldsymbol{P}, \boldsymbol{Q})) = \mathrm{H}(\mathrm{Hom}_R(\boldsymbol{P}, N)) = \mathrm{Ext}_R(M, N),$$

as graded abelian groups. Moreover, we will use the important subcomplex of $\mathrm{Hom}_R(\boldsymbol{A}, \boldsymbol{B})$ defined as

$$\overline{\mathrm{Hom}}_R(\boldsymbol{A}, \boldsymbol{B})_n = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_R(A_i, B_{i+n}) = \overline{\mathrm{Hom}}_R(\boldsymbol{A}, \boldsymbol{B})^{-n}.$$

For two R-modules M and N, we can define the following additional graded abelian group.

Definition 3.23. The *bounded cohomology of M with coefficients in N over R* is the graded group

$$\overline{\mathrm{Ext}}_R(M, N) = \mathrm{H}(\overline{\mathrm{Hom}}_R(\boldsymbol{P}, \boldsymbol{Q})),$$

where $\boldsymbol{P} \to M$ and $\boldsymbol{Q} \to N$ are two projective resolutions of M and N respectively.

The choice of the term 'bounded' comes from the fact that every DG-morphism α_i vanishes when $|i|$ is big enough.

Denoting by $\widehat{\mathrm{Hom}}_R(\boldsymbol{P},\boldsymbol{Q})$ the quotient $\overline{\mathrm{Hom}}_R(\boldsymbol{P},\boldsymbol{Q})/\mathrm{Hom}_R(\boldsymbol{P},\boldsymbol{Q})$, we can now follow Vogel.

Definition 3.24. Given a ring R and two R-modules M and N, we call *stable cohomology* the graded abelian group

$$\widehat{\mathrm{Ext}}_R(M,N) = \mathrm{H}(\widehat{\mathrm{Hom}}_R(\boldsymbol{P},N)),$$

where $\boldsymbol{P}$ is any projective resolution of M.

Given a group G and a commutative ring K, for every $K[G]$-module M, we call *Tate-Vogel cohomology groups of* M the collection

$$\hat{\mathrm{H}}^n_K(G,M) = \widehat{\mathrm{Ext}}^n_{K[G]}(K,M) = \mathrm{H}^n(\mathrm{Hom}_{K[G]}(\boldsymbol{P},M)),$$

where $\boldsymbol{P} \to K$ is a $K[G]$-projective resolution of K.

Remark 3.25. **(a)** The same process can be carried on with tensor product to develop homology. This is actually the original approach of Goichot [24]. Beware: the notation in [3] is highly inconsistent with that of [24].

(b) To define $\widehat{\mathrm{Ext}}_R(M,N)$, one could of course also use an injective resolution $N \to \boldsymbol{I}$ and compute $\mathrm{H}(\widehat{\mathrm{Hom}}_R(M,\boldsymbol{I}))$.

(c) In case G has finite virtual cohomological dimension, we recover the already defined Farrell-Tate cohomology (cf. [24], Theorem 3.1).

By construction, we have the following short exact sequence

$$0 \longrightarrow \overline{\mathrm{Hom}}_R(\boldsymbol{A},\boldsymbol{B}) \longrightarrow \mathrm{Hom}_R(\boldsymbol{A},\boldsymbol{B}) \longrightarrow \widehat{\mathrm{Hom}}_R(\boldsymbol{A},\boldsymbol{B}) \longrightarrow 0$$

that yields a link among the three cohomology theories just defined through the long exact sequence similar to (3.2):

$$\cdots \longrightarrow \mathrm{Ext}^n_R(M,N) \longrightarrow \widehat{\mathrm{Ext}}^n_R(M,N) \longrightarrow \overline{\mathrm{Ext}}^{n+1}_R(M,N) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Ext}^{n+1}_R(M,N) \longrightarrow \widehat{\mathrm{Ext}}^{n+1}_R(M,N) \longrightarrow \overline{\mathrm{Ext}}^{n+2}_R(M,N) \longrightarrow \cdots.$$

Hence, this suggests that, *whenever the suitable resolutions exist*, there is an isomorphism

$$\overline{\mathrm{Ext}}^n_R(M,N) \simeq \mathrm{Ext}^n_{\mathcal{R}}(M,N).$$

We conclude this section with the following criterion given by Tate-Vogel cohomology to detect Gorenstein rings.

Theorem 3.26 (Corollary 6.3 and Theorem 6.4 in [3]). *Let R be a commutative local ring with maximal ideal $\mathfrak{m}$ and residue field k. If the rank of $\widehat{\mathrm{Ext}}^n_R(k,k)$ is finite for some integer n, then R is Gorenstein.*

3.6 Complete resolutions versus totally acyclic approximations

So far, we have discussed complete resolutions and see how they are related to various generalizations of Tate cohomology.

In general, the issues with the concept of a complete resolution (of projectives) are two. The first and most difficult is of course its existence. The second is how to ensure that the cohomology defined through it does not depend on the choice of a particular one. The latter is easier to address and it has been settled in [17] by Cornikh and Kropholler, for modules, and by Veliche, for complexes of modules, with the following two generalized definitions.

Definition 3.27. Let R be a ring and M an R-module. A *complete resolution of M* is a morphism of complexes $\boldsymbol{T} \xrightarrow{\tau} \boldsymbol{P}$ where

(1) the complex $\boldsymbol{T}$ is an exact complex of R-projective modules such that $\mathrm{Hom}_R(\boldsymbol{T},Q)$ is still exact for every projective R-module Q;

(2) the complex $\boldsymbol{P}$ is an R-projective resolution of M;

(3) the morphism τ_n is an isomorphism for n big enough.

A complex satisfying condition (1) is said to be *totally acyclic*.

Definition 3.28. Let $\boldsymbol{M}$ be a complex over a ring R.

(1) A complex $\boldsymbol{P}$ is *semiprojective* if P_i is projective for every i and $\mathrm{Hom}_R(\boldsymbol{P}, -)$ preserves quasi-isomorphisms.

(2) A *semiprojective resolution of* $\boldsymbol{M}$ is any quasi-isomorphism $\varphi : \boldsymbol{P} \to \boldsymbol{M}$, where $\boldsymbol{P}$ is semiprojective.

(3) A *complete resolution of* $\boldsymbol{M}$ is a diagram $\boldsymbol{T} \xrightarrow{\tau} \boldsymbol{P} \xrightarrow{\pi} \boldsymbol{M}$, where $\boldsymbol{T}$ is a totally acyclic complex, $\boldsymbol{P}$ is a semiprojective resolution of $\boldsymbol{M}$, and τ_n is an isomorphism for n big enough.

Remark 3.29. The latter consistently includes the former, as semiprojectivity reduces, for complexes concentrated in 0, to preserving common isomorphisms.

The additional requirements, compared for example to Definition 2.2, ensure the independence of Tate cohomology from the complete resolution chosen for the computation. In the case of a finite group, of a group with finite virtual cohomological dimension, or over a Gorenstein ring, they are not explicitly necessary, because they are automatically verified. More precisely, Iyengar and Krause proved the following.

Theorem 3.30 (Corollary 5.5 in [31]). *Let R be a Noetherian commutative ring with a dualizing complex (see Definition 3.31). Then the following are equivalent:*

(1) R is Gorenstein;

(2) every acyclic complex of projective R-modules is totally acyclic;

(3) every acyclic complex of injective R-modules is totally acyclic.

Definition 3.31. Let R be a commutative Noetherian ring. A complex $\boldsymbol{D}$ is a *dualizing complex* if the cohomology of $\boldsymbol{D}$ is bounded and finitely generated over R, the complex $\boldsymbol{D}$ has finite injective dimension, and the canonical homomorphism $R \to R\mathrm{Hom}_R(\boldsymbol{D}, \boldsymbol{D})$ is a quasi-isomorphisms, hence an isomorphism in $\mathcal{D}(R)$.

Remark 3.32. If π is a finite abelian group and k is a field, then $k[\pi]$ is a dualizing complex. The hypothesis of being abelian can be removed, see Setup 1.4' and Proposition 3.8 in [32]. This also makes evident that this definition cannot be confused with Definition 3.3.

We now turn our attention to the existence of a complete resolution. We have already seen the definition of the Gorenstein dimension for modules. As explained by Veliche, there is an equivalent formulation, where the connection to complete resolutions is more evident.

Definition 3.33. Let R be a ring.

(1) An R-module G is said to be *Gorenstein projective* if there exists a totally acyclic complex $(\boldsymbol{T}, d^{\boldsymbol{T}})$ such that $G = \operatorname{coker}(d_1)$.

(2) The *Gorenstein projective dimension* $\operatorname{Gpd}_R(M)$ of an R-module M is the minimum integer g such that there exists an exact sequence

$$0 \longrightarrow G_g \longrightarrow G_{g-1} \longrightarrow \dots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

where G_i is Gorenstein projective for every i.

(3) The *Gorenstein projective dimension* $\operatorname{Gpd}_R(\boldsymbol{M})$ of a complex of left R-modules $\boldsymbol{M}$ is the minimum integer g such that there exists a complete resolution $\boldsymbol{T} \xrightarrow{\tau} \boldsymbol{P} \longrightarrow \boldsymbol{M}$ for which τ_n is an isomorphism for every $n \geq g$.

Remark 3.34. For all R-modules M, we have $G\dim_R(M) = \operatorname{Gpd}_R(M)$, see (2.4.1) in [51]. Moreover, if $\boldsymbol{M}$ is concentrated in 0, the notion of Gorenstein projective dimension for modules and complexes coincide, see Corollary 3.6 in [51].

Theorem 3.35 (Theorem 3.4 in [51])**.** *Let g be an integer and $\boldsymbol{M}$ be a complex of left R-modules. We have $\operatorname{Gpd}_R \boldsymbol{M} < g$ if and only if for every semiprojective resolution $\boldsymbol{P} \longrightarrow \boldsymbol{M}$, there exists a (surjective) complete resolution $\boldsymbol{T} \xrightarrow{\tau} \boldsymbol{P} \longrightarrow M$ such that $\tau_n = \mathbb{1}_{T_n}$ for every $n \geq g$.*

Remark 3.36. A commutative ring R is Gorenstein if and only if every R-module has finite Gorenstein projective dimension. In this case, we recover the same result of Section 3.4.

If, for example, a group G has finite virtual cohomological dimension n, then for every G-module M we have $\operatorname{Gpd}_{\mathbb{Z}[G]}(M) \leq n$.

From a different point of view, in [32] Jørgensen argued that a generalization of Tate cohomology, in order to be called such, should be bound to some sort of complex. Hence, from this perspective, the Tate-Vogel cohomology should then be only called stable cohomology. In the same vein, he suggested a different approach that we now outline.

Given a ring R, we denote by $\mathcal{K}_{\text{tac}}(\mathcal{P}(R))$ the full subcategory of $\mathcal{K}(\mathcal{P}(R))$ consisting of totally acyclic complexes of projective left R-modules (we are adopting here the notation of [31]), while in [32] the same category is denoted by $\mathrm{E}(R)$) and by $e_* : \mathcal{K}_{\text{tac}}(\mathcal{P}(R)) \to \mathcal{K}(\mathcal{P}(R))$ the natural embedding. To avoid confusion, in this paper we work with the following concept.

Definition 3.37. Assume that e_* has a right adjoint $e^!$. We call the image $e^!(\boldsymbol{P})$ of a complex in $\mathcal{K}(\mathcal{P}(R))$ a *totally acyclic projective approximation of* $\boldsymbol{P}$.

We are now ready for a different generalization of Tate cohomology.

Definition 3.38. Let R be a ring and assume that e_* has a right adjoint $e^!$. Then, given two left R-modules M and N, we define the Tate cohomology groups of M and N as

$$\widehat{\mathrm{Ext}}^n(M, N) = \mathrm{H}^n(\mathrm{Hom}_R(e^!(\boldsymbol{P}), N));$$

where $\boldsymbol{P}$ is any projective resolution of M.

The advantage of considering totally acyclic projective approximations rather than complete projective resolutions is that they exist for a wider range of cases. In the same paper, Jørgensen proves for example the following.

Theorem 3.39 (Theorem 1.10 in [32])**.** *In any of the following two cases*

- *R is a commutative noetherian ring with a dualizing complex,*
- *R is a left-coherent and right noetherian k-algebra over a field k for which there exists a left-noetherian k-algebra S and a dualizing complex ${}_BD_A$,*

the functor $e_* : \mathcal{K}_{\mathrm{tac}}(\mathcal{P}(R)) \to \mathcal{K}(\mathcal{P}(R))$ *has a right adjoint.*

The two notions coincide if a complete resolution exists.

Theorem 3.40 (Lemma 3.6 in [32]). *Let R be a ring such that e_* has a right adjoint $e^!$. Let M be a left R-module such that a complete resolution $\boldsymbol{T} \to \boldsymbol{P} \to M$ exists. Then we have*

$$e^!(\boldsymbol{P}) \simeq \boldsymbol{T}.$$

Outside of this setting, though, the cohomology theories do differ, as Iyengar and Krause showed with the following example.

Example 3.41 (§1 in [31]). Let A be a commutative local ring, with maximal ideal $\mathfrak{m}$ and residue field k, such that $\mathfrak{m}^2 = 0$ and $\mathrm{rank}_k(\mathfrak{m}) \geq 2$. This ring is not Gorenstein, but the injective hull of k is a dualizing complex for A: hence an adjoint $e^! : \mathcal{K}(\mathcal{P}(R)) \to \mathcal{K}_{\mathrm{tac}}(\mathcal{P}(R))$ exists.

By Proposition 6.1 in [31], any totally acyclic complex in $\mathcal{C}(A)$ is homotopically trivial, that is, corresponds to 0 in $\mathcal{K}(A)$. As a consequence, the Tate cohomology $\widehat{\mathrm{Ext}}(M, N)$, in the sense of Jørgensen, is trivial for any pair of A-modules M and N. On the contrary, this cannot be the stable $\widehat{\mathrm{Ext}}$ computed following [24] or [3], since, by Theorem 3.26, the ring A would then be a Gorenstein ring.

3.7 Products in negative cohomology

Tate cohomology can be seen as an extension of the regular cohomology to negative degrees. We have reported that the cup-product can be also extended, with the same notation, to a product

$$\cup : \hat{\mathrm{H}}^p(G, A) \otimes \hat{\mathrm{H}}^q(G, B) \longrightarrow \hat{\mathrm{H}}^{p+q}(G, A \otimes B),$$

for any pair of integers p and q. A natural curiosity could then be to investigate the nature of these *mixed* ($pq < 0$) or *negative* ($p, q < 0$) products. It should be noted that, even if in principle we have already shown such a case since the beginning (see for example Section 1.2), in the previous situations the cohomology itself was periodic; hence, the cup-product was just a shift of the classical one and we are not really in the presence of new phenomena.

In [5], Benson and Carlson have shown interesting results on this topic, in the specific case of $K[G]$-modules, where K is a field of positive characteristic and G is a finite group. The computational advantages in this context are that

- $K[G]$-modules are in fact vector spaces over K, with a G-action,
- the ring $K[G]$ is itself self-injective (see for example [15]; more in general, $K[G]$ is a Frobenius K-algebra),
- projective $K[G]$-modules are also injective.

Before restricting to these conditions, a generalized definition for Tate cohomology is proposed in [5], for any commutative ring R and any group G, as follows.

Definition 3.42. Let R be a commutative ring and G a group. Given two $R[G]$-modules M and N and projective resolutions $\boldsymbol{P} \xrightarrow{\varepsilon} M$ and $\boldsymbol{Q} \xrightarrow{\eta} N$, an *almost-chain map* μ *of degree* n *from* $(\boldsymbol{P}, \varepsilon)$ *to* $(\boldsymbol{Q}, \eta)$ is a family $\{\mu_i\}_{i\in\mathbb{Z}}$ of $R[G]$-homomorphisms $\mu_i : P_{i+n} \to Q_i$, such that for all but a finite number of indices the diagram

$$\begin{tikzcd} P_{i+n} \arrow[r, "d_{i+n}"] \arrow[d, "\mu_i"'] & P_{i+n-1} \arrow[d, "\mu_{i-1}"] \\ Q_i \arrow[r, "d_i"'] & Q_{i-1} \end{tikzcd}$$

commutes. The $R[G]$-modules P_i and Q_i are assumed to be trivial for $i < 0$, so that no restriction is required on the indexes.

Two almost-chain maps μ and ν are *almost-chain homotopic* if there exists a family $\{\sigma_i\}_{i\in\mathbb{Z}}$ of $R[G]$-homomorphisms such that $\mu_i - \nu_i = d_i \circ \sigma_i + \sigma_{i+1} \circ d_{i+n+1}$ for all but a finite number of indices.

Observe that the composition of two almost-chain maps of degree m and n is an almost-chain map of degree $m+n$ and almost chain-homotopy gives an equivalence relation. The following definition can then be given.

Definition 3.43. Let R be a commutative ring, G be a group, and M and N be $R[G]$-modules. We set

$$\widehat{\mathrm{Ext}}^n(M,N) = \left\{ \begin{array}{c} \text{almost-chain homotopy classes of} \\ \text{almost-chain maps } (\boldsymbol{P},\varepsilon) \to (\boldsymbol{Q},\eta) \text{ of degree } n \end{array} \right\}.$$

In particular, taking R as M, we define

$$\hat{\mathrm{H}}^n(G,M) = \widehat{\mathrm{Ext}}^n_{R[G]}(R,M) = \widehat{\mathrm{Ext}}^n_G(\mathbb{Z},M).$$

We can now observe that if G is finite (or of finite virtual cohomological dimension, respectively), we recover the classical Tate (or Farrell) cohomology. The usage of complete resolutions is avoided by allowing some of the morphisms to be in some sense ignored in finitely many degrees, relying on the fact that in order to compute $\hat{\mathrm{H}}^\bullet(G.M)$ only the leftmost part of each resolution is relevant.

The Yoneda products

$$\widehat{\mathrm{Ext}}^p_{R[G]}(M_2,M_3) \otimes \widehat{\mathrm{Ext}}^q_{R[G]}(M_1,M_2) \longrightarrow \widehat{\mathrm{Ext}}^{p+q}_{R[G]}(M_1,M_3)$$

are also directly defined using the already mentioned fact that the composition of almost-chain maps is still an almost-chain map of the required degree. (For a full definition of the cup-products, see §2 in [5].)

After having generalized Tate cohomology and cup-products, Benson and Carlson proceed with their program, proving among other results the following.

Proposition 3.44 ([5], Lemma 2.2)**.** *If K is a field of positive characteristic p and G is a finite group such that its Sylow p-subgroup is neither cyclic nor the generalized quaternion group, then*

$$\hat{\mathrm{H}}^m(G,K) \cdot \hat{\mathrm{H}}^{-m}(G,K) = 0.$$

Theorem 3.45 ([5], Theorem 3.1 and 3.3). *Let the p-rank of G be greater than 1. Then if the cohomology ring* $\mathrm{H}(G, M)$ *is Cohen-Macaulay, we have*

$$\hat{\mathrm{H}}^m(G, K) \cdot \hat{\mathrm{H}}^n(G, K) = 0,$$

for every m and n both negative.

If, viceversa, there exist negative integers m and n such that

$$\hat{\mathrm{H}}^m(G, K) \cdot \hat{\mathrm{H}}^n(G, K) \neq 0,$$

then $\mathrm{H}(G, K)$ *has depth 1, and the center of any Sylow p-subgroup of G has rank one.*

Theorem 3.46 ([5], Theorem 4.1). *If G has p-rank 2 and* $\mathrm{H}(G, K)$ *is not Cohen-Macaulay, then there exist negative integers m and n such that*

$$\hat{\mathrm{H}}^m(G, K) \cdot \hat{\mathrm{H}}^n(G, K) \neq 0.$$

The conditions in Proposition 3.44 will be the same as in Proposition 3.48, where conditions under which the Tate cohomology is periodic are discussed. This supports the general idea that products in negative cohomology are somehow exceptional. In this perspective, the proof itself of Theorem 3.46 provides a way to find non-trivial examples. For instance, let $G = \mathrm{SD}_{16}$ be the semi-dihedral group of order 16. We have then

$$\mathrm{H}(G.K) = \frac{K[x, y, z, w]}{(x^3, xy, xz, z^2 - y^2 w)},$$

where the respective degrees are 1, 1, 3, and 4. Being each of the $\mathrm{H}^n(G, K)$ a K-vector space, their dimensions can be used to control their size. Fitting them in a spectral sequence converging to 0 and using this data, Benson and Carlson obtain that there must exist an element $u \in \hat{\mathrm{H}}^{-3}(G, K)$ such that $uw = x$. Using Tate duality (or Lemma 2.1 in [5]), it is finally obtained an element $v \in \hat{\mathrm{H}}^{-2}(G, K)$ subject to $uv \neq 0$, as wanted.

3.8 Periodicity and topological aspects

Another peculiarity of Tate cohomology is the phenomenon of periodicity. We have already seen an example at the end of Section 2.3 in the case of a finite cyclic group π.

Another example from §5 of Chapter XII in [13] is provided by the family of groups called *generalized quaternion groups*, that can be presented by $\langle x, y \mid x^t = y^2, xyx = y\rangle$ for some fixed integer t. In this case Tate cohomology has period 4.

In general, the study of periodicity has tight bounds both to the cup product and to the structure of the group π itself, as we can see from the following two propositions.

Proposition 3.47 (Proposition 11.1, XII in [13]). *Let π be a finite group of order k. For each $\gamma \in \hat{\mathrm{H}}^q(\pi, \mathbb{Z})$, the following are equivalent:*

- *the order of γ is k, and hence it generates $\hat{\mathrm{H}}^q(\pi, \mathbb{Z})$;*
- *there is an element $\gamma^{-1} \in \hat{\mathrm{H}}^{-q}(\pi, \mathbb{Z})$ such that $\gamma^{-1} \cup \gamma = 1$;*
- *the cup product $\cup\gamma$ induces an isomorphism*

$$\hat{\mathrm{H}}^n(\pi, A) \xrightarrow{\sim} \hat{\mathrm{H}}^{n+q}(\pi, A),$$

for every π-module A and every n.

Local Class Field Theory (as we have seen in Definition 2.4) is a clear example of this setting.

The second proposition is the following.

Proposition 3.48 (Theorem 11.6, XII in [13]). *For each finite group π the following conditions are equivalent:*

- *the Tate cohomology of π has a positive period;*
- *every abelian subgroup of π is cyclic;*
- *every p-subgroup of π is either cyclic or a generalized quaternion group;*
- *every Sylow subgroup of π is either cyclic or a generalized quaternion group.*

Similar statements are still valid also for Farrell-Tate cohomology (see [10], Chapter X.4). Examples in this sense come mainly from topology. To give an idea of the kind of results that can be obtained, in [16] it is proven the following.

Theorem 3.49 ([16] Corollary 1.4)**.** *If G is a group with finite virtual cohomological dimension, then G acts freely and properly discontinuously on $\mathbb{R}^m \times \mathbb{S}^{n-1}$ if and only if G is countable and the Farrell-Tate cohomology $\hat{\mathrm{H}}^n(G, \mathbb{Z})$ is periodic.*

Further works that study similar problems are, among the others, [36] and [47].

3.9 Profinite groups

Finite groups appear in Class Field Theory as Galois groups of number fields or finite extensions of finite fields. In this context, they are viewed as objects of the category of profinite groups. We recall that a group is profinite if it is the projective limit of an inverse system of finite groups. Hence, finite groups are trivially profinite. Therefore, it may look somehow natural to extend Tate cohomology in this direction too. The category under consideration is *the category of G-modules with the discrete topology and a continuous action by G*, often denoted by $\mathcal{C}_G$. In this setting, $\mathcal{C}_G$ does still have enough injectives, hence cohomology can be defined, but does not have enough projectives, whenever G is infinite. Nevertheless, in [43] Scheiderer found a way to partially overcome this problem. The main remark is that the functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ gives an equivalence of categories, called Pontryagin duality, between the category of profinite G-modules and the category of torsion discrete G-modules. Since the latter still has enough injectives, the former does have enough projectives. Obviously, this constrains the coefficients of the cohomology.

After Scheiderer, Studer de Boer, in her Ph.D. thesis (cf. [12]), defines another generalization of the Farrell-Tate cohomology for profinite groups with finite virtual cohomological dimension through the completion of the $\mathrm{H}^n(G,-)$'s functors, as in Section 3.8 but with respect to injectives, following the approach of [41].

4. Tate cohomology for algebras

Let us fix a commutative ring K and consider a K-algebra Λ. Denote by Λ^{op} the opposite algebra of Λ.

Definition 4.1. The *enveloping algebra of the K-algebra* Λ is the K-algebra $\Lambda \otimes_K \Lambda^{\mathrm{op}}$. It is denoted by Λ^e.

A two-sided Λ-module, or a $\Lambda\Lambda$-bimodule A, is equivalent to a left Λ^e-module via the product $(\lambda \otimes \mu)a = \lambda a \mu$. In particular, Λ itself is a left Λ^e-module. By defining the K-module homomorphism $\rho : \lambda \otimes \mu \mapsto \lambda \cdot \mu$ we obtain the augmented triple $(\Lambda^e, \rho, \Lambda)$. The corresponding augmentation ideal J is then generated by the elements of the form $\lambda \otimes 1 - 1 \otimes \lambda$.

We are now ready, again through Definition 2.1, for the following definition.

Definition 4.2. Let Λ be K-algebra and A a two-sided Λ-module.

- The *homology of the algebra* Λ *with coefficients in* A is

$$\mathrm{H}_n(\Lambda, A) = \mathrm{Tor}_n^{\Lambda^e}(A, \Lambda).$$

- The *cohomology of the algebra* Λ *with coefficients in* A is

$$\mathrm{H}^n(\Lambda, A) = \mathrm{Ext}^n_{\Lambda^e}(\Lambda, A).$$

These groups, that are actually K-modules, are called *Hochschild homology and cohomology groups for* Λ. They are a generalization of the (co)homology of algebras defined by Hochschild [27]; they coincide when K is a field.

We shall remark that in general, given a supplemented algebra, the (co)homology computed as an algebra or an augmented ring may differ. In the case of group rings, they coincide (see [13], X.6.1 and following).

As for group cohomology, also in the context of algebras, we find some generalizations. Nakayama already showed in [37] how to build a complete cohomology theory for a Frobenius algebra. The two approaches he proposes are again either by a correspondent of the norm homomorphism or by the equivalent of a complete resolution. After that paper, to our knowledge, not much have been done on this topic until recently. New developments, that we summarize now, are on the other hand, for example, published in [6] or [40]. We use the setting of Bergh and Jørgersen, since it is more modern and it includes what has been done by Nakayama in the same spirit of the generalization of Farrell.

In order to recover the results of [2] we consider a k-algebra Λ such that the enveloping algebra Λ^e is two-sided, Noetherian, and Gorenstein, that is, it has finite injective dimension over itself. Then, Λ has a *complete resolution* $\boldsymbol{T}$, in the sense of Farrell, but with the additional property that its dual $T^* = \mathrm{Hom}_\Lambda(\boldsymbol{T}, \Lambda)$ is still exact.

Definition 4.3. Let A be a bimodule.

- The *Tate-Hochschild homology of* Λ *with coefficients in* A is

$$\widehat{\mathrm{HH}}_n(\Lambda, A) = \widehat{\mathrm{Tor}}_n^{\Lambda^e}(A, \Lambda) = \mathrm{H}_n(A \otimes_{\Lambda^e} \boldsymbol{T}).$$

- The *Tate-Hochschild cohomology of* Λ *with coefficients in* A is

$$\widehat{\mathrm{HH}}^n(\Lambda, A) = \widehat{\mathrm{Ext}}_{\Lambda^e}^n(\Lambda, A) = \mathrm{H}^n(\mathrm{Hom}_{\Lambda^e}(\boldsymbol{T}, A)).$$

As for the Farrell cohomology, for $n > d$, these groups coincide respectively with the Hochschild homology and cohomology of Λ as in Definition 4.2.

In their paper [6], Bergh and D. Jorgensen study the properties of these groups. For example, if Λ has finite dimension over K and A is finitely generated, both its homology and cohomology are finite dimensional K-vector spaces. Another noteworthy aspect relates to the dualities that can be found.

Theorem 4.4 (Theorem 2.4 in [6]). *Let Λ be a finite dimensional Gorenstein algebra over K. Consider M and L finitely generated left modules and N a finitely generated right module. If the Gorenstein dimension of Λ is at most d, we have isomorphisms of finite dimensional K-vector spaces*

$$\widehat{\mathrm{Tor}}^{\Lambda}_{n}(N, M) \simeq \widehat{\mathrm{Tor}}^{\Lambda}_{-(n-d+1)}(\Omega^d_\Lambda(M)^*, D(N)),$$

$$\widehat{\mathrm{Ext}}^{n}_{\Lambda}(N, M) \simeq \widehat{\mathrm{Ext}}^{-(n-d+1)}_{\Lambda}(L, D(\Omega^d_\Lambda(M))^*),$$

for every n in $\mathbb{Z}$.

Here we have $D(-) = \mathrm{Hom}_k(-, K)$ and $\Omega^d_\Lambda(-)$ is the n-th syzygy of a minimal resolution of M. We refer to the paper for details.

The case of the Frobenius and quasi-Frobenius algebras is the original setting studied by Nakayama in [37] and [38], and it is also an interesting special case for [6]. Indeed, in this situation we have a cleaner duality

$$\widehat{\mathrm{Ext}}^{n}_{\Lambda}(M, L) \simeq \widehat{\mathrm{Ext}}^{-n-1}_{\Lambda}(L, {}_{\nu}M);$$

here ν is the *Nakayama automorphism* given by the Frobenius structure. A similar statement holds for $\widehat{\mathrm{Tor}}$.

Specializing the former formula to the (co)homology of Λ, we have the following.

Theorem 4.5 (Theorem 3.7 in [6]). *There are isomorphisms*

$$\widehat{\mathrm{HH}}_n(\Lambda, \Lambda) \simeq \widehat{\mathrm{HH}}_{-n-1}(\Lambda, \Lambda), \qquad \widehat{\mathrm{HH}}^{n}(\Lambda, \Lambda) \simeq \widehat{\mathrm{HH}}^{-n-1}(\Lambda, {}_{\nu^2}\Lambda).$$

Other applications of this type of generalization are in the same [6] for a quantum complete intersection, in [40] for Hopf algebras, and in [20] for Calabi-Yau Frobenius algebras. In this last paper, the Hochschild-Tate cohomology is introduced as a stable cohomology, in the same style as Vogel or [3].

References

[1] Auslander, Maurice and Bridger, Mark. *Stable module theory.* Memoirs of the American Mathematical Society, No. 94. American Mathematical Society, Providence, R.I., 1969.

[2] Avramov, Luchezar L. and Martsinkovsky, Alex. Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension. *Proc. London Math. Soc. (3)*, 85(2):393–440, 2002.

[3] Avramov, Luchezar L. and Veliche, Oana. Stable cohomology over local rings. *Adv. Math.*, 213(1):93–139, 2007.

[4] Bass, Hyman. On the ubiquity of Gorenstein rings. *Math. Z.*, 82:8–28, 1963.

[5] Benson, D. J. and Carlson, Jon F. Products in negative cohomology. *J. Pure Appl. Algebra*, 82(2):107–129, 1992.

[6] Bergh, Petter Andreas and Jorgensen, David A. Tate-Hochschild homology and cohomology of Frobenius algebras. *J. Noncommut. Geom.*, 7(4):907–937, 2013.

[7] Bergh, Petter Andreas and Jorgensen, David A. and Oppermann, Steffen. The negative side of cohomology for Calabi-Yau categories. *Bull. Lond. Math. Soc.*, 46(2):291–304, 2014.

[8] Bieri, Robert and Eckmann, Beno. Groups with homological duality generalizing Poincaré duality. *Invent. Math.*, 20:103–124, 1973.

[9] Brown, Kenneth S. High dimensional cohomology of discrete groups. *Proc. Nat. Acad. Sci. U. S. A.*, 73(6):1795–1797, 1976.

[10] Brown, Kenneth S. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1982.

[11] Bruns, Winfried and Herzog, Jürgen. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[12] Carla Studer-de Boer. *Tate Cohomology for Profinite Groups*. PhD thesis, ETH - Zurich, 2001.

[13] Cartan, Henri and Eilenberg, Samuel. *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.

[14] Christensen, Lars Winther and Jorgensen, David A. Tate (co)homology via pinched complexes. *Trans. Amer. Math. Soc.*, 366(2):667–689, 2014.

[15] Connell, Ian G. On the group ring. *Canad. J. Math.*, 15:650–685, 1963.

[16] Connolly, Francis X. and Prassidis, Stratos. Groups which act freely on $\mathbf{R}^m \times S^{n-1}$. *Topology*, 28(2):133–148, 1989.

[17] Cornick, Jonathan and Kropholler, Peter H. On complete resolutions. *Topology Appl.*, 78(3):235–250, 1997.

[18] Eisenbud, David. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[19] Emmanouil, Ioannis. On certain cohomological invariants of groups. *Adv. Math.*, 225(6):3446–3462, 2010.

[20] Eu, Ching-Hwa and Schedler, Travis. Calabi-Yau Frobenius algebras. *J. Algebra*, 321(3):774–815, 2009.

[21] Farrell, F. Thomas. An extension of Tate cohomology to a class of infinite groups. *J. Pure Appl. Algebra*, 10(2):153–161, 1977/78.

[22] Gedrich, T. V. and Gruenberg, K. W. Complete cohomological functors on groups. *Topology Appl.*, 25(2):203–223, 1987. Singapore topology conference (Singapore, 1985).

[23] Gelfand, Sergei I. and Manin, Yuri I. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[24] Goichot, François. Homologie de Tate-Vogel équivariante. *J. Pure Appl. Algebra*, 82(1):39–64, 1992.

[25] Grothendieck, Alexander. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[26] Heller, Alex. The loop-space functor in homological algebra. *Trans. Amer. Math. Soc.*, 96:382–394, 1960.

[27] Hochschild, G. On the cohomology groups of an associative algebra. *Ann. of Math. (2)*, 46:58–67, 1945.

[28] Iacob, Alina. Absolute, Gorenstein, and Tate torsion modules. *Comm. Algebra*, 35(5):1589–1606, 2007.

[29] Ikenaga, Bruce M. Homological dimension and Farrell cohomology. *J. Algebra*, 87(2):422–457, 1984.

[30] Ikenaga, Bruce M. Generalized cohomological dimension. *J. Pure Appl. Algebra*, 40(2):125–134, 1986.

[31] Iyengar, Srikanth and Krause, Henning. Acyclicity versus total acyclicity for complexes over Noetherian rings. *Doc. Math.*, 11:207–240 (electronic), 2006.

[32] Jørgensen, Peter. Existence of Gorenstein projective resolutions and Tate cohomology. *J. Eur. Math. Soc. (JEMS)*, 9(1):59–76, 2007.

[33] Kashiwara, Masaki and Schapira, Pierre. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[34] Kropholler, Peter H. Hierarchical decompositions, generalized Tate cohomology, and groups of type $(\mathrm{FP})_\infty$. In *Combinatorial and geometric group theory (Edinburgh, 1993)*, volume 204 of *London Math. Soc. Lecture Note Ser.*, pages 190–216. Cambridge Univ. Press, Cambridge, 1995.

[35] Mislin, Guido. Tate cohomology for arbitrary groups via satellites. *Topology Appl.*, 56(3):293–300, 1994.

[36] Mislin, Guido and Talelli, Olympia. On groups which act freely and properly on finite dimensional homotopy spheres. In *Computational and geometric aspects of modern algebra (Edinburgh, 1998)*, volume 275 of *London Math. Soc. Lecture Note Ser.*, pages 208–228. Cambridge Univ. Press, Cambridge, 2000.

[37] Nakayama, Tadasi. On the complete cohomology theory of Frobenius algebras. *Osaka Math. J.*, 9:165–187, 1957.

[38] Nakayama, Tadasi. Note on complete cohomology of a quasi-Frobenius algebra. *Nagoya Math. J.*, 13:115–121, 1958.

[39] Neukirch, Jürgen. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[40] Nguyen, Van C. Tate and Tate-Hochschild cohomology for finite dimensional Hopf algebras. *J. Pure Appl. Algebra*, 217(10):1967–1979, 2013.

[41] Nucinkis, Brita E. A. Complete cohomology for arbitrary rings using injectives. *J. Pure Appl. Algebra*, 131(3):297–318, 1998.

[42] R.-O. Buchweitz. Maximal cohen-macaulay modules and tate cohomology over gorenstein rings. *preprint*, 1986.

[43] Scheiderer, Claus. Farrell cohomology and Brown theorems for profinite groups. *Manuscripta Math.*, 91(2):247–281, 1996.

[44] Serre, Jean-Pierre. Cohomologie des groups discrets. 1971.

[45] Serre, Jean-Pierre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[46] Serre, Jean-Pierre. Cohomologie et arithmétique. In *Séminaire Bourbaki, Vol. 2*, pages Exp. No. 77, 263–269. Soc. Math. France, Paris, 1995.

[47] Talelli, Olympia. Periodicity in group cohomology and complete resolutions. *Bull. London Math. Soc.*, 37(4):547–554, 2005.

[48] Talelli, Olympia. On complete resolutions. In *Geometric and cohomological methods in group theory*, volume 358 of *London Math. Soc. Lecture Note Ser.*, pages 291–305. Cambridge Univ. Press, Cambridge, 2009.

[49] Tate, John. The higher dimensional cohomology groups of class field theory. *Ann. of Math. (2)*, 56:294–297, 1952.

[50] Toën, Bertrand. Lectures on dg-categories. In *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 243–302. Springer, Berlin, 2011.

[51] Veliche, Oana. Gorenstein projective dimension for complexes. *Trans. Amer. Math. Soc.*, 358(3):1257–1283 (electronic), 2006.

[52] Weibel, Charles A. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[53] Weibel, Charles A. History of homological algebra. In *History of topology*, pages 797–836. North-Holland, Amsterdam, 1999.

Resumen

Este artículo es una revisión del desarrollo y generalizaciones de la cohomología de Tate. El número de tales generalizaciones es alto y la literatura en torno a muchas de ellas es vasta. Por consiguiente, no pretendemos dar un recuento completo de las ramas que se desprenden de las ideas originales de Tate; esto más bien representa un bosquejo de cómo estas ideas se han ido desarrollando.

Palabras clave: Cohomología de Tate, dimensión de Gorenstein, resoluciones completas, cohomología estable.

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