# A short proof of Jung's theorem

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***Abstract***

We give a short and elementary proof of Jung's theorem, which states that for a field $K$ of characteristic zero the automorphisms of $K[x, y]$ are generated by elementary automorphisms and linear automorphisms.

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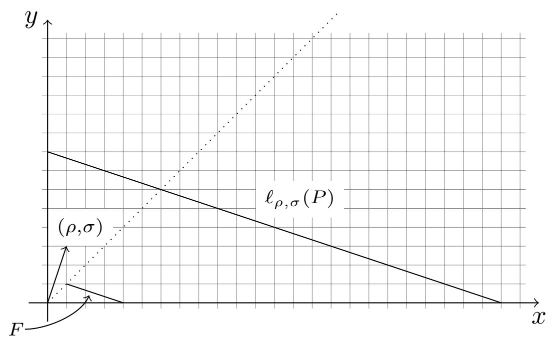
# Introduction

The theorem of Jung [5] states that if $K$ is a field of characteristic zero, then any automorphism of $L = K[x, y]$ is the finite composition of elementary automorphisms (given by $x \mapsto x,\ y \mapsto y + p(x)$ or by $y \mapsto y,\ x \mapsto x + p(y)$) and linear automorphisms. Many authors have given proofs of this fact, for example [1], [3], [7], [8], [9], [10] and [11]. The last and very short and elegant proof is given in [6], proof that works in every algebraically closed field of characteristic zero. The key step in the proof of [6] is the same as in ours: In the situation of the figure




there exists a polynomial $F$ (called $\zeta$ in [6]) such that $F = \mu x(y + \lambda x^{\sigma})$ with $[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P)$. Then we apply $\varphi$ given by $\varphi(x) = x$ and $\varphi(y) = y - \lambda x^{\sigma}$ and obtain $\deg(\varphi(P)) < \deg(P)$. Here $[P, Q]$ stands for the determinant of the jacobian matrix of two polynomials $P, Q$ and $\ell_{\rho,\sigma}(P)$ is the leading form of $P$ with respect to the weight $(\rho, \sigma)$.

To our knowledge our proof is the shortest and simplest (except for that of [6]), and Theorem 1.5 is the only fact we use that is not straightforward nor elementary. The element $F$ can be traced back to 1975 in [4]. In order to obtain a proof for a field that is not necessarily algebraically closed, we have to prove that for a polynomial automorphism there can be only one point at infinity, which we do in Proposition 2.2.

## 1. Preliminaries

We first gather notation and results of [2]. We define the *set of directions* by

$$\mathfrak{V} = \{(\rho, \sigma) \in \mathbb{Z}^2 : \gcd(\rho, \sigma) = 1\}.$$

For all $(\rho, \sigma) \in \mathfrak{V}$ and $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ we write $v_{\rho,\sigma}(i, j) = \rho i + \sigma j$ and for $P = \sum a_{i,j} x^i y^j \in L \setminus \{0\}$, we define

- the *support* of $P$ as $\operatorname{Supp}(P) = \{(i, j) : a_{i,j} \neq 0\}$;
- the $(\rho, \sigma)$-*degree* of $P$ as $v_{\rho,\sigma}(P) = \max\{v_{\rho,\sigma}(i, j) : a_{i,j} \neq 0\}$;
- the $(\rho, \sigma)$-*leading term* of $P$ as $\ell_{\rho,\sigma}(P) = \displaystyle\sum_{\{\rho i+\sigma j=v_{\rho,\sigma}(P)\}} a_{i,j} x^i y^j$.

We also set $\ell_{\rho,\sigma}(0) = 0$.

We say that $P \in L$ is $(\rho, \sigma)$-*homogeneous* if $P = \ell_{\rho,\sigma}(P)$. We assign to each direction its corresponding unit vector in $S^1$, and we define an *interval* in $\mathfrak{V}$ as the preimage under this map of an arc of $S^1$ that is not the whole circle. We consider each interval endowed with the order that increases counterclockwise.

For each $P \in L \setminus \{0\}$, we let $H(P)$ denote the *Newton polygon of* $P$. It is evident that each one of its edges is the convex hull of the support of $\ell_{\rho,\sigma}(P)$, where $(\rho, \sigma)$ is orthogonal to the given edge and points outside of $H(P)$. These directions form the set

$$\operatorname{Dir}(P) = \{(\rho, \sigma) \in \mathfrak{V} : \# \operatorname{Supp}(\ell_{\rho,\sigma}(P)) > 1\}.$$

**Notation 1.1.** Let $(\rho, \sigma) \in \mathfrak{V}$ arbitrary. We let $\operatorname{st}_{\rho,\sigma}(P)$ and $\operatorname{en}_{\rho,\sigma}(P)$ denote the first and the last point that we find on $H(\ell_{\rho,\sigma}(P))$ when we run counterclockwise along the boundary of $H(P)$. Note that these points coincide when $\ell_{\rho,\sigma}(P)$ is a monomial.

We say that two vectors $A, B \in \mathbb{R}^2$ are *aligned*, and write $A \sim B$, if we have $0 = A \times B = \det\left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right)$, where $A = (a_1, a_2)$ and $B = (b_1, b_2)$.

**Proposition 1.2** (Proposition 1.13 in [2]). *Let $P, Q \in L \setminus \{0\}$ and let $(\rho,\sigma)$ be a direction. If $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] \neq 0$ then $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = \ell_{\rho,\sigma}([P,Q])$.* □

The next result is in the spirit of [2, Proposition 2.4].

**Proposition 1.3.** *Let $P, Q \in L \setminus \{0\}$.*

*If $\mathrm{st}_{1,1}(P) \nsim \mathrm{st}_{1,1}(Q)$ then $\mathrm{st}_{1,1}(P) + \mathrm{st}_{1,1}(Q) - (1,1) = \mathrm{st}_{1,1}([P,Q])$.*

*If $\mathrm{en}_{1,1}(P) \nsim \mathrm{en}_{1,1}(Q)$ then $\mathrm{en}_{1,1}(P) + \mathrm{en}_{1,1}(Q) - (1,1) = \mathrm{en}_{1,1}([P,Q])$.*

*Proof.* We only prove the first statement, the other one is similar. Write

$$\ell_{1,1}(P) = a_0 x^r y^s + a_1 x^{r-1} y^{s+1} + \cdots + a_n x^{r-n} y^{s+n}$$

and

$$\ell_{1,1}(Q) = b_0 x^u y^v + \cdots + b_m x^{u-m} y^{v+m},$$

with $a_0, b_0 \neq 0$. Then $\mathrm{st}_{1,1}(P) = (r,s)$, $\mathrm{st}_{1,1}(Q) = (u,v)$ and

$$[\ell_{1,1}(P), \ell_{1,1}(Q)] = a_0 b_0 (rv - us) x^{r+u-1} y^{s+v-1} + \\ + \alpha_1 x^{r+u-2} y^{s+v} + \alpha_2 x^{r+u-3} y^{s+v+1} + \cdots,$$

for some $\alpha_j$. If $\mathrm{st}_{1,1}(P) \nsim \mathrm{st}_{1,1}(Q)$, then $rv - us \neq 0$ and from Proposition 1.2 we obtain $\ell_{1,1}([P,Q]) = [\ell_{1,1}(P), \ell_{1,1}(Q)]$. Consequently we get

$$\mathrm{st}_{1,1}([P,Q]) = (r+u-1, s+v-1) = \mathrm{st}_{1,1}(P) + \mathrm{st}_{1,1}(Q) - (1,1),$$

as desired. □

*Remark* 1.4. Let $(\rho,\sigma) \in \mathfrak{V}$ and let $P, F \in L \setminus \{0\}$ be $(\rho,\sigma)$-homogeneous such that $[F,P] = P$. If $F$ is a monomial, then $F = \lambda xy$ with $\lambda \in K^{\times}$, and either we have $\rho + \sigma = 0$ or $P$ is also a monomial.

The following theorem is an important tool in the constructions of [2]. It is the only result we use that is not straightforward.

**Theorem 1.5** (Theorem 2.6 in [2]). *Let $P \in L$ and let $(\rho, \sigma) \in \mathfrak{V}$ be such that $\rho + \sigma > 0$ and $v_{\rho,\sigma}(P) > 0$. If $[P, Q] \in K^{\times}$ for some $Q \in L$, then there exists a $(\rho, \sigma)$-homogeneous element $F \in L$ such that*

$$v_{\rho,\sigma}(F) = \rho + \sigma \quad \text{and} \quad [F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P).$$

□

If $I$ is an interval in $\mathfrak{V}$ and if there is no closed half circle contained in $I$, then for all $(\rho, \sigma), (\rho_1 \sigma_1) \in I$ we have

$$(\rho_1, \sigma_1) < (\rho, \sigma) \text{ if and only if } (\rho_1, \sigma_1) \times (\rho, \sigma) > 0.$$

**Proposition 1.6** (Proposition 3.6 in [2]). *Let $P \in L \backslash \{0\}$ and let $(\rho_1, \sigma_1)$ and $(\rho_2, \sigma_2)$ be consecutive elements in $\mathrm{Dir}(P)$. Then we have*

$$\mathrm{en}_{\rho_1,\sigma_1}(P) = \mathrm{Supp}(\ell_{\rho,\sigma}(P)) = \mathrm{st}_{\rho_2,\sigma_2}(P)$$

*for each $(\rho, \sigma)$ such that $(\rho_1, \sigma_1) < (\rho, \sigma) < (\rho_2, \sigma_2)$.* □

**Proposition 1.7** (Proposition 3.10 in [2]). *Let $P, Q \in L$ and $\varphi \colon L \to L$ be an algebra morphism. Then*

$$[\varphi(P), \varphi(Q)] = \varphi([P, Q])[\varphi(x), \varphi(y)].$$

□

## 2. Jung's Theorem

Now we start our proof of Jung's theorem.

**Lemma 2.1.** *Take $f \in \mathrm{Aut}(L)$ and set $P = f(x)$. If*

$$(a, b) \in \{\mathrm{en}_{1,1}(P), \mathrm{st}_{1,1}(P)\},$$

*then $a = 0$ or $b = 0$.*

*Proof.* We will prove only the case $(a,b) = \mathrm{st}_{1,1}(P)$, since the argument in the other case is the same. Assume $a \geq b > 0$. We set $R_0 = x$ and $R_j = [R_{j-1}, P]$ for $j > 0$. Then we have

$$\mathrm{st}_{1,1}(R_0) = (1,0) \nsim (a,b) = \mathrm{st}_{1,1}(P),$$

and so, by Proposition 1.3(1) we also have

$$\mathrm{st}_{1,1}(R_1) = (1,0) + (a,b) - (1,1) = (a, b-1),$$

which is not aligned to $(a,b)$. Increasing $k$ and using Proposition 1.3(1) again and again one obtains inductively

$$\mathrm{st}_{1,1}(R_k) = (ka - k + 1, kb - k),$$

since $(ka-k+1, kb-k) \nsim (a,b)$ holds for all $k \geq 1$. Hence, we conclude $R_k \neq 0$ for all $k$. But this is impossible by the following argument. since $x \in K[P,Q]$, we can write $x = \sum_{i,j} a_{i,j} P^i Q^j$. For $\lambda = [Q,P] \in K^{\times}$, we have $R_1 = \lambda \sum_{i,j} j a_{i,j} P^i Q^{j-1}$, so the maximal power of $Q$ decreases. Eventually it is zero for some $R_k$, and then $R_{k+1} = 0$.

If $b \geq a > 0$, then we set $R_0 = y$ and $R_{j+1} = [R_j, P]$, and the same argument yields a contradiction. Hence we must have $a = 0$ or $b = 0$, as claimed. □

The following proposition shows that for an automorphism $f$ there can be only one factor at infinity, or equivalently, that $\ell_{1,1}(f(x))$ is the power of one linear factor.

**Proposition 2.2.** *Let $f : L \to L$ be an automorphism and set $P = f(x)$. Then we have either $\mathrm{Supp}(\ell_{1,1}(P)) = \{(a,0)\}$ or $\mathrm{Supp}(\ell_{1,1}(P)) = \{(0,a)\}$ or $\ell_{1,1}(P) = \mu(x - \lambda y)^a$; here $a = v_{1,1}(P)$ and $\mu, \lambda \in K^{\times}$.*

*Proof.* Without loss of generality we assume that $K$ is algebraically closed. Suppose $a = v_{1,1}(P) > 0$ and write $\ell_{1,1}(P) = x^a p(z)$, where $z = x^{-1}y$ and $p(z) \in K[z]$. Let $b$ denote $\deg(p(z))$. If $0 < b < a$, then $\mathrm{en}_{1,1}(P) = (a,0) + b(-1,1) = (a-b, b)$, which contradicts Lemma 2.1.

On the other hand, if $\deg(p) = 0$, then we get $\mathrm{Supp}(\ell_{1,1}(P)) = \{(a, 0)\}$. So we are reduced to consider the case $\deg(p(z)) = a$. If we have neither

$$\mathrm{Supp}(\ell_{1,1}(P)) = \{(0, a)\} \quad \text{nor} \quad \ell_{1,1}(P) = \mu(x - \lambda y)^a,$$

then $p(z) = \mu \prod_{i=1}^{k} (z - \lambda_i)^{m_i}$ has a root $\lambda_{i_0}$ with multiplicity $0 < m_{i_0} < a$. But then the automorphism $\varphi$ given by $\varphi(x) = x$ and $\varphi(y) = y + \lambda_{i_0} x$ yields

$$\ell_{1,1}(\varphi(P)) = \varphi(\ell_{1,1}(P)) = x^a p(z + \lambda_{i_0}) = \mu x^a z^{m_{i_0}} \prod_{\substack{i=1 \\ i \neq i_0}}^{k} (z - \overline{\lambda}_i)^{m_i},$$

where $\overline{\lambda}_i = \lambda_i - \lambda_{i_0}$ and $\prod_{\substack{i=1 \\ i \neq i_0}}^{k} \overline{\lambda}_i^{m_i} \neq 0$. This implies

$$\mathrm{st}_{1,1}(\varphi(P)) = (a, 0) + m_{i_0}(-1, 1) = (a - m_{i_0}, m_{i_0}),$$

where $a - m_{i_0} \neq 0$ and $m_{i_0} \neq 0$, which contradicts Lemma 2.1 and concludes the proof. $\square$

**Theorem 2.3.** *Each automorphism $f : L \to L$ is a composition of elementary automorphisms and linear automorphisms.*

*Proof.* Set $P = f(x)$. If $\deg(P) = 1$, then we can assume $P = x$, and then we have $f(y) = \lambda y + q(x)$ since $[f(x), f(y)] \in K^{\times}$. It follows that $f$ is the composition of elementary automorphisms and linear automorphisms. Therefore it suffices to prove that if $\deg(P) > 1$, then there exists a map $\varphi$, which is a composition of elementary automorphisms, such that $\deg(\varphi(P)) < \deg(P)$. By Proposition 2.2 we have either $\mathrm{Supp}(\ell_{1,1}(P)) = \{(a, 0)\}$, $\mathrm{Supp}(\ell_{1,1}(P)) = \{(0, a)\}$ or $\ell_{1,1}(P) = \mu(x - \lambda y)^a$, here $a = \deg(P)$ and $\mu, \lambda \in K^{\times}$. Actually we can assume (and we do it) that we have

$$\mathrm{Supp}(\ell_{1,1}(P)) = \{(a, 0)\}. \tag{2.1}$$

In fact, if $\mathrm{Supp}(\ell_{1,1}(P)) = \{(0,a)\}$, then we apply the automorphism given by $x \mapsto y$ and $y \mapsto -x$, which is a composition of elementary automorphisms, and if $\ell_{1,1}(P) = \mu(x-\lambda y)^a$, we apply the elementary automorphism given by $x \mapsto x+\lambda y$ and $y \mapsto y$.

Moreover, since we have $[P, f(y)] \in K^{\times}$, it is impossible to have $P = \mu X^a$, and so $P$ is not a monomial.

Let $(\rho,\sigma)$ be the successor of $(1,1)$, which is the first element of $\mathrm{Dir}(P)$ that one encounters starting from $(1,1)$ and running counter-clockwise.

If $(\rho,\sigma) \geq (0,1)$, then from Proposition 1.6 we obtain $(a,0) = \mathrm{st}_{0,1}(P)$, and then for all $(i,j) \in \mathrm{Supp}(P)$ we have $j = v_{0,1}(i,j) \leq v_{0,1}(a,0) = 0$, which implies $P \in K[x]$. Hence, since $[P, f(y)] \in K^{\times}$, we have $\deg(P) = 1$.

It remains to consider the case $(1,1) < (\rho,\sigma) < (0,1)$, or, equivalently, $\sigma > \rho > 0$. By Theorem 1.5 we know that there exists a $(\rho,\sigma)$-homogenous element $F \in K[x,y]$ such that

$$[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P) \quad \text{and} \quad v_{\rho,\sigma}(F) = \rho + \sigma.$$

For all $(i,j) \in \mathrm{Supp}(F)$ we have $\rho i + \sigma j = \rho + \sigma$ and so

$$(1-i)\rho = (j-1)\sigma. \tag{2.2}$$

Hence $j > 1$ is impossible and if $j = 1$, then $i = 1$. Since $\ell_{\rho,\sigma}(P)$ is not a monomial, we know from Remark 1.4 that $F$ has at least two points in its support. Hence we get $(1,1) \in \mathrm{Supp}(F)$ and there must be a point of the form $(i,0) \in \mathrm{Supp}(F)$. Using this and Equality (2.2), we obtain $\sigma = (i-1)\rho$, which implies $\rho = 1$, since $\rho$ and $\sigma$ are coprime. Hence $F = \mu x(y + \lambda x^{\sigma})$ for some $\mu, \lambda \in K^{\times}$. Moreover, since $\mathrm{st}_{\rho,\sigma}(P) = (a,0)$, there exists $p(z) \in K[z]$ such that $\ell_{\rho,\sigma}(P) = x^a p(z)$, where $z = yx^{-\sigma}$. Note also that $\deg(p(z)) > 0$ is satisfied, since $\ell_{\rho,\sigma}(P)$ is not a monomial.

Consider now the elementary automorphism $\varphi$ given by $\varphi(x) = x$ and $\varphi(y) = y - \lambda x^{\sigma}$. Since $\varphi$ is $(\rho,\sigma)$-homogenous we have

$$\ell_{\rho,\sigma}(\varphi(P)) = \varphi(\ell_{\rho,\sigma}(P)) = \varphi(x^a p(z)) = x^a p(z-\lambda). \tag{2.3}$$

On the other hand, by Proposition 1.7, we get

$$[\varphi(F), \ell_{\rho,\sigma}(\varphi(P))] = [\varphi(F), \varphi(\ell_{\rho,\sigma}(P))] = \varphi(\ell_{\rho,\sigma}(P)) = \ell_{\rho,\sigma}(\varphi(P)).$$

Since $\varphi(F) = \mu xy$, from Remark 1.4, it follows that $\ell_{\rho,\sigma}(\varphi(P))$ is a monomial. Hence, by (2.3), we have

$$\ell_{\rho,\sigma}(\varphi(P)) = \mu_p x^a z^N,$$

and so, also $(a, 0) \notin \mathrm{Supp}(\varphi(P))$. Now, for $(i, j) \in \mathrm{Supp}(\varphi(P))$, we have

$$\begin{aligned} v_{1,1}(i,j) &= i + j \\ &\le i + \sigma j = v_{\rho,\sigma}(i,j) \\ &\le v_{\rho,\sigma}(\varphi(P)) = v_{\rho,\sigma}(P) = v_{\rho,\sigma}(a,0) = a = v_{1,1}(P), \end{aligned}$$

where the last equality follows from (2.1). Furthermore, the equality would be possible only if $j = 0$ and $i = a$, but just prove above that $(i, j) \neq (a, 0)$ is satisfied. Hence we get $v_{1,1}(\varphi(P)) < v_{1,1}(P)$, as desired. □

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**Resumen**

Presentaremos una prueba corta y elemental del teorema de Jung. Este teorema establece que para un cuerpo $K$ de característica cero los automorfismos de $K[x, y]$ son generados por automorfismos lineales y los llamados elementales.

**Palabras clave:** Conjetura del jacobiano, teorema de Jung.

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