A large deviation principle for a natural sequence of point processes on a Riemannian two-dimensional manifold

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Abstract

We follow the techniques of Paul Dupuis, Vaios Laschos, and Kavita Ramanan in [8] to prove a large deviation principle for a sequence of point processes defined by Gibbs measures on a compact orientable two-dimensional Riemannian manifold. We see that the corresponding sequence of empirical measures converges to the solution of a partial differential equation and, in some cases, to the volume form of a constant curvature metric.

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1 Model and results

Let (M, g) be a compact oriented two-dimensional Riemannian manifold of genus different from one. Denote by *vol* the normalized volume form associated to g and the orientation. Define the 2-form

$$\Lambda = \frac{\operatorname{Ric} g}{2\pi\chi(M)}, \tag{1.1}$$

where $\operatorname{Ric} g$ is the Ricci curvature seen as a 2-form and $\chi(M)$ denotes the Euler characteristic. More precisely, write $\operatorname{Ric} g = K_g vol$, where K_g is the Gaussian curvature of g, while the usual symmetric Ricci curvature would be $K_g\, g$. We shall think of Λ as a signed measure.

It is known that there exists a continuous symmetric function

$$G : M \times M \to \mathbb{R} \cup \{\infty\}$$

such that the function $G_x : M \to \mathbb{R} \cup \{\infty\}$ defined by $G_x(y) = G(x, y)$ is integrable and satisfies

$$\Delta G_x = -\delta_x + \Lambda \tag{1.2}$$

for every $x \in M$. More precisely, for every $f \in C^\infty(M)$ and $x \in M$, we have

$$\int_M G(x, y)\Delta f(y) = -f(x) + \int_M f(y) d\Lambda(y), \tag{1.3}$$

here $\Delta : C^\infty(M) \to \Omega^2(M)$ is the usual Laplacian, i.e. $\Delta = d * d$, where $*$ is the Hodge star operator and d is the exterior derivative. Moreover, such G is unique up to an additive constant and we can choose G such that

$$\int_M G(x, y) d\Lambda(y) = 0 \tag{1.4}$$

for every $x \in M$. See [6] for a proof and more information.

Take an integer $n \geq 2$. We consider a system of n indistinguishable particles with total charge 1 interacting via the electrostatic force. In other words, each particle has charge $1/n$ and the two-particle interaction potential is G. This means that the total energy will be $H_n : M^n \to \mathbb{R} \cup \{\infty\}$, defined by

$$H_n(x_1, ..., x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

Choose a sequence of positive numbers $\{\beta_n\}_{n\geq 2}$ and a positive number $\beta > 0$ such that $\beta_n \to \beta$. We define the **Gibbs non-normalized measure associated to** H_n **and** β_n as the finite measure γ_n on M^n given by

$$d\gamma_n = \exp\left(-n\beta_n H_n\right)\, dvol^{\otimes n}.$$

The **Gibbs probability measure** will be the probability measure

$$\mathbb{P}_n = \frac{\gamma_n}{Z_n},$$

where $Z_n = \gamma_n(M^n)$ is called the **partition function**. The probability measure $\mathbb{P}_n$ describes a system of n particles with Hamiltonian H_n and inverse temperature $n\beta_n$.

For any metrizable compact space E we endow $\mathcal{P}(E)$, the space of probability measures on E, with the smallest topology such that for every continuous function $f : E \to \mathbb{R}$ the application $\mu \to \int_E f d\mu$ is continuous. We can see that this is again a metrizable compact space. See Appendix for a short proof, and [5] for extra information. Furthermore, a sequence of probability measures on E, say $\{\mu_n\}_{n\in\mathbb{N}}$, converges to $\mu \in \mathcal{P}(E)$ if and only if $\int_E f d\mu_n$ converges to $\int_E f d\mu$ for every continuous function $f : E \to \mathbb{R}$. In fact, it is enough to verify that $\int_E f d\mu_n$ converges to $\int_E f d\mu$ for f belonging to a countable dense family of the space of continuous functions with the uniform topology.

The space M^n is to be 'injected' in $\mathcal{P}(M)$ by means of the continuous application

$$i_n : M^n \to \mathcal{P}(M)$$

$$(x_1, ..., x_n) \mapsto \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i},$$

and we will study the limit of the sequence of probabilites $i_n(\mathbb{P}_n)$, the **pushforward laws of** $\mathbb{P}_n$.

Define the **macroscopic energy** as

$$\begin{aligned} W : \mathcal{P}(M) &\to \mathbb{R} \cup \{\infty\} \\ \mu &\mapsto \int_{M\times M} G(x,y)\, d\mu(x)\, d\mu(y), \end{aligned}$$

and the **free energy** as

$$\begin{aligned} F : \mathcal{P}(M) &\to \mathbb{R} \cup \{\infty\} \\ \mu &\mapsto \frac{\beta}{2} W(\mu) + D\left(\mu \,\|\, vol\right), \end{aligned}$$

where $D\left(\mu \,\|\, vol\right)$ denotes the relative entropy of μ with respect to vol, also known as the **Kullback - Leibler divergence**, defined by

$$D(\mu \| vol) = \int_M \log\left(\frac{d\mu}{dvol}\right) d\mu$$

if μ is absolutely continuous with respect to vol, and $D(\mu\|vol) = \infty$ otherwise. Now we can state our main result.

Theorem 1.1 (Laplace principle)**.** *For every continuous $f : \mathcal{P}(M) \to \mathbb{R}$ we have the convergence*

$$\frac{1}{n} \log \int_{M^n} e^{-nf\circ i_n} d\gamma_n \xrightarrow[n\to\infty]{} -\inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

To state a large deviation principle as an easy corollary we need to define first the function

$$\begin{aligned} I : \mathcal{P}(M) &\to \mathbb{R} \cup \{\infty\} \\ \mu &\mapsto F(\mu) - \inf F. \end{aligned}$$

Corollary 1.2 (A large deviation principle)**.** *For every closed set $C \subset \mathcal{P}(M)$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq - \inf_{x\in C} I(x),$$

and for every open set $O \subset \mathcal{P}(M)$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(O)) \geq - \inf_{x\in O} I(x).$$

This tells us that to understand the limiting behavior of $i_n(\mathbb{P}_n)$ we must study the free energy F. The first two main properties will be studied in Section 2.

Proposition 1.3 (Convexity and lower semicontinuity of F)**.** *The function F is strictly convex and lower semicontinuous.*

Thus, F achieves its minimum at only one point. The following theorem characterizes this minimum.

Theorem 1.4 (Minimum of F)**.** *The function F achieves its minimum at a probability measure μ_{eq} that is absolutely continuous with respect to vol and such that $\rho = \dfrac{d\mu_{eq}}{dvol}$ is a C^∞ strictly positive everywhere function that satisfies the differential equation*

$$\Delta \log \rho = \beta\, \mu_{eq} - \beta\Lambda. \tag{1.5}$$

Remark 1.5 (Equivalent formulation: equation on the Ricci curvature)**.** If we define the metric $\bar{\omega} = \rho\, g$, we have that μ_{eq} is the volume form associated to $\bar{\omega}$, and Equation 1.5 can be written as

$$Ric\, \bar{\omega} = \left(2\pi\chi(M) + \frac{\beta}{2}\right) Ric\, g - \frac{\beta}{2}\mu_{eq}$$

(because of the identity $\Delta \log \rho = 2Ric\, g - 2Ric\, \bar{\omega}$).

Finally, by Corollary 1.2 and an application of the Borel-Cantelli lemma we get the following.

Corollary 1.6 (Convergence of the empirical measures)**.** *If $\{X_n\}_{n\geq 2}$ is a sequence of random elements in $\mathcal{P}(M)$ such that $X_n \sim i_n(\mathbb{P}_n)$, then*

$$X_n \xrightarrow{\text{a.s.}} \mu_{eq},$$

where μ_{eq} is the unique minimizer of F.

2 Lower semicontinuity and convexity of I

In this section we prove Proposition 1.3. It is well known that $D(\cdot\|vol)$ is lower semicontinuous and strictly convex (see [9, Lemma 1.4.3]). What we need to establish is lower semicontinuity and convexity for W.

Proof of the lower semicontinuity of W. For positive m set $G_m(x,y) = G(x,y) \wedge m = \min\{G(x,y), m\}$. Then

$$\mu \mapsto \int_{M\times M} G_m(x,y)\, d\mu(x)\, d\mu(y)$$

is a continuous function of μ. As W is the increasing limit of functions as m tends to infinity, we get that W is lower semicontinuous. □

Proof of the convexity of W. To prove convexity it is enough to show that for every $\mu, \nu \in \mathcal{P}(M)$ we have

$$W\left(\frac{1}{2}\mu + \frac{1}{2}\nu\right) \leq \frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) \tag{2.1}$$

due to the lower semicontinuity of W. Inequality 2.1 is equivalent to

$$\frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) \geq \int_{M\times M} G(x,y)\, d\mu(x)\, d\nu(y). \tag{2.2}$$

This inequality is easy to verify if μ and ν are differentiable, *i.e.*, given by differentiable forms. Indeed, in that case, the functions

$$f(x) = \int_M G(x,y)\, d\mu(y) \quad \text{and} \quad g(x) = \int_M G(x,y)\, d\nu(y)$$

satisfy

$$\Delta f = -\mu + \Lambda \quad \text{and} \quad \Delta g = -\nu + \Lambda$$

because of 1.3, are differentiable due to the ellipticity of the Laplacian, and have zero integral with respect to Λ because of 1.4. So, in terms of f and g Inequality 2.2 reads

$$-\frac{1}{2}\int_M f\,\Delta f\, - \frac{1}{2}\int_M g\,\Delta g \geq -\int_M f\,\Delta g,$$

which is equivalent to

$$\int_M \|\nabla(f-g)\|^2\, dvol \geq 0.$$

For the general case we need two lemmas.

Lemma 2.1. *Let μ be a continuous probability measure, i.e., given by a continuous 2-form. Then there exists a sequence μ_n of differentiable probability measures such that*

$$\mu_n \to \mu \qquad \textit{and} \qquad W(\mu_n) \to W(\mu).$$

Proof. As μ is continuous, we can write $d\mu = \rho\, dvol$ with ρ continuous. Take a sequence of differentiable functions $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\rho_n \to \rho$ uniformly. We can assume $\rho_n \geq 0$ (redefine $\rho_n = \rho_n + \|\rho - \rho_n\|_\infty$) and $\int_M \rho_n\, dvol = 1$ (because $\int_M \rho_n dvol \to \int_M \rho\, dvol$). Define μ_n by means of $d\mu_n = \rho_n dvol$. We notice that

$$\mu_n \to \mu$$

holds due to the uniform convergence and, as $\rho_n \otimes \rho_n \to \rho \otimes \rho$ uniformly, we obtain

$$\int_{M\times M} G(x,y)\rho_n(x)\rho_n(y)\, dvol(x)\, dvol(y) \to \int_{M\times M} G(x,y)\rho(x)\rho(y)\, dvol(x)\, dvol(y)$$

by the dominated convergence theorem outside the diagonal (because G is $vol \otimes vol$ - integrable and the sequence ρ_n is uniformly bounded). □

To approximate arbitrary probability measures with continuous ones we refer to [12, Lemma 6.3.1]. It states the following.

Lemma 2.2. *Let μ be any probability measure such that $W(\mu) < \infty$. Then there exists a sequence μ_n of continuous probability measures such that*

$$\mu_n \to \mu \qquad \text{and} \qquad W(\mu_n) \to W(\mu).$$

□

To complete the proof of the convexity, let $\mu, \nu \in \mathcal{P}(M)$. Take two sequences $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ of differentiable measures such that $\mu_n \to \mu$, $W(\mu_n) \to W(\mu)$, and $\nu_n \to \nu$, $W(\nu_n) \to W(\nu)$. We want to take the lower limit to the sequence of inequalities

$$\frac{1}{2}W(\mu_n) + \frac{1}{2}W(\nu_n) \geq \int_{M\times M} G(x,y)\, d\mu_n(x) d\nu_n(y).$$

For this, notice that $(\mu,\nu) \mapsto \int_{M\times M} G(x,y)\, d\mu(x)d\nu(y)$ is lower semicontinuous. This can be seen as a consequence of the fact that it can be reexpresed as the increasing limit as m goes to infinity of the continuous functions $(\mu,\nu) \mapsto \int_{M\times M} G_m(x,y)\, d\mu(x)d\nu(y)$ where $G_m(x,y) = G(x,y) \wedge m$. Then, we get

$$\begin{aligned}\frac{1}{2}W(\mu) + \frac{1}{2}W(\nu) &\geq \liminf_{n\to\infty} \int_{M\times M} G(x,y)\, d\mu_n(x) d\nu_n(y) \\ &\geq \int_{M\times M} G(x,y)\, d\mu(x) d\nu(y),\end{aligned}$$

and the proof is complete. □

3 The minimum of F

Now we prove Theorem 1.4. Let $\rho \in C^\infty(M)$ be a differentiable positive solution of the equation (see [7])

$$\Delta \log \rho = \beta\, \mu_{eq} - \beta\Lambda,$$

where $d\mu_{eq} = \rho\, dvol$. We will prove that the functional F achieves its minimum at μ_{eq}. For this we shall calculate the derivative of F at μ_{eq} and prove that it is zero. We start with the following result.

Lemma 3.1 (Derivative of the energy)**.** *Let μ_0 and μ_1 be two probability measures. Define $\mu_t = t\mu_1 + (1-t)\mu_0$, for $t \in [0,1]$. If $W(\mu_0) < \infty$ and $W(\mu_1) < \infty$ then $W(\mu_t)$ is differentiable at $t = 0$, and satisfies*

$$\frac{d}{dt} W(\mu_t)\,|_{t=0} \; = 2 \int_{M\times M} G(x,y)\, d\mu_0(x)\ (d\mu_1(y) - d\mu_0(y))\,.$$

Proof. As $W(\mu_0)$ and $W(\mu_1)$ are finite, due to the convexity of W we have that

$$\begin{aligned} W(\mu_t) = t^2 &\int_{M\times M} G(x,y) d\mu_1(x)\, d\mu_1(y) + \\ &+ 2t(1-t) \int_{M\times M} G(x,y) d\mu_0(x)\, d\mu_1(y) + \\ &+ (1-t)^2 \int_{M\times M} G(x,y) d\mu_0(x)\, d\mu_0(y) \end{aligned}$$

is finite. The linear term (in the variable t) is given by

$$2 \int_{M\times M} G(x,y)\, d\mu_0(x)\ (d\mu_1(y) - d\mu_0(y))\,,$$

which is the sought derivative. □

Lemma 3.2 (Derivative of the entropy)**.** *Let μ_0 and μ_1 be two probability measures. Define $\mu_t = t\mu_1 + (1-t)\mu_0$, for $t \in [0,1]$. If $D(\mu_0 \| vol) < \infty$, $D(\mu_1 \| vol) < \infty$ and $\int_M \left| \log\left(\frac{d\mu_0}{d\,vol}\right)\right| d\mu_1 < \infty$, then $D(\mu_t \| vol)$ is differentiable at $t = 0$, and satisfies*

$$\frac{d}{dt} D(\mu_t \| vol)\,|_{t=0} \; = \int_M \log\left(\frac{d\mu_0}{d\,vol}(y)\right) (d\mu_1(y) - d\mu_0(y))\,.$$

Proof. We use the notation

$$\rho_0 = \frac{d\mu_0}{d\,vol} \quad \text{and} \quad \rho_1 = \frac{d\mu_1}{d\,vol}.$$

As $D(\mu_0\|vol)$ and $D(\mu_1\|vol)$ are finite, by the convexity of the entropy we get that $D(\mu_t\|vol)$ is also finite. In particular, we have

$$\int_M |\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)| d\mu_t < \infty$$

and, as $\mu_t = t\mu_1 + (1-t)\mu_0$, if $0<t<1$, we get

$$\int_M |\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)| d\mu_0 < \infty$$

and

$$\int_M |\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)| d\mu_1 < \infty.$$

Keeping this in mind it makes sense to write

$$\begin{aligned} D(\mu_t\|vol) = & \int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right) t\rho_1(x) dvol(x) + \\ & + \int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)(1-t)\rho_0(x) dvol(x) \\ = & \, t\int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)(\rho_1(x)-\rho_0(x))\, dvol(x) + \\ & + \int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)\rho_0(x)\, dvol(x), \end{aligned}$$

which together with

$$D(\mu_0\|vol) = \int_M \log(\rho_0(x))\,\rho_0(x)\, dvol(x)$$

yields

$$\begin{aligned} \frac{1}{t}\left(D(\mu_t\|vol) - D(\mu_0\|vol)\right) = & \int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right) d\mu_1(x) + \\ & - \int_M \log\left(t\rho_1(x)+(1-t)\rho_0(x)\right) d\mu_0(x) \\ & + \int_M \frac{1}{t}\left[\log\left(t\rho_1(x)+(1-t)\rho_0(x)\right)\right]\rho_0(x) dvol(x) \\ & - \int_M \frac{1}{t}\log\rho_0(x)\rho_0(x)\, dvol(x). \end{aligned}$$

For the first two terms we notice that, as $t \to 0$, we get

$$\log\left(\, t\rho_1(x) + (1-t)\rho_0(x)\,\right) \to \log \rho_0(x),$$

for every $x \in M$. We know that $|\log\left(t\rho_1(x) + (1-t)\rho_0(x)\,\right)|$ is bounded by $\left|\log\left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_0\,\right)\right| + |\log \rho_0(x)|$, for $0 < t \leq \frac{1}{2}$, and we can use the dominated convergence theorem to reach

$$\int_M \log\left(t\rho_1(x) + (1-t)\rho_0(x)\,\right) d\mu_1(x) \to \int_M \log \rho_0(x)\, d\mu_1(x)$$

and

$$\int_M \log\left(t\rho_1(x) + (1-t)\rho_0(x)\,\right) d\mu_0(x) \to \int_M \log \rho_0(x)\, d\mu_0(x).$$

Finally, we notice the convergence

$$\frac{1}{t}\left[\log\left(t\rho_1(x) + (1-t)\rho_0(x)\,\right) - \log \rho_0(x)\right] \rho_0(x) \uparrow \left[\rho_1(x) - \rho_0(x)\right]$$

as $t \downarrow 0$ for every $x \in M$, and since each term is integrable, we can use the monotone convergence theorem to obtain

$$\int_M \frac{1}{t}\left[\log\left(t\rho_1(x) + (1-t)\rho_1(x)\,\right) - \log \rho_0(x)\right] \rho_0(x)\, dvol(x) \to 0.$$

□

Now we are in position to prove Theorem 1.4.

Proof of Theorem 1.4. Let μ be any probability measure different from μ_{eq} and such that $F(\mu) < \infty$. Define $\mu_t = t\mu + (1-t)\mu_{eq}$, for $t \in [0,1]$. Multiply the equality

$$\Delta \log \rho = \beta\, \rho\, vol - \beta\Lambda,$$

by $G(x,y)$ and integrate in one variable to get

$$-\log \rho(y) + \int_M \log \rho(x)\, d\Lambda(x) = \beta \int_M G(x,y)\rho(x)\, dvol(x)$$

for every $y \in M$. Then, by Lemma 3.1 and 3.2, we have

$$\begin{aligned}
\frac{d}{dt}F(\mu_t)|_{t=0} &= \beta \int_{M\times M} G(x,y)d\mu_{eq}(x)\ (d\mu(y) - d\mu_{eq}(y)) + \\
&\quad + \int_{M\times M} \log\rho(y)\,(d\mu(y) - d\mu_{eq}(y)) \\
&= \int_M \left(\beta \int_M G(x,y)\rho(x)\,dvol(x) + \log\rho(y)\right)(d\mu(y) - d\mu_{eq}(y)) \\
&= \int_M \left(\int_M \log\rho(x)\,d\Lambda(x)\right)(d\mu(y) - d\mu_{eq}(y)) \\
&= \left(\int_M \log\rho(x)\,d\Lambda(x)\right)\int_M (d\mu(y) - d\mu_{eq}(y)) = 0.
\end{aligned}$$

This implies, due to the strict convexity of $F(\mu_t)$, the inequality

$$F(\mu_{eq}) > F(\mu).$$

□

4 Laplace principle

Theorem 1.1 will be proved in this section. For this, we first understand some limiting properties of the energy. Write

$$W_n(x_1,...,x_n) = 2H_n(x_1,...,x_n) = \frac{1}{n^2}\sum_{i\neq j} G(x_i,x_j).$$

The easiest property we need to establish is the following.

Proposition 4.1. *For $\mu \in \mathcal{P}(M)$, we have*

$$\int_{M^n} W_n d\mu^{\otimes n} \to W(\mu).$$

Proof. We integrate to get

$$\int_{M^n} W_n d\mu^{\otimes n} = \frac{n(n-1)}{n^2}\int_{M\times M} G(x,y)\,d\mu(x)\,d\mu(y).$$

Then we take limits to complete the proof. □

For more general $\tau_n \in \mathcal{P}(M^n)$ (not necessarily of the form $\mu^{\otimes n}$) we can obtain a bound for below of the lim inf.

Proposition 4.2. *For each n choose $\tau_n \in \mathcal{P}(M^n)$. Suppose there exists a probability distribution on $\mathcal{P}(M)$, say ζ (that is $\zeta \in \mathcal{P}(\mathcal{P}(M))$), such that $i_n(\tau_n) \to \zeta$. Then we have*

$$\int_{\mathcal{P}(M)} W d\zeta \leq \liminf_{n\to\infty} \int_{M^n} W_n d\tau_n.$$

Proof. As usual, for each $m \geq 0$ define $G_m(x,y) = G(x,y) \wedge m$. For each n take a random element $(X_1^n, ..., X_n^n) \in M^n$ with law τ_n and $\mu \in \mathcal{P}(M)$ with law ζ. Define $\mu_n = i_n(X_1^n, ..., X_n^n)$. We have then

$$\begin{aligned} \int_{M\times M} G_m(x,y) d\mu_n(x)\, d\mu_n(y) &= \frac{1}{n^2} \sum_{i\neq j} G_m(X_i^n, X_j^n) + \frac{m}{n} \\ &\leq \frac{1}{n^2} \sum_{i\neq j} G(X_i^n, X_j^n) + \frac{m}{n}, \end{aligned}$$

from which, taking expected values, we obtain

$$\mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu_n(x)\, d\mu_n(y)\right] \leq \mathbb{E}\left[W_n(X_1^n, ..., X_n^n)\right] + \frac{m}{n}. \quad (4.1)$$

We have thus

$$\mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu_n(x)\, d\mu_n(y)\right] \to \mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu(x)\, d\mu(y)\right]$$

by the continuity of G_m. So, by letting $n \to \infty$ in Inequality 4.1, we reach

$$\mathbb{E}\left[\int_{M\times M} G_m(x,y) d\mu(x)\, d\mu(y)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[W_n(X_1^n, ..., X_n^n)\right].$$

By letting $m \to \infty$, we finally conclude

$$\mathbb{E}\left[\int_{M\times M} G(x,y) d\mu(x)\, d\mu(y)\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[W_n(X_1^n, ..., X_n^n)\right]$$

by the monotone convergence theorem. □

Remark 4.3. In the previous proposition we may choose a sequence of increasing integers n_k and for each k a measure $\tau_k \in \mathcal{P}(M^{n_k})$ such that $i_{n_k}(\tau_k) \to \zeta$, and get the same result:

$$\int_{\mathcal{P}(M)} W d\zeta \leq \liminf_{k\to\infty} \int_{M^{n_k}} W_{n_k} d\tau_k.$$

Now we can start proving Theorem 1.1.

Proof of Theorem 1.1. Take $f : \mathcal{P}(M) \to \mathbb{R}$ continuous. Because of the identity

$$\frac{1}{n}\log \int_{M^n} e^{-nf\circ i_n} d\gamma_n = \frac{1}{n}\log \int_{M^n} e^{-n\left(f\circ i_n + \frac{\beta_n}{2}W_n\right)} dvol^{\otimes n},$$

we only need to prove

$$\frac{1}{n}\log \int_{M^n} e^{-n\left(f\circ i_n + \frac{\beta_n}{2}W_n\right)} dvol^{\otimes n} \to -\inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

For that we use the following result (see [9, Proposition 4.5.1]).

Lemma 4.4 (Variational formulation)**.** *Let E be a Polish space, μ a probability measure on E and $g : E \to \mathbb{R}\cup\{\infty\}$ a measurable function bounded from below. Under those hypothesis, the relation*

$$\log \int_E e^{-g} d\mu = -\inf_{\tau\in\mathcal{P}(E)} \left\{\int_E g\, d\tau + D(\tau\|\mu)\right\}.$$

holds □

In our case, we have

$$\frac{1}{n}\log \int_{M^n} e^{-n\left(f\circ i_n + \frac{\beta_n}{2}W_n\right)} dvol^{\otimes n} =$$
$$= -\inf_{\tau\in\mathcal{P}(M^n)} \left\{\int_{M^n} f\circ i_n\, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n\, d\tau + \frac{1}{n}D(\tau\|vol^{\otimes n})\right\}.$$

Let us start with an upper limit inequality. More precisely, we prove the relation

$$\limsup_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| vol^{\otimes n}) \right\} \leq \inf_{\mu\in\mathcal{P}(M)} \{ f(\mu) + F(\mu) \}. \tag{4.2}$$

For this, we need to see that for every probability measure $\mu \in \mathcal{P}(M)$ we get

$$\limsup_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| vol^{\otimes n}) \right\} \leq f(\mu) + F(\mu). \tag{4.3}$$

It will be enough to find, for every $n \geq 2$, a probability measure $\tau_n \in \mathcal{P}(M^n)$ such that

$$\limsup_{n\to\infty} \left\{ \int_{M^n} f\circ i_n \, d\tau_n + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n \| vol^{\otimes n}) \right\} \leq f(\mu) + F(\mu).$$

We choose the simplest one: $\tau_n = \mu^{\otimes n}$. If so, by the law of large numbers in the compact space M, we have

$$i_n(\tau_n) \to \delta_\mu.$$

Indeed, take a sequence $\{X_k\}_{k\in\mathbb{N}}$ of independent and identically distributed random elements of M with law μ and take any continuous function $g : M \to \mathbb{R}$. Then, $\{g(X_k)\}_{k\in\mathbb{N}}$ is a sequence of independent and identically distributed bounded random variables. By the strong law of large numbers we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} g(X_k) = \mathbb{E}[g(X_1)]$$

almost surely. This can be written as

$$\lim_{n\to\infty} \int_M g \; d[i_n(X_1, ..., X_n)] = \int_M g \, d\mu,$$

and taking a countable dense family of functions we get

$$\lim_{n\to\infty} i_n(X_1, ..., X_n) = \mu$$

almost surely. By the dominated convergence theorem, the almost sure convergence implies the convergence of their laws, and so, as the law of $i_n(X_1, ..., X_n)$ is $i_n(\tau_n)$ and μ is deterministic (of law δ_μ), we obtain

$$i_n(\tau_n) \to \delta_\mu.$$

Hence, we get

$$\lim_{n\to\infty} \int_{M^n} f \circ i_n \, d\tau_n = f(\mu).$$

The second term has already been studied in Proposition 4.1: we have

$$\lim_{n\to\infty} \int_{M^n} W_n d\tau_n = W(\mu).$$

Finally, we use

$$D(\tau_n \| vol^{\otimes n}) = nD(\mu \| vol)$$

to get

$$\begin{aligned} \lim_{n\to\infty} & \left\{ \int_{M^n} f \circ i_n \, d\tau_n + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n \| vol^{\otimes n}) \right\} \\ & = f(\mu) + \frac{\beta}{2} W(\mu) + D(\mu \| vol). \end{aligned}$$

The second and final step is to prove the **lower bound**

$$\begin{aligned} \liminf_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} & \left\{ \int_{M^n} f \circ i_n \, d\tau + \frac{\beta_n}{2} \int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau \| vol^{\otimes n}) \right\} \\ & \geq \inf_{\mu\in\mathcal{P}(M)} \{ f(\mu) + F(\mu) \}. \end{aligned} \tag{4.4}$$

We proceed by contradiction. Suppose this is not true, *i.e.* we have

$$\liminf_{n\to\infty} \inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau\|vol^{\otimes n}) \right\} < \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + F(\mu)\}.$$

Then we can find $C \in \mathbb{R}$ subject to

$$\inf_{\tau\in\mathcal{P}(M^n)} \left\{ \int_{M^n} f\circ i_n \, d\tau + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau + \frac{1}{n} D(\tau\|vol^{\otimes n}) \right\} < C < \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + F(\mu)\}$$

for every n along a subsequence. For each of those n we pick $\tau_n \in \mathcal{P}(M^n)$ such that

$$\int_{M^n} f\circ i_n \, d\tau_n + \frac{\beta_n}{2}\int_{M^n} W_n \, d\tau_n + \frac{1}{n} D(\tau_n\|vol^{\otimes n}) < C$$

The idea now is to take the limit (or just the limit of a subsequence) and derive a contradiction. To achieve that we use the following lemma.

Lemma 4.5. *There exists a subsequence of $\{\tau_n\}$, that we will still call $\{\tau_n\}$ for ease of notation, and a probability distribution ζ (i.e. $\zeta \in \mathcal{P}(\mathcal{P}(M))$) on $\mathcal{P}(M)$, such that $i_n(\tau_n) \to \zeta$ and*

$$\int_{\mathcal{P}(M)} D(\cdot\|vol) \, d\zeta \le \liminf_{n\to\infty} \frac{1}{n} D(\tau_n\|vol^{\otimes n}).$$

Proof. Given a probability measure $\tau_n \in \mathcal{P}(M^n)$ we can construct a n-tuple of random probabilities in M by means of marginals. More precisely, there exists a random variable $(\mathcal{T}_n^1, \mathcal{T}_n^2, ..., \mathcal{T}_n^n)$ on $\mathcal{P}(M)^n$ and a random variable $(X_1, ..., X_n) \in M^n$ with law τ_n, such that

$$\int_M g \, d\mathcal{T}_n^i = \mathbb{E}[g(X_i)|X_1, ..., X_{i-1}],$$

for every continuous function $g : M \to \mathbb{R}$.

We can prove (see Proposition 7.2 in the Appendix for an idea of the proof, or see [9, Theorem C.3.1] for a complete proof) that

$$D(\tau_n \| vol^{\otimes n}) = \mathbb{E}\left[\sum_{i=1}^{n} D(\mathcal{T}_n^i \| vol)\right]$$

holds. So, by the convexity of $D(\cdot\|vol)$ we get

$$\mathbb{E}\left[D\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i \middle\| vol\right)\right] \leq \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} D(\mathcal{T}_n^i\|vol)\right] = \frac{1}{n}D(\tau_n\|vol^{\otimes n}).$$

The compactness of $\mathcal{P}(\mathcal{P}(M)\times\mathcal{P}(M))$ allows us to extract a subsequence of $\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i, \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}\right) \in \mathcal{P}(M)\times\mathcal{P}(M)$ such that $\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i, \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}\right)$ converges in law to, say, $(\chi,\tilde{\chi})$. Then, we get $\chi = \tilde{\chi}$ almost surely (see Proposition 7.4 in the Appendix or [8, Lemma 3.5]). Denote by ζ the common law of χ and $\tilde{\chi}$. The fact that $D(\cdot\|vol)$ is lower semicontinuous and bounded from below implies that it can be written as an increasing pointwise limit of bounded continuous functions, and then the function $\alpha \mapsto \int_{\mathcal{P}(M)} D(\cdot\|vol)d\alpha$ is also lower semicontinuous. In particular, we get

$$\int_{\mathcal{P}(M)} D(\cdot\|vol)\, d\alpha \leq \liminf \mathbb{E}\left[D\left(\frac{1}{n}\sum_{i=1}^{n}\mathcal{T}_n^i \middle\| vol\right)\right].$$

□

We can now complete the proof by noticing that Lemma 4.5 and Proposition 4.2 imply

$$\int_{\mathcal{P}(M)} \left(f + \frac{\beta}{2}W + D(\cdot\|vol)\right) d\zeta \leq C < \inf_{\mu\in\mathcal{P}(M)}\{f(\mu)+F(\mu)\},$$

or, equivalently,

$$\int_{\mathcal{P}(M)} (f+F)\, d\zeta \leq C < \inf_{\mu\in\mathcal{P}(M)}\{f(\mu)+F(\mu)\},$$

which is impossible. □

5 Convergence of $i_n(\mathbb{P}_n)$

We prove the corollaries in this section: Corollary 1.2, about the large deviation principle, and Corollary 1.6, about the convergence of the empirical measures.

Proof of Corollary 1.2. By [9, Theorem 1.2.3] and the fact that I is lower semicontinuous the following Laplace principle implies the large deviation principle: for every continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n}\log\int_{M^n} e^{-nf\circ i_n} d\mathbb{P}_n \overset{n\to\infty}{\longrightarrow} -\inf_{\mu\in\mathcal{P}(M)}\{f(\mu)+I(\mu)\}.$$

Using the measures γ_n and the definition of I it is enough to prove that for every continuous function $f : \mathcal{P}(M) \to \mathbb{R}$ we have

$$\frac{1}{n}\log\int_{M^n} e^{-nf\circ i_n} \frac{d\gamma_n}{Z_n} \overset{n\to\infty}{\longrightarrow} -\inf_{\mu\in\mathcal{P}(M)}\{f(\mu)+F(\mu)-\inf F\}.$$

However, by Theorem 1.1 applied to the function $f=0$, we get

$$\frac{1}{n}\log Z_n \underset{n\to\infty}{\longrightarrow} -\inf F,$$

and combining this with the same theorem for general f, we get

$$\frac{1}{n}\log\int_{M^n} e^{-nf\circ i_n} d\gamma_n \overset{n\to\infty}{\longrightarrow} -\inf_{\mu\in\mathcal{P}(M)}\{f(\mu)+F(\mu)\},$$

and the proof is finished. □

Proof of Corollary 1.6 . Take random probabilities $\{X_n\}_{n\geq 2}$ coupled in any way but such that $X_n \sim i_n(\mathbb{P}_n)$. For any closed set C that does not contain μ_{eq}, we have $\inf_{x\in C} I(x) > 0$ due to the semicontinuity of I. The property

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}_n(i_n^{-1}(C)) \leq -\inf_{x\in C} I(x)$$

implies that there exists $A > 0$ and $N \in \mathbb{N}$ such that

$$\frac{1}{n} \log \mathbb{P}_n(i_n^{-1}(C)) \leq -A$$

for every $n > N$. Hence we have

$$\mathbb{P}_n(i_n^{-1}(C)) \leq e^{-nA}$$

for every $n > N$, which yields

$$\sum_{n=1}^{\infty} \mathbb{P}_n(i_n^{-1}(C)) < \infty.$$

By the Borel-Cantelli lemma we get then

$$\mathbb{P}(\text{there exists } M \in \mathbb{N} \text{ such that } i > M \text{ implies } X_i \notin C) = 1.$$

Take a countable local base $\{O_i\}_{i \in \mathbb{N}}$ around μ_{eq} and apply the previous argument for every $C = O_i^c$ to obtain almost sure convergence. □

6 Final comments

This work has been inspired on the article by Robert Berman [4] where a slightly different model is treated. Our proof of the large deviation principle is an adaptation of the article by Paul Dupuis, Vaios Laschos, and Kavita Ramanan [8] to the case of compact manifolds.

Here we have studied just one kind of limiting behavior for a sequence of point processes on a surface. There are two main issues that, to our knowledge, are still open: the **fluctuations** and the **local behaviour**.

By **fluctuations** we mean the following. Take $f \in C^{\infty}(M)$ and μ_n a sequence with law $i_n(\mathbb{P}_n)$. We have proved, in Corollary 1.6, the convergence

$$\int f d\mu_n \to \int f \, d\mu_{eq},$$

what we could rewrite as

$$\int f d\mu_n = \int f \, d\mu_{eq} + o(1).$$

The idea is to find the next order terms (to prove a central limit type theorem). More precisely, to find a sequence $\alpha_n \to \infty$ such that

$$\alpha_n \left(\int f d\mu_n - \int f \, d\mu_{eq} \right)$$

converges weakly, and describe such limit.

When we talk about **local behavior** we take $x \in M$ and a chart

$$\phi : U \to T_x M$$

such that $\phi(x) = 0$ and $d\phi_x = id|_{T_x M}$. We fix n points $(X_1, ..., X_n)$ distributed according to $\mathbb{P}_n$. We get a point process in $T_x M$ with points $\phi(X_1), ..., \phi(X_n)$ (when $X_i \in U$). We then scale this point process by $\sqrt{n}$ and find the limit (in some sense) point process. We ask how this point process depends on $x \in M$.

These questions are already answered in the case of some determinantal point processes (see [1] and [3]) and in the one dimensional case (see [10]). Very recent results about fluctuations on $\mathbb{R}^2$ can be found in [2] and [11].

7 Appendix

Here we deal with several tools used along this paper.

Proposition 7.1. *Let E be a compact metrizable space. Then $\mathcal{P}(E)$, the space of probability measures on E, is a compact metrizable space.*

Proof. By the Stone-Weierstrass theorem we know that the space of continuous functions on E is separable in the topology of uniform convergence. Choose a dense countable set $\{f_m\}_{m \in \mathbb{N}}$.

Let d be a metric in E that induces its topology. Define $\bar{d} : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}$ by

$$\bar{d}(\mu, \nu) = \sum_{m \in \mathbb{N}} \frac{1}{2^m} \wedge \left| \int_E f_m d\mu - \int_E f_m d\nu \right| .$$

We can see that the topology induced by $\bar{d}$ is the smallest topology such that $\mu \mapsto \int_E f_m d\mu$ is continuous for every $m \in \mathbb{N}$. But by density and uniform convergence the functional $\mu \mapsto \int_E f_m d\mu$ is continuous for every $m \in \mathbb{N}$ if and only if $\mu \mapsto \int_E f d\mu$ is continuous for any continuous function $f : E \to \mathbb{R}$. So, the topology induced by $\bar{d}$ is the weak topology of $\mathcal{P}(E)$.

To see that $\mathcal{P}(E)$ is compact it is enough to show that it is sequentially compact. Take a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on E. By a diagonal procedure we can choose a subsequence $\{\mu_{n_i}\}_{i\in\mathbb{N}}$ such that $\int_E f_m d\mu_{n_i}$ converges as i goes to infinity for every $m \in \mathbb{N}$. This implies that $\int_E f d\mu_{n_i}$ converges as i goes to infinity for every continuous function $f : E \to \mathbb{R}$. Indeed, we can prove that $\{\int_E f d\mu_{n_i}\}_{i\in\mathbb{N}}$ is Cauchy. For this, take $\epsilon > 0$ and choose $m \in \mathbb{N}$ such that $\|f_m - f\| < \epsilon/3$. Take a number M such that if $i, j > M$ then $|\int_E f_m d\mu_{n_i} - \int_E f_m d\mu_{n_j}| < \epsilon/3$. Then, whenever $i, j > M$, we have

$$\begin{aligned} \left| \int_E f d\mu_{n_i} - \int_E f d\mu_{n_j} \right| \leq & \left| \int_E f d\mu_{n_i} - \int_E f_m d\mu_{n_i} \right| + \\ & + \left| \int_E f_m d\mu_{n_i} - \int_E f_m d\mu_{n_j} \right| + \\ & + \left| \int_E f_m d\mu_{n_j} - \int_E f_m d\mu_{n_j} \right| \\ < & \epsilon \end{aligned}$$

Define $\Lambda : C(E) \to \mathbb{R}$ as $\Lambda(f) = \lim_{i\to\infty} \int_E f d\mu_{n_i}$. Then Λ is a positive linear functional and so, there exists a positive measure μ on E such that $\Lambda(f) = \int_E f d\mu$ for every $f \in C(E)$. As $\Lambda(1) = \lim_{i\to\infty} \int_E 1 d\mu_{n_i} = 1$, we obtain $\mu \in \mathcal{P}(E)$. In this way, we have extracted a subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$ that converges. □

In what follows, instead of writing $d\mu(x)$ we write $\mu(dx)$.

As in the proof of Lemma 4.5, given a probability measure $\mu \in \mathcal{P}(M^n)$ we can construct a n-tuple of random probabilities $(\mu_1, \mu_2, ..., \mu_n)$ in $\mathcal{P}(M)^n$ and a random element $(X_1, ..., X_n) \in M^n$ with law μ such that

$$\int_M f d\mu_i = \mathbb{E}[f(X_i)|X_1, ..., X_{i-1}]$$

holds.

Proposition 7.2 (Chain rule)**.** *We have*

$$D(\mu \| vol^{\otimes_n}) = \mathbb{E}\left[\sum_{i=1}^{n} D(\mu_i \| vol)\right].$$

Sketch of the proof. We will give an idea of the proof ignoring issues of measurability and finiteness of the entropy. For extra details we refer to [9, Theorem C.3.1].

We consider M^n with a probability measure μ. In this case the random element with law μ is $(X_1, ..., X_n)$ where $X_i : M^n \to M$ is the projection onto the i-th coordinate. Suppose that

$$\tilde{\mu}_k : M^{k-1} \to \mathcal{P}(M)$$

is a transition kernel from $(X_1, ..., X_{k-1})$ to X_k.

If we define

$$\mu_k = \tilde{\mu}_k \circ \pi_{k-1},$$

where $\pi_{k-1} : M^n \to M^{k-1}$ is the projection onto the first $k-1$ coordinates, we see that $(\mu_1, ..., \mu_n)$ satisfies the properties of the definition. If we assume all entropies are finite, we get

$$
\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^{n} D(\mu_k \| vol)\right] &= \sum_{k=1}^{n} \mathbb{E}\left[D(\mu_k \| vol)\right] \\
&= \sum_{k=1}^{n} \int_{M^{k-1}} D(\tilde{\mu}_k(x) \| vol)\, [\pi_{k-1}(\mu)](dx) \\
&= \sum_{k=1}^{n} \int_{M^{k-1}} \left(\int_M \log\left(\frac{\tilde{\mu}_k(x,dy)}{vol(dy)}\right) \mu_k(x,dy) \right) [\pi_{k-1}(\mu)](dx) \\
&= \sum_{k=1}^{n} \int_{M^{k-1}\times M} \log\left(\frac{\tilde{\mu}_k(x,dy)}{vol(dy)}\right) [\pi_k(\mu)]\,(dx,dy) \\
&= \sum_{k=1}^{n} \int_{M^n} \log\left(\rho_k(x)\right) \mu(dx),
\end{aligned}
$$

where $\rho_k : M^n \to [0,\infty]$ is equal to $\rho_k = \frac{\tilde{\mu}_k(x,dy)}{vol(dy)} \circ \pi_k$.

Then we just have to notice the equality

$$
\prod_{i=1}^{n} \rho_i(x) = \frac{\mu(dx)}{vol^{\otimes_n}(dx)},
$$

that follows from the definition.

□

Lemma 7.3. *Let $(X_1, ..., X_n) \in M^n$ and $(\mu_1, ..., \mu_n) \in \mathcal{P}(M)^n$ be random elements as before. Consider the random measures*

$$
\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \mu_i \quad , \quad \hat{\nu} = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}.
$$

Then we have

$$
\mathbb{P}\left(\left| \int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy) \right| > \epsilon \right) \leq 4\frac{\|f\|_\infty^2}{n\epsilon^2}.
$$

Proof. By Chebyshev's inequality, we need to understand the quantity

$$\mathrm{Var}\ \left(\int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy)\right).$$

The first term is

$$\int_M f(y)\hat{\mu}(dy) = \frac{1}{n}\sum_{k=1}^{n}\int_M f(y)\,\mu_k(dy) = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}[f(X_k)|X_1,...,X_{k-1}]$$

while the second is

$$\int_M f(x)\hat{\nu}(dx) = \frac{1}{n}\sum_{i=1}^{n} f(X_i).$$

We can see that both have the same expected value, and if $i < j$, we have

$$\mathbb{E}\Big[\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)\mathbb{E}[f(X_j)|X_1,...,X_{j-1}]\Big] =$$

$$= \mathbb{E}\Big[\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)f(X_j)\Big]$$

because $\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)$ is $(X_1,...,X_{j-1})$ measurable. Then we get

$$\mathbb{E}\Big[\Big(f(X_i)-\mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)\Big(f(X_j)-\mathbb{E}[f(X_j)|X_1,...,X_{j-1}]\Big)\Big] = 0.$$

So we have

$$\begin{aligned}\mathrm{Var}\ &\left(\int_M f(x)\hat{\mu}(dx) - \int_M f(y)\hat{\nu}(dy)\right) = \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[\Big(f(X_i) - \mathbb{E}[f(X_i)|X_1,...,X_{i-1}]\Big)^2\right] \\ &\leq \frac{1}{n^2}\sum_{i=1}^{n} 4\|f\|_\infty^2 = \frac{1}{n}4\|f\|_\infty^2,\end{aligned}$$

and by Chebyshev's inequality we conclude our claim. □

Proposition 7.4. *Using the notation of the proof in Lemma 4.5, if we have*

$$(\hat{\mu}_n, \hat{\nu}_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \tau_n^i, \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \right) \to (\chi, \tilde{\chi})$$

in law, then we have $\chi = \tilde{\chi}$ almost surely.

Proof. For any continuous $f : M \to \mathbb{R}$, the function

$$T_f : \mathcal{P}(M) \times \mathcal{P}(M) \to \mathbb{R}$$
$$(\mu, \nu) \mapsto \int_M f(x)\mu(dx) - \int_M f(y)\nu(dy)$$

is continuous. By Lemma 7.3, for every continuous f, we get

$$\mathbb{P}\left(|T_f(\hat{\mu}_n, \hat{\nu}_n)| > \epsilon\right) \leq 4\frac{\|f\|_\infty^2}{n\epsilon^2},$$

and, by the Portmanteau theorem (taking the lower limit on both sides), we reach

$$\mathbb{P}\left(|T_f(\chi, \tilde{\chi})| > \epsilon\right) = 0$$

for every $\epsilon > 0$. Thus we have

$$\mathbb{P}\left(|T_f(\chi, \tilde{\chi})| = 0\right) = 1.$$

Next, choose a dense sequence $\{f_m\}_{m\in\mathbb{N}}$ in the space of continuous functions on M endowed with the topology of uniform convergence in order to obtain

$$\mathbb{P}\left(|T_{f_m}(\chi, \tilde{\chi})| = 0 \text{ for all } m\right) = 1.$$

But, by density, we have

$$\{|T_{f_m}(\chi, \tilde{\chi})| = 0 \text{ for all } m\} = \{|T_f(\chi, \tilde{\chi})| = 0 \text{ for all continuous } f\},$$

which means $\chi = \tilde{\chi}$ almost surely. □

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Resumen

Siguiendo las técnicas desarrolladas por Paul Dupuis, Vaios Laschos y Kavita Ramanan en [8], se establecerá un principio de grandes desviaciones para una secuencia de procesos puntuales definidos por medidas de Gibbs en una variedad riemanniana bidimensional compacta y orientable. Veremos que la correspondiente secuencia de medidas empíricas converge a la solución de una ecuación diferencial parcial y, en ciertos casos, a la forma de volumen de una métrica de curvatura constante.

Palabras clave: Medidas de Gibbs; gas de Coulomb; medida empírica; principio de grandes desvíos; sistemas de partículas interactuantes; potencial singular; variedad de Einstein 2-dimensional; entropía relativa.

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