Notes on flat pseudo-Riemannian manifolds

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(Traslated into English by Elizabeth Gasparim[2]***)***

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Abstract

In these notes we survey basic concepts of affine geometry and their interaction with Riemannian geometry. We give a characterization of affine manifolds which has as counterpart those pseudo-Riemannian manifolds whose Levi-Civita connection is flat. We show that no connected semisimple Lie group admits a left invariant flat affine connection. We characterize flat pseudo-Riemannian Lie groups. For a flat left-invariant pseudo-metric on a Lie group, we show the equivalence between the completeness of the Levi-Civita connection and unimodularity of the group. We emphasize the case of flat left invariant hyperbolic metrics on the cotangent bundle of a simply connected flat affine Lie group. We also discuss Lie groups with bi-invariant pseudo-metrics and the construction of orthogonal Lie algebras.

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1 Introduction

A real smooth manifold M of dimension n is called an **affine manifold** if it admits a maximal atlas whose change of coordinates are restrictions of affine transformations of $\mathbb{R}^n$. Having an affine structure over M is equivalent to having a flat and torsion free linear connection ∇ on TM (see Theorem 2.5). A pair (M, ∇), where ∇ is a flat affine connection (i.e. ∇ is a flat and torsion free linear connection) on M, is called a **flat affine manifold**. When $M = G$ is a Lie group and ∇ is a left invariant flat affine connection, the pair (G, ∇) is called a **flat affine Lie group**. If g is a pseudo-metric on M (respectively μ is a left invariant pseudo-metric on G) such that the Levi-Civita connection associated to g has vanishing curvature tensor, the pair (M, g) (respectively (G, μ)) is called a flat pseudo-Riemannian manifold (respectively flat pseudo-Riemannian Lie group).

These notes are organized as follows. The first two sections are devoted to the study of flat affine manifolds. Theorem 3.2 is essential because it gives a characterization of flat affine Lie groups that we will use throughout these notes. We show that no connected semisimple real Lie group admits a left invariant flat affine connection (Theorem 3.6). In Section 3 we introduce some basic concepts of Riemannian geometry and exhibit examples of flat affine structures compatible with pseudo-metrics. Section 4 is dedicated to the study of flat pseudo-Riemannian Lie groups. We give a characterization of such Lie groups and we show that the left-invariant affine structure defined by the Levi-Civita connection is geodesically complete if and only if the group is unimodular (Theorem 5.3). We also show that the cotangent bundle of a simply connected flat affine Lie group is endowed with an affine Lie group structure and a left invariant flat hyperbolic metric (Proposition 6.1). In the Section 7 we study **orthogonal Lie groups**, that is, Lie groups endowed with bi-invariant metrics. To study properties of orthogonal Lie groups we introduce the notion of **orthogonal Lie algebra**, which will be used in the method of **double orthogonal extension**. As an application, we describe how to construct the oscillator Lie algebra of the oscillator Lie group which appear in several branches of Physics and Mathematical-Physics and give rise to particular solutions of the Einstein-Yang-Mills equations. Finally, we present another characterization of flat Riemannian Lie groups using some consequences of the presence of an orthogonal structure in a Lie algebra (Theorem 7.9).

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2 Flat affine manifolds

In what follows M will denote a connected paracompact real smooth manifold of dimension n. We will denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields over M and by $C^\infty(M)$ the associative algebra of functions on M with values in $\mathbb{R}$.

The objects of study of these notes are flat affine paracompact manifolds. In particular, we study flat affine structures that are compatible with pseudo-Riemannian metrics. A good understanding of the category of Lagrangian submanifolds requires a good knowledge of the category of flat affine manifolds (see [26, Thm 7.8]). Also, flat affine manifolds with holonomy reduced to $\mathrm{Gl}(n,\mathbb{Z})$ appear naturally in integrable systems and Mirror symmetry (see [12]). Further applications of flat affine manifolds appear in the study of Hessian structures and Information Geometry (see [23, c. 6]).

Let V be a real finite dimensional vector space. The space of **affine transformations of** V is the Lie group $\mathrm{Aff}(V) = V \rtimes_{Id} \mathrm{GL}(V)$ determined by the semi-direct product of the Abelian Lie group $(V,+)$ with the Lie group $\mathrm{GL}(V)$ via the identity representation. Its Lie algebra is the product vector space $\mathfrak{aff}(V) = V \rtimes_{id} \mathfrak{gl}(V)$ with Lie bracket given by

$$[(x,t),(y,s)] = (t(y) - s(x), [t,s]_{\mathfrak{gl}(V)}),$$

for all $x, y \in V$ and $t, s \in \mathfrak{gl}(V)$.

We say that M **admits an affine structure** if there exists an maximal atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in J}$ of M having change of coordinates that are restrictions of affine transformations of $\mathbb{R}^n$, that is, for each $\alpha, \beta \in J$ with $U_\alpha \cap U_\beta \neq \emptyset$, there exists $\sigma_{\alpha\beta} \in \mathrm{Aff}(\mathbb{R}^n)$ such that

$$\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)} = \sigma_{\alpha\beta}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)}.$$

If G is a discrete Lie subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ that acts freely and properly discontinuously over $\mathbb{R}^n$, then the quotient manifold $\mathbb{R}^n/G$ admits an affine structure such that the coordinate changes are restrictions of elements of G (see [22, p. 349]).

Example 2.1. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, $U_1 = S^1 - \{(1,0)\}$ and $U_2 = S^1 - \{(0,1)\}$. If $\varphi_1 : U_1 \to (0, 2\pi)$ and $\varphi_2 : U_2 \to \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$ are defined respectively by

$$z \mapsto \arg(z) \quad \text{and} \quad z \mapsto \begin{cases} \arg(z) - \dfrac{\pi}{2}, & \text{if} \quad \arg(z) \in \left(\dfrac{\pi}{2}, 2\pi\right) \\ \arg(z) + 3\dfrac{\pi}{2}, & \text{if} \quad \arg(z) \in \left(0, \dfrac{\pi}{2}\right) \end{cases},$$

then the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ determines an affine structure for S^1.

Example 2.2. *Hopf manifolds.* Let $\lambda > 1$ be a fixed real number. Denote by G the group of transformations of $\mathbb{R}^n\backslash\{0\}$ defined by

$$\begin{aligned} T_n : \mathbb{R}^n\backslash\{0\} &\to \mathbb{R}^n\backslash\{0\} \\ x &\mapsto T_n(x) = \lambda^n \cdot x, \end{aligned}$$

for all $n \in \mathbb{Z}$. The set G is a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ that acts freely and properly discontinuously over $\mathbb{R}^n\backslash\{0\}$. Therefore $\mathbb{R}^n\backslash\{0\}/G$ is an affine manifold called a **Hopf manifold** which we will denote by $\mathrm{Hopf}(\lambda, n)$. Topologically these manifolds are either the disjoint union of two Hopf circles $\mathbb{R}^+/G$, when $n = 1$, or diffeomorphic to $S^{n-1} \times S^1$ when $n > 1$.

Recall that a **linear connection** on a smooth manifold M is an $\mathbb{R}$-bilinear map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ that is $C^\infty(M)$-linear on the first component and satisfies

$$\nabla_X fY = X(f)Y + f\nabla_X Y,$$

for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. The **torsion tensor** T_∇ and **curvature tensor** R_∇ associated to a linear connection ∇ are defined respectively by

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{2.1}$$

and

$$R_\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. When $T_\nabla = 0$ and $R_\nabla = 0$ we say that ∇ is a **flat affine connection** and the pair (M, ∇) is called a **flat affine manifold**.

Remark 2.3. The pair (M, ∇) is a flat affine manifold if and only if there exists an atlas for M such that the Christoffel symbols associated to ∇ vanish identically on all charts (see [5, p. 108]).

Example 2.4. If $(x^1, \cdots, x^n)$ are the usual coordinates in $\mathbb{R}^n$, the **usual linear connection** ∇^0 on $\mathbb{R}^n$ is defined as

$$\nabla^0_X Y = \sum_{j=1}^{n} X(f^j)\frac{\partial}{\partial x^j}, \quad \text{where} \quad Y = \sum_{j=1} f^j \frac{\partial}{\partial x^j}.$$

It is simple to verify that ∇^0 is a flat affine connection on $\mathbb{R}^n$ and that we have $\Gamma^k_{ij} = 0$ for all $i, j, k = 1, \cdots, n$.

From now on, by smooth manifolds we mean real manifolds that are C^∞ differentiable. The following characterization of affine manifolds appears in [2].

Theorem 2.5 (Auslander-Markus)**.** *A real smooth manifold M has an affine structure if and only if there exists a flat affine connection on M.*

Proof. Suppose that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in J}$ is an affine structure for M. For each $\alpha \in J$, we endow the open set $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ with the usual linear connection ∇^0. The pullback of $\nabla^0\big|_{\varphi_\alpha(U_\alpha)}$ by the diffeomorphism φ_α defines a flat affine connection ∇_α over U_α. We choose ∇ over M as the linear connection subjected to $\nabla|_{U_\alpha} = \nabla_\alpha$ for all $\alpha \in J$. To verify that ∇ is well defined observe that, for $\alpha, \beta \in J$ with $U_\alpha \cap U_\beta \neq \emptyset$, setting $\varphi_\alpha = (x^1, \cdots, x^n)$ and $\varphi_\beta = (y^1, \cdots, y^n)$ on $U_\alpha \cap U_\beta$, we have

$$(\Gamma^i_{jk})_\alpha = \sum_{l=1}^{n} \frac{\partial^2 y^l}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^l} + \sum_{l,m,q=1}^{n} (\Gamma^l_{mq})_\beta \frac{\partial y^m}{\partial x^j}\frac{\partial y^q}{\partial x^k}\frac{\partial x^i}{\partial y^l}. \tag{2.3}$$

Given that the Christoffel symbols of ∇^0 vanish, we obtain $(\Gamma^l_{mq})_\beta = 0$ on U_β. Moreover we have $\dfrac{\partial^2 y^l}{\partial x^j \partial x^k} = 0$, since $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha\cap U_\beta)}$ is the restriction of an element of $\mathrm{Aff}(\mathbb{R}^n)$, and hence we obtain $(\Gamma^i_{jk})_\alpha = 0$, showing that ∇ is well defined. Furthermore, using equation (2.3) we can verify that ∇ is the unique flat affine connection that can be obtained in this fashion.

Reciprocally, suppose that ∇ is a flat affine connection on M. For each $p \in M$ there exists a neighborhood V_p of 0 in T_pM and a neighborhood U_p of p in M such that the exponential map associated to ∇, denoted by $\exp_p\colon V_p \to U_p$, is a diffeomorphism (see [11, p. 148]). Given a basis $\{X_1, \cdots, X_n\}$ for the tangent space T_pM, we define local charts on U_p by

$$x^i \left(\exp_p \left(\sum_{j=1}^{n} a^j X_j \right) \right) = a^i,$$

if $\sum_{j=1}^{n} a^j X_j \in V_p$ for all $i = 1, \cdots, n$. Since ∇ is flat affine, there exists an atlas over M with respect to which we have $\Gamma_{ij}^k = 0$ for every chart. The computation of geodesic curves γ in a chart $(U, (y^1, \cdots, y^n))$ of such an atlas amounts to solving the system of ordinary differential equations $\dfrac{d^2 y^i(\gamma(t))}{dt^2} = 0$ for $i = 1, \cdots, n$, whose solution, for a fixed initial condition $\left(p, \sum_{j=1}^{n} a^j X_j\right) \in TM$, is unique. Therefore, setting $\dfrac{\partial^2 x*^l}{\partial x^j \partial x^k} = 0$ on each intersection, these normal coordinates $(U_p, (x^1, \cdots, x^n))$ generate a unique atlas over M for which the changes of coordinates are restrictions of elements of $\mathrm{Aff}(\mathbb{R}^n)$. $\square$

In general, determining whether a smooth manifold admits a flat affine structure or not is a difficult question, and there are obstructions for the existence of said structures.

Example 2.6. We list some manifolds that do not admit flat affine structures:

- Compact simply connected manifolds (see [7]).
- Compact manifolds with finite fundamental group (see [2]).
- In particular for $n > 1$ the real n-sphere S^n, the real projective space $\mathbb{RP}^n$ and the group of rotations $O(n)^+$ do not admit flat affine structures.

Further topological obstructions for the existence of a flat affine structures are listed in [24].

Remark 2.7. There is no direct relation between the notion of affine variety as given in algebraic geometry (namely a set cut out by polynomial equations) and the definition of affine manifold in the way that we present it (when a manifold admits an affine structure). For example, for $n > 1$ the n-dimensional real sphere S^n is an affine algebraic variety but is not a flat affine manifold in the sense of our definition.

3 Flat affine Lie groups

In what follows G denotes a connected real Lie group. For each $\sigma \in G$, we denote by $L_\sigma : G \to G$ the map left multiplication by σ in G, that is, the map defined by $\tau \mapsto L_\sigma(\tau) = \sigma\tau$. The tangent space $T_\epsilon G$ of G at the identity and the Lie algebra of left invariant vector fields $\mathfrak{X}_l(G)$ on G are isomorphic vector spaces as follows. For each $x \in T_\epsilon G$, we associate the left invariant vector field x^+ defined by

$$x^+_\sigma = (L_\sigma)_{*,\epsilon}(x) = \frac{d}{dt}\bigg|_{t=0} (\sigma \cdot \exp_G(tx)),$$

for all $\sigma \in G$. Under this isomorphism we give a structure of Lie algebra to $\mathfrak{g} = T_\epsilon G$ and call it the Lie algebra of G.

A linear connection ∇ on G is called **left invariant** if L_σ is an affine transformation of (G, ∇) for all $\sigma \in G$. More precisely, we must have

$$(L_{\sigma^{-1}})_* \left(\nabla_{(L_\sigma)_* X}(L_\sigma)_* Y\right) = \nabla_X Y,$$

for all $X, Y \in \mathfrak{X}(G)$ and $\sigma \in G$. From this definition it follows immediately that a connection ∇ on G is left invariant if and only if for all $x^+, y^+ \in \mathfrak{X}_l(G)$ we have $\nabla_{x^+} y^+ \in \mathfrak{X}_l(G)$.

Lemma 3.1. *There exists a bijective correspondence between left invariant linear connections on G and bilinear maps on $\mathfrak{g}$.*

Proof. If ∇ is a left invariant linear connection on G, then the assignment $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, given by $(x, y) \mapsto x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$, defines a bilinear map on $\mathfrak{g}$. Conversely, suppose that $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a bilinear map on $\mathfrak{g}$. Define ∇ on G as the linear connection such that $\nabla_{x^+} y^+ = (x \cdot y)^+$ verifies

$$\nabla_{fx^+} y^+ = f.(x \cdot y)^+ \qquad \text{and} \qquad \nabla_{x^+} f y^+ = x^+(f) y^+ + f.(x \cdot y)^+, \tag{3.1}$$

for all $x, y \in \mathfrak{g}$ and $f \in C^\infty(G)$. Since the left invariant vector fields determine an absolute parallelism over G, we have that $\mathfrak{X}_l(G)$ generates $\mathfrak{X}(G)$ as a $C^\infty(G)$-module. Hence, each smooth vector field over G can be written as a $C^\infty(G)$-linear combination of left invariant vector fields. Using this fact together with the identities exhibited in (3.1) we can easily conclude that ∇ is a left invariant linear connection on G. $\square$

When there exists a left invariant flat affine connection ∇ on G, the pair (G, ∇) is called a **flat affine Lie group**. To characterize flat affine Lie groups and to study their structure is an open problem which was proposed by J. Milnor in [20]. The following characterization of flat affine Lie groups was given in [10] and [15].

Theorem 3.2 (Koszul and Medina)**.** *Let G be a connected n-dimensional real Lie group, $\mathfrak{g}$ its Lie algebra and $\widetilde{G}$ its universal covering Lie group. Then, the following are equivalent.*

1. *There exists a left invariant flat affine connection on G.*
2. *There exists a bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ such that*
$$[x, y] = x \cdot y - y \cdot x \tag{3.2}$$
and
$$L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}, \tag{3.3}$$
for all $x, y \in \mathfrak{g}$, here $L_x : \mathfrak{g} \to \mathfrak{g}$ is the map defined by $y \mapsto L_x(y) = x \cdot y$.
3. *There exists a real n-dimensional vector space V and a Lie group homomorphism $\rho\colon \widetilde{G} \to \mathit{Aff}(V)$ such that the left action of $\widetilde{G}$ over V defined by $\sigma \cdot v = \rho(\sigma)(v)$ for all $(\sigma, v) \in \widetilde{G} \times V$, allows a point having open orbit and discrete isotropy.*

Proof. We first show that 1 implies 2. Let ∇ be a left invariant flat affine connection on G. By Lemma 3.1 we have that $L_x(y) = x{\cdot}y = (\nabla_{x^+}y^+)(\epsilon)$ defines a bilinear map on $\mathfrak{g}$. Substituting this equality into the formulas of torsion and curvature (2.1)-(2.2) for ∇, we obtain identities (3.2) and (3.3), respectively.

To get 2 implies 3, suppose that there exists a bilinear map $\cdot$: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ satisfying (3.2) and (3.3), where $L_x : \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $y \mapsto L_x(y) = x \cdot y$, for all $x, y \in \mathfrak{g}$. Then the map $\theta : \mathfrak{g} \to \mathfrak{aff}(\mathfrak{g})$, defined by $x \mapsto (x, L_x)$, is a well defined Lie algebra homomorphism. This follows from the fact that (3.2) and (3.3) imply that the map $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, defined by $x \mapsto L_x$, is a well defined Lie algebra homomorphism which satisfies $[x, y] = L_x(y) - L_y(x)$ for all $x, y \in \mathfrak{g}$. On the other hand, for $0 \in \mathfrak{g}$ the map $\psi_0 : \mathfrak{g} \to \mathfrak{g}$ given by $x \mapsto x + L_x(0) = x$ is a linear isomorphism. Thus, by means of the exponential map of G, we obtain a homomorphism of Lie groups $\rho : \widetilde{G} \to \mathrm{Aff}(\mathfrak{g})$ given by $\sigma \to (Q(\sigma), F_\sigma)$, where for $\sigma = \exp_G(x)$ we have

$$Q(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k!}(L_x)^{k-1}(x) \quad \text{and} \quad F_\sigma = \mathrm{Exp}(L_x) = \sum_{k=0}^{\infty} \frac{1}{k!}(L_x)^k.$$

Since ψ_0 is surjective, the orbit of $0 \in \mathfrak{g}$ by the left action of $\widetilde{G}$ over $\mathfrak{g}$, defined by $\sigma \cdot 0 = Q(\sigma) + F_\sigma(0) = Q(\sigma)$ for all $\sigma \in \widetilde{G}$, is open. Moreover, by the injectivity of ψ_0 it follows that the isotropy of $0 \in \mathfrak{g}$ by the given action is dicrete. The latter implies that the orbital map $\pi : \widetilde{G} \to \mathrm{Orb}(0)$, given by $\sigma \mapsto Q(\sigma)$, is a local diffeomorphism and hence a covering map (see [15]).

Finally let us show that 3 implies 1. Let V be a real vector space of dimension n and assume that there exists a Lie group homomorphism $\rho : \widetilde{G} \to \mathrm{Aff}(V)$, defined by $\sigma \to (Q(\sigma), F_\sigma)$, which admits a point $v \in V$ with open orbit and discrete isotropy for the action of $\widetilde{G}$ on the left over V induced by ρ. The latter implies that the map $F : \widetilde{G} \to \mathrm{GL}(V)$ defined by $\sigma \mapsto F_\sigma$, is a Lie group homomorphism and $Q : \widetilde{G} \to V$, given by $\sigma \mapsto Q(\sigma)$, is a smooth map that satisfies

$$Q(\sigma\tau) = Q(\sigma) + F_\sigma(Q(\tau)),$$

for all $\sigma, \tau \in \widetilde{G}$. Moreover, the orbital map $\pi : \widetilde{G} \to \mathrm{Orb}(v)$ given by $\sigma \mapsto Q(\sigma) + F_\sigma(v)$ is a local diffeomorphism. Differentiating at the identity of $\widetilde{G}$, we obtain the Lie algebra homomorphism $\theta : \mathfrak{g} \to \mathfrak{aff}(V)$ given by $x \mapsto (q(x), f_x)$ for all $x \in \mathfrak{g}$, where the map $f : \mathfrak{g} \to \mathfrak{gl}(V)$ defined by $x \mapsto f_x$, is a Lie algebra homomorphism and $q : \mathfrak{g} \to V$, given by $x \mapsto q(x)$, is the linear map

$$q([x, y]) = f_x(q(y)) - f_y(q(x)), \tag{3.4}$$

for all $x, y \in \mathfrak{g}$. Moreover, the map $\psi_v : \mathfrak{g} \to V$ defined by $x \mapsto q(x) + f_x(v)$ is a linear isomorphism. Now, for each $x \in \mathfrak{g}$ we define

$$L_x = \psi_v^{-1} \circ f_x \circ \psi_v.$$

Since $f : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism, we have $L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}$ for all $x, y \in \mathfrak{g}$. On the other hand, since $q : \mathfrak{g} \to V$ satisfies (3.4), we conclude $[x, y] = L_x(y) - L_y(x)$ for all $x, y \in \mathfrak{g}$. Therefore, using Lemma 3.1, we obtain that the linear connection ∇ defined by

$$\nabla_{x^+} y^+ = (x \cdot y)^+ = (L_x(y))^+,$$

for all $x, y \in \mathfrak{g}$, is a left invariant flat affine connection on G. Using the linear isomorphism ψ_v, it can be easily drawn that the Lie algebra homomorphisms in $\mathfrak{g} \to \mathfrak{aff}(V)$ defined by $x \mapsto (x, L_x)$ and $x \to (q(x), f_x)$ are isomorphic. □

Example 3.3. *Dimension 2.* Recall that the Lie group of affine transformations of the real line is given by the product manifold $\mathrm{Aff}(\mathbb{R}) = \mathbb{R}^* \times \mathbb{R}$, with product $(a, b) \cdot (c, d) = (ac, ad + b)$. Its Lie algebra is identified with the vector space $\mathfrak{aff}(\mathbb{R}) = \mathrm{Vect}_{\mathbb{R}}\{e_1, e_2\}$ with Lie bracket $[e_1, e_2] = e_2$. Next we introduce is a family of left invariant flat affine connections on $\mathrm{Aff}(\mathbb{R})$ which are not isomorphic. For α real, set

$$\nabla_{e_1^+} e_1^+ = \alpha e_1^+, \quad \nabla_{e_1^+} e_2^+ = e_2^+, \quad \nabla_{e_2^+} e_1^+ = \nabla_{e_2^+} e_2^+ = 0.$$

Here $e_1^+ = x\dfrac{\partial}{\partial x}$ and $e_2^+ = x\dfrac{\partial}{\partial y}$ are the left invariant vector fields associated to e_1 and e_2, respectively. A description of left invariant flat affine structures over $\mathrm{Aff}(\mathbb{R})$ can be found in [18].

Example 3.4. *Dimension 3.* The **Heisenberg** Lie group of dimension 3 is given by the set of matrices

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \; : \; x, y, z \in \mathbb{R} \right\}.$$

The Lie algebra of H_3 is identified with $\mathfrak{h}_3 = \mathrm{Vect}_{\mathbb{R}}\{e_1, e_2, e_3\}$ with Lie bracket $[e_1, e_2] = e_3$. The following is a left invariant flat affine connection on H_3:

$$\nabla_{e_1^+} = \nabla_{e_3^+} = 0, \quad \nabla_{e_2^+} e_1^+ = -e_3, \quad \nabla_{e_2^+} e_2^+ = e_1^+, \quad \nabla_{e_2^+} e_3^+ = 0.$$

The vector fields $e_1^+ = \dfrac{\partial}{\partial x}$, $e_2^+ = \dfrac{\partial}{\partial y} + x\dfrac{\partial}{\partial z}$, and $e_3^+ = \dfrac{\partial}{\partial z}$, denote the left invariant vector fields associated to e_1, e_2 and e_3, respectively.

Example 3.5. *Dimension 4.* The product manifold $\mathbb{R} \ltimes_\rho \mathbb{R}^3$ has the structure of a Lie group given by the semidirect product of the Abelian Lie group $(\mathbb{R}^3, +)$ with $(\mathbb{R}, +)$ via the Lie group homomorphism

$$\begin{aligned} \rho : \mathbb{R} &\longrightarrow \mathrm{GL}(\mathbb{R}^3) \\ t &\longmapsto \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Next we introduce a family of flat left invariant affine connections on $\mathbb{R} \ltimes_\rho \mathbb{R}^3$:

$$\nabla_{e_2^+} = \nabla_{e_3^+} = \nabla_{e_4^+} = 0, \quad \nabla_{e_1^+} e_1^+ = \alpha e_1^+,$$
$$\nabla_{e_2^+} e_2^+ = e_2^+, \quad \nabla_{e_1^+} e_3^+ = -e_3^+, \quad \nabla_{e_1^+} e_4^+ = 0,$$

for all $\alpha \in \mathbb{R}$. The vector fields

$$e_1^+ = \frac{\partial}{\partial t}, \quad e_2^+ = e^t \frac{\partial}{\partial x}, \quad e_3^+ = e^{-t} \frac{\partial}{\partial y}, \quad e_4^+ = \frac{\partial}{\partial z},$$

determine a basis for $\mathfrak{X}_l(\mathbb{R} \ltimes_\rho \mathbb{R}^3)$.

Recall that a Lie group G is called **semisimple** if its Lie algebra decomposes into a direct sum of simple Lie algebras. An interesting result, due to C. Chevalley and S. Eilenberg (see [6]) states that a Lie algebra $\mathfrak{g}$ is semisimple if and only if we have $H^1(\mathfrak{g}, \theta) = 0$ for every real representation θ of $\mathfrak{g}$ over a finite dimensional vector space. Accordingly, we have the following beautiful result of A. Bon-Yau Chu in [4].

Theorem 3.6 (Bon-Yau Chu)**.** *Let G be a real semisimple Lie group. Then G does not admit a left invariant flat affine connection.*

Proof. Let G be a semisimple real Lie group of dimension n and $\mathfrak{g}$ its Lie algebra. Since $\mathfrak{g}$ is semisimple, its derived ideal satisfies $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. This implies that every linear representation θ of $\mathfrak{g}$ on a finite dimensional vector space has trace $\mathrm{tr}(\theta(x)) = 0$ for all $x \in \mathfrak{g}$. Suppose that there exists a left invariant flat affine connection ∇ on G. Then, by Theorem 3.2, the map $L : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto L_x$, where $L_x : \mathfrak{g} \to \mathfrak{g}$ is the linear map given by $y \mapsto L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$, is a linear representation of $\mathfrak{g}$ on the vector space $\mathfrak{g}$. We denote by $C^p(\mathfrak{g}, L)$ and $H^p(\mathfrak{g}, L)$ the spaces of p-cochains and the p-th cohomology group of $\mathfrak{g}$ associated to the linear representation L, respectively. We define $\gamma \in C^1(\mathfrak{g}, L)$ by $\gamma(x) = x$ for all $x \in \mathfrak{g}$. Then, since ∇ is torsion free and left invariant, we have

$$\mathrm{d}\gamma(x, y) = L_x(\gamma(y)) - L_y(\gamma(x)) - \gamma([x, y]) = x \cdot y - y \cdot x - [x, y] = 0,$$

for all $x, y \in \mathfrak{g}$. Therefore, we have $\mathrm{d}\gamma = 0$. Since $\mathfrak{g}$ is semisimple, we obtain $H^1(\mathfrak{g}, L) = 0$. Hence, there exists $z \in C^0(\mathfrak{g}, L) = \mathfrak{g}$ such that $x = \gamma(x) = \mathrm{d}z(x) = L_x(z)$ for all $x \in \mathfrak{g}$. Once again, since the torsion tensor of ∇ is null we reach

$$x = L_x(z) = x \cdot z = z \cdot x - [z, x] = (L_z - \mathrm{ad}_z)(x),$$

which implies the relation $L_z = I + \mathrm{ad}_z$, where I and ad are the identity map and the adjoint representation of $\mathfrak{g}$, respectively. Since L and ad are linear representations of $\mathfrak{g}$, we obtain $0 = \mathrm{tr}(L_z) = \mathrm{tr}(I) + \mathrm{tr}(\mathrm{ad}_z) = \dim(\mathfrak{g}) = n$, which is a contradiction. $\square$

Example 3.7. The special linear group $\mathrm{SL}(n, \mathbb{R})$, the special orthogonal group $\mathrm{SO}(n, \mathbb{R})$ and the symplectic linear group $\mathrm{Sp}(n, \mathbb{R})$ do not allow a structure of flat affine Lie group, given that they are semisimple.

4 Flat pseudo-Riemannian manifolds

Our next objective is to study left invariant flat affine structures over Lie groups in the case when these structures are compatible with a pseudo-Riemannian metric. To do so, we introduce the following structures from Riemannian geometry. Let M be a smooth connected paracompact manifold of real dimension n. For each $p \in M$, we denote by $L^2(T_pM, \mathbb{R})$ the set of all bilinear maps $\beta : T_pM \times T_pM \to \mathbb{R}$. Recall that the **index** ν of a symmetric bilinear form β on a real finite-dimensional vector space V is the largest integer that is the dimension of a subspace $W \subset V$ on which $\beta|_W$ is negative definite. Equivalently, if β is also non-degenerate, the index ν of V is the number of -1 in the diagonal of the matrix representation of β with respect to any orthonormal basis of V.

A **pseudo-metric** g on M is an assignment $p \mapsto g_p \in L^2(T_pM, \mathbb{R})$ such that the following conditions are met:

1. $g_p(X_p, Y_p) = g_p(Y_p, X_p)$ for all $X_p, Y_p \in T_pM$,
2. g_p is non-degenerate for all $p \in M$,
3. if $(U, (x^1, \cdots, x^n))$ is a chart of M, the coefficients g_{ij} of the local representation
$$g_p = \sum_{i,j=1}^{n} g_{ij}(p) \cdot dx^i|_p \otimes dx^j|_p,$$
are smooth functions,
4. the index of g_p is the same for all $p \in M$.

In other words, a pseudo-metric is a field of tensors of type $(0, 2)$ that is symmetric, non-degenerate and of constant index. The pair (M, g), where g is a pseudo-metric on M, is called a **pseudo-Riemannian manifold**.

The common index ν of g_p in a pseudo-Riemannian manifold (M, g) is the **index** of M. When $\nu = 0$ we say that (M, g) is a **Riemannian manifold**. In such case g_p determines an inner product over T_pM for all $p \in M$. On the other hand, when $\nu = 1$ and $n \geq 2$ the pair (M, g) is called a **Lorentzian manifold**. In the first case, the signature of g is $(0, n)$ while in the second case $(1, n)$. A bilinear form over a finite dimensional real vector space that satisfies the first two conditions of our definition is called a **scalar product**. An inner product is a scalar product that is positive definite.

A linear connection ∇ on a pseudo-Riemannian manifold (M, g) is said to be **compatible with the pseudo-metric structure** of M if it satisfies $\nabla g = 0$, that is, if

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{4.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. The following result is usually called the **Fundamental theorem of pseudo-Riemannian Geometry**.

Theorem 4.1 (Levi-Civita). *Given (M, g) a pseudo-Riemannian manifold, there exists a unique linear connection ∇ on M that is compatible with the pseudo-metric structure of M and has vanishing torsion tensor. Such a linear connection is characterized by the* **Koszul formula**

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y) + \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

The linear connection of Theorem 4.1 is called the **Levi-Civita connection**. It is important to observe that the Koszul formula implies that the Christoffel symbols associated to the Levi-Civita connection satisfy the relation

$$\sum_{l=1}^{n} g_{lk} \Gamma^l_{ji} = \frac{1}{2} \left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^k} \right), \tag{4.2}$$

for all $i, j, k = 1, \cdots, n$. When the curvature tensor of the Levi-Civita connection ∇ associated to a pseudo-Riemannian manifold (M, g) vanishes, the pseudo-metric g is called **flat**, and the pair (M, g) is a **flat pseudo-Riemannian manifold**.

The basic model of flat pseudo-Riemannian manifolds is the space $(\mathbb{R}^n_\nu, g^\nu_0, \nabla^0)$ where $\mathbb{R}^n_\nu$ equals $\mathbb{R}^n$ with pseudo-metric g^ν_0 of index ν with $0 \leq \nu \leq n$, defined by

$$g^\nu_0 = -\sum_{j=1}^{\nu} dx^j \otimes dx^j + \sum_{j=\nu+1}^{n} dx^j \otimes dx^j.$$

A simple computation shows that the usual linear connection ∇^0 of $\mathbb{R}^n$ is the Levi-Civita connection associated to g^ν_0. When $\nu = 0$, the pseudo-Riemannian manifold $\mathbb{R}^n_\nu$ reduces to $\mathbb{R}^n$. On the other hand, for $\nu = 1$ and $n \geq 2$, the manifold $\mathbb{R}^n_1$ is known as the **n-dimensional Minkowski space**. The Lorentzian manifold $(\mathbb{R}^4_1, g^1_0)$ is the basic model for relativistic space-time.

An **isometry** between two pseudo-Riemannian manifolds (M_1, g_1) and (M_2, g_2) is a diffeomorphism $f : M_1 \to M_2$ satisfying $f^*g_2 = g_1$, that is,

$$(g_2)_{F(p)}(F_{*,p}(X_p), F_{*,p}(Y_p)) = (g_1)_p(X_p, Y_p),$$

for all $X_p, Y_p \in T_pM_1$ with $p \in M_1$.

Remark 4.2. If (M, g) is a pseudo-Riemannian manifold and $f : M \to M$ is an isometry, the uniqueness of the Levi-Civita connection ∇ associated to g implies that f is an affine transformation of (M, ∇). More precisely, we have

$$f_*^{-1}(\nabla_{f_*X} f_*Y) = \nabla_X Y, \tag{4.3}$$

for all $X, Y \in \mathfrak{X}(M)$ (see [11, p. 161]).

If $O(n, \mathbb{R})$ denotes the linear orthogonal group, the **group of isometries** of $(\mathbb{R}^n, g_0)$ is the Lie group $\mathrm{OAff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes_{Id} O(n, \mathbb{R})$ determined by the semi-direct product of the Abelian Lie group $(\mathbb{R}^n, +)$ and the orthogonal group $O(n, \mathbb{R})$ via the identity representation. An important consequence of Theorem 2.5 to the case of Riemannian manifolds, which can be proven in a similar way, is the following result (see for instance [20]).

Proposition 4.3. *A real smooth manifold M of dimension n admits a flat Riemannian metric if and only if there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in J}$ of M for which the changes of coordinates are restrictions of the elements of the group of isometries of $(\mathbb{R}^n, g_0)$; that is, for each $\alpha, \beta \in J$ with $U_\alpha \cap U_\beta \neq \emptyset$, there exists $\sigma_{\alpha\beta} \in OAff(\mathbb{R}^n)$ such that*

$$\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha\cap U_\beta)} = \sigma_{\alpha\beta}\big|_{\varphi_\alpha(U_\alpha\cap U_\beta)} .$$

□

For the next examples, we denote by G_j the discrete subgroup of $\mathrm{OAff}(\mathbb{R}^2)$ which acts freely and properly discontinuously over $\mathbb{R}^2$, for $j = 1, 2, 3, 4$. Recall that in such a case the quotient manifold $\mathbb{R}^2/G_j$ admits an affine structure whose changes of coordinates are restrictions of elements of G_j (see [22, p. 349]). There exists four types of flat complete 2-dimensional Riemannian manifolds other than $(\mathbb{R}^2, g_0)$ they are given in the following example. See for instance [11, p. 209-224] for further details.

Example 4.4. *Ordinary cylinder.* Let G_1 be the set of transformations of $\mathbb{R}^2$ defined by

$$C_n(x, y) = (x + n, y), \quad \text{for all} \quad n \in \mathbb{Z}.$$

The quotient manifold $\mathbb{R}^2/G_1$ determined by the action of G_1 over $\mathbb{R}^2$ is diffeomorphic to the ordinary cylinder $S^1 \times \mathbb{R}$.

Example 4.5. *Ordinary torus.* Consider the set G_2 of transformations of $\mathbb{R}^2$ given by

$$T^{a,b}_{n,m}(x, y) = (x + ma + n, y + mb),$$

for all $n, m \in \mathbb{Z}$ and $a, b \in \mathbb{R}$, $b \neq 0$. The quotient manifold $\mathbb{R}^2/G_2$ determined by the action of G_2 over $\mathbb{R}^2$ is diffeomorphic to the ordinary torus.

Example 4.6. *Infinite Möbius band.* We denote by G_3 the set of transformations of $\mathbb{R}^2$ defined by

$$M_n(x,y) = (x+n, (-1)^n y),$$

for all $n \in \mathbb{Z}$. The quotient manifold $\mathbb{R}^2/G_3$ determined by the action of G_3 over $\mathbb{R}^2$ is diffeomorphic to the infinite Möbius band.

Example 4.7. *Klein bottle.* Let G_4 be the set of transformations of $\mathbb{R}^2$ given by

$$K^b_{n,m}(x,y) = (x+n, (-1)^n y + bm),$$

for all $n, m \in \mathbb{Z}$ and $b \in \mathbb{R}\backslash\{0\}$. The quotient manifold $\mathbb{R}^2/G_4$ determined by the action of G_4 over $\mathbb{R}^2$ is diffeomorphic to the Klein bottle.

The existence of partitions of unity for M helps us guaranty the existence of Riemannian metrics on M. Nevertheless, partitions of unity do not allow us to prove the existence of pseudo-metrics on M with index at least 1. In fact, there are topological obstructions to the existence of such pseudo-metrics. For example, a compact manifold M admits a Lorentzian metric if and only if its Euler characteristic $\chi(M)$ is equal to zero. This because in such cases we can guaranty the existence of a nowhere vanishing vector field on M (see [14] or [25, p. 207]). The only compact two dimensional surfaces satisfying this condition are the torus and the Klein bottle.

5 Flat pseudo-Riemannian Lie groups

Let G be a real connected Lie group of dimension n and $\mathfrak{g}$ its Lie algebra. The goal of this section is to discuss the open problem proposed by J. Milnor in [20] of describing left invariant flat affine structures in the case when G admits left invariant flat pseudo-metrics.

A pseudo-metric μ on G is called **left invariant** if $L^*_\sigma \mu = \mu$ for all $\sigma \in G$. In other words, μ is left invariant if L_σ is an isometry of (G,μ) for all $\sigma \in G$. The pair (G,μ), where μ is a left invariant pseudo-metric on G, is called a **pseudo-Riemannian Lie group**.

There is a faithful correspondence between left invariant pseudo-metrics on G and scalar products on $\mathfrak{g}$, depicted as follows. If μ is a left invariant pseudo-metric on G, then $\mu_\epsilon : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defines a scalar product on $\mathfrak{g}$. On the other hand, given a scalar product $\mu_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ for $\mathfrak{g}$, as a consequence of the chain rule, we can define a left invariant pseudo-metric μ on G by the formula

$$\mu_\sigma(X_\sigma, Y_\sigma) = \mu_0((L_{\sigma^{-1}})_{*,\sigma}(X_\sigma), (L_{\sigma^{-1}})_{*,\sigma}(Y_\sigma)), \tag{5.1}$$

for all $X_\sigma, Y_\sigma \in T_\sigma G$ with $\sigma \in G$. If (G, μ) is a pseudo-Riemannian Lie group, identity (4.3) implies that the Levi-Civita connection ∇ associated to μ is a left invariant linear connection. On the other hand, since μ is a left invariant pseudo-metric we get

$$\mu_\sigma(x^+_\sigma, y^+_\sigma) = \mu_\sigma((L_\sigma)_{*,\epsilon}(x), (L_\sigma)_{*,\epsilon}(y)) = \mu_\epsilon(x, y),$$

for all $x, y \in \mathfrak{g}$. This implies that the map $\mu(x^+, y^+) : G \to \mathbb{R}$ defined by $\sigma \mapsto \mu_\sigma(x^+_\sigma, y^+_\sigma)$, is constant for all $x^+, y^+ \in \mathfrak{X}_l(G)$. Therefore, putting $x \cdot y = L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ for $x, y \in \mathfrak{g}$, we have

$$[x, y] = x \cdot y - y \cdot x \quad \text{and} \tag{5.2}$$

$$\mu_\epsilon(L_x(y), z) + \mu_\epsilon(y, L_x(z)) = 0, \tag{5.3}$$

for $x, y, z \in \mathfrak{g}$. The bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by $x \cdot y = L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ is called the **Levi-Civita product**. The Koszul formula implies that the Levi-Civita product is characterized by the expression

$$\mu_\epsilon(L_x(y), z) = \frac{1}{2}(\mu_\epsilon([x, y], z) - \mu_\epsilon([y, z], x) + \mu_\epsilon([z, x], y)), \tag{5.4}$$

where $x, y, z \in \mathfrak{g}$.

A pseudo-Riemannian Lie group (G, μ) is called **flat** if the curvature tensor of the Levi-Civita connection associated to μ is identically zero.

Let (V, μ_0) be a real finite-dimensional vector space with a scalar product μ_0. The group of **orthogonal transformations** of (V, μ_0), denoted by $O(V, \mu_0)$, is defined as the set of transformations $T : V \to V$

which satisfy $\mu_0(T(x), T(y)) = \mu_0(x, y)$ for all $x, y \in V$. It is a Lie group whose Lie algebra is the set $\mathfrak{o}(V, \mu_0)$ of endomorphisms $t : V \to V$ verifying the identity $\mu_0(t(x), y) + \mu_0(x, t(y)) = 0$ for all $x, y \in V$. The **group of isometries** of (V, μ_0), denoted by $\mathrm{OAff}(V)$, is defined as the semi-direct product $V \rtimes_{Id} O(V, \mu_0)$ of the Abelian Lie group $(V, +)$ and $O(V, \mu_0)$ via the identity representation.

Remark 5.1. If (G, μ) is a flat pseudo-Riemannian Lie group and ∇ is the Levi-Civita connection associated to μ, the map $L \colon \mathfrak{g} \to \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ defined by $x \mapsto L_x$, where $L_x \colon \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$, is a well defined Lie algebra homomorphism.

We can now have a first characterization of flat pseudo-Riemannian Lie groups as given by A. Aubert and A. Medina in [1].

Proposition 5.2 (Aubert-Medina)**.** *Let G be a real connected Lie group of dimension n, $\mathfrak{g}$ its Lie algebra, and $\widetilde{G}$ its universal covering Lie group. Then, the following are equivalent.*

1. *There exists a left invariant flat pseudo-metric on G.*
2. *There exist a scalar product $\mu_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and a bilinear map $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ over $\mathfrak{g}$ such that* (5.2) *and* (5.3) *are satisfied together with $L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}$, for all $x, y \in \mathfrak{g}$.*
3. *There exist a real n-dimensional vector space (V, μ_0) together with a scalar product μ_0 and a Lie group homomorphism $\rho : \widetilde{G} \to OAff(V, \mu_0)$ such that the left action of $\widetilde{G}$ over V defined by $\sigma \cdot v = \rho(\sigma)(v)$ for all $(\sigma, v) \in \widetilde{G} \times V$ admits a point with open orbit and discrete isotropy.*

Proof. We first prove that 1 implies 2. If (G, μ) is a flat pseudo - Riemannian Lie group and ∇ is the Levi-Civita connection associated to μ, from our previous observations we know that μ_ϵ and the Levi-Civita product associated to ∇ satisfy the required identities.

To see 2 implies 3 recall the proof of Theorem 3.2. By hypothesis, the map $\theta\colon \mathfrak{g} \to \mathfrak{g} \rtimes_{id} \mathfrak{o}(\mathfrak{g}, \mu_0)$ defined by $x \mapsto (x, L_x)$ for all $x \in \mathfrak{g}$ is a well defined Lie algebra homomorphism. Therefore, using the exponential map of G, we obtain a homomorphism of Lie groups $\rho\colon \widetilde{G} \to \mathrm{OAff}(\mathfrak{g}, \mu_0)$ for which $0 \in \mathfrak{g}$ is a point with open orbit and discrete isotropy for the left action of $\widetilde{G}$ over $\mathfrak{g}$ defined by $\sigma \cdot x = \rho(\sigma)(x)$ for all $(\sigma, x) \in \widetilde{G} \times \mathfrak{g}$.

Finally 3 implies 1. Let $\rho\colon \widetilde{G} \to \mathrm{OAff}(V, \mu_0)$ be a homomorphism of Lie groups, defined by $\sigma \mapsto (Q(\sigma), F_\sigma)$ for all $\sigma \in \widetilde{G}$, where (V, μ_0) is a real vector space of dimension n together with a scalar product μ_0 such that the orbital map $\pi\colon \widetilde{G} \to \mathrm{Orb}(v)$ defined by $\sigma \mapsto Q(\sigma) + F_\sigma(v)$ is a local diffeomorphism for some $v \in V$. Differentiating on the identity of $\widetilde{G}$, we obtain a Lie algebra homomorphism $\theta\colon \mathfrak{g} \to \mathfrak{g} \rtimes_{id} \mathfrak{o}(V, \mu_0)$ given by $x \mapsto (q(x), f_x)$, where the linear map $\psi_v\colon \mathfrak{g} \to V$ defined by $x \mapsto q(x) + f_x(v)$ is an isomorphism. Now, define on $\mathfrak{g}$ the scalar product $\widetilde{\mu}_0$ and the bilinear map $\cdot$ respectively by

$$\widetilde{\mu_0}(x, y) = \mu_0(\psi_v(x), \psi_v(y))$$

and

$$L_x = \psi_v^{-1} \circ f_x \circ \psi_v, \quad y \mapsto x \cdot y = L_x(y),$$

for all $x, y \in \mathfrak{g}$. If μ denotes the left invariant pseudo-metric on G induced by $\widetilde{\mu_0}$ through Formula (5.1), it is easy to check that $\cdot$ is the Levi-Civita product associated to the Levi-Civita connection determined by μ, given that we have $f_x \in \mathfrak{o}(V, \mu_0)$ for all $x \in \mathfrak{g}$. Therefore, as in the proof of Theorem 3.2 we conclude that μ is a left invariant flat pseudo-metric on G. $\square$

The following result allows us to determine when the Levi-Civita connection associated to a left invariant flat pseudo-metric is geodesically complete. Recall that a linear connection ∇ over a smooth manifold M is **geodesically complete** if for any initial condition $(p, X_p) \in TM$ its geodesics are defined for all $t \in \mathbb{R}$. If (G, ∇) is a flat affine Lie group, J. Helmstetter showed in [8] that ∇ is geodesically complete if and only if $\mathrm{tr}(R_x) = 0$ for all $x \in \mathfrak{g}$, here $R_x\colon \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by

$R_x(y) = (\nabla_{y^+} x^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$. On the other hand, a Lie group G is called **unimodular** if its left invariant Haar measure is also right invariant. J. Milnor showed in [19] that a Lie group G is unimodular if and only if $\det(\mathrm{Ad}_\sigma) = \pm 1$ for all $\sigma \in G$. If G is connected, this is equivalent to requiring $\mathrm{tr}(\mathrm{ad}_x) = 0$ for all $x \in \mathfrak{g}$ (compare [19]). For the next result see [1].

Theorem 5.3 (Aubert-Medina). *Let (G, μ) be a connected flat pseudo-Riemannian Lie group. Then the Levi-Civita connection associated to μ is geodesically complete if and only if G is unimodular.*

Proof. Let ∇ be the Levi-Civita connection associated to the left invariant flat pseudo-metric μ. We denote by $L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ the Levi-Civita product on $\mathfrak{g}$ associated to ∇. If μ_ϵ is the scalar product on $\mathfrak{g}$ induced by μ, we have

$$\mu_\epsilon(L_x(y), z) + \mu_\epsilon(y, L_x(z)) = 0,$$

for all $x, y, z \in \mathfrak{g}$. This implies that L_x is antisymmetric with respect to μ_ϵ. Therefore, if $L_x^* \colon \mathfrak{g} \to \mathfrak{g}$ denotes the adjoint operator of L_x with respect to μ_ϵ, we have $L_x^* = -L_x$ for all $x \in \mathfrak{g}$. On the other hand, if $\mathfrak{g}^*$ denotes the dual space associated to $\mathfrak{g}$ and ${}^tL_x \colon \mathfrak{g}^* \to \mathfrak{g}^*$ is the transposed of the linear map L_x, the linear isomorphism $\varphi \colon \mathfrak{g} \to \mathfrak{g}^*$ defined by $\varphi(x) = \mu_\epsilon(x, \cdot)$ where $\varphi(x)(y) = \mu_\epsilon(x, y)$ for all $x, y \in \mathfrak{g}$, fits into the following commutative diagram for all $x \in \mathfrak{g}$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "L_x^*"] \arrow[d, "\varphi"'] & \mathfrak{g} \arrow[d, "\varphi"] \\
\mathfrak{g}^* \arrow[r, "{}^tL_x"'] & \mathfrak{g}^*;
\end{tikzcd}$$

thus, we have $L_x^* = \varphi^{-1} \circ^t L_x \circ \varphi$ and so

$$-\mathrm{tr}(L_x) = \mathrm{tr}(L_x^*) = \mathrm{tr}(\varphi^{-1} \circ^t L_x \circ \varphi) = \mathrm{tr}({}^tL_x) = \mathrm{tr}(L_x),$$

for all $x \in \mathfrak{g}$. This implies $\mathrm{tr}(L_x) = 0$ for $x \in \mathfrak{g}$.

Now suppose that ∇ is geodesically complete. Since this is a left invariant flat affine connection, we have $\mathrm{tr}(R_x) = 0$, where $R_x : \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by $R_x(y) = y \cdot x$ for all $x, y \in \mathfrak{g}$. On the other hand, identity (5.2) implies $\mathrm{ad}_x = L_x - R_x$, and therefore we get

$$\mathrm{tr}(\mathrm{ad}_x) = \mathrm{tr}(L_x - R_x) = \mathrm{tr}(L_x) - \mathrm{tr}(R_x) = 0,$$

for all $x \in \mathfrak{g}$, which shows that G is a unimodular Lie group.

Reciprocally, if G is a unimodular Lie group, we have $\mathrm{tr}(\mathrm{ad}_x) = 0$ for all $x \in \mathfrak{g}$. As we have $\mathrm{ad}_x = L_x - R_x$ and $\mathrm{tr}(L_x) = 0$, we obtain $\mathrm{tr}(R_x) = 0$ for all $x \in \mathfrak{g}$. From the fact that ∇ is a left invariant flat affine connection and $\mathrm{tr}(R_x) = 0$, we get that it is geodesically complete. $\square$

Example 5.4. The group of affine transformations of the line $\mathrm{Aff}(\mathbb{R})$ has a natural left invariant flat Lorentzian metric given by $\mu = \frac{1}{x^2}(dx \otimes dy + dy \otimes dx)$. The Levi-Civita connection associated to μ is determined by the rules

$$\nabla_{e_1^+} e_1^+ = -e_1^+, \quad \nabla_{e_1^+} e_2^+ = e_2^+, \quad \nabla_{e_2^+} e_1^+ = \nabla_{e_2^+} e_2^+ = 0.$$

Since $\mathrm{Aff}(\mathbb{R})$ is not unimodular, we have that ∇ is not geodesically complete. The natural left invariant Riemannian metric on $\mathrm{Aff}(\mathbb{R})$ given by $\overline{\mu} = \frac{1}{x^2}(dx \otimes dx + dy \otimes dy)$ is also not flat. The Levi-Civita connection associated to $\overline{\mu}$ is determined by

$$\overline{\nabla}_{e_1^+} e_1^+ = \overline{\nabla}_{e_1^+} e_2^+ = 0, \quad \overline{\nabla}_{e_2^+} e_1^+ = -e_2^+, \quad \overline{\nabla}_{e_2^+} e_2^+ = e_1^+.$$

It is easy to verify that the curvature tensor of $\overline{\nabla}$ is not identically zero. As a consequence of Theorem 7.9 (of next section) it is possible to show that there does not exist left invariant flat Riemannian metrics on $\mathrm{Aff}(\mathbb{R})$.

Example 5.5. Over the Heisenberg group H_3, we can define a left invariant flat Lorentzian metric by

$$\mu = dx \otimes dz + dy \otimes dy + dz \otimes dx - x(dx \otimes dy + dy \otimes dx).$$

The Levi-Civita connection associated to μ is determined by

$$\nabla_{e_1^+} = \nabla_{e_3^+} = 0, \quad \nabla_{e_2^+} e_1^+ = -e_3, \quad \nabla_{e_2^+} e_2^+ = e_1^+, \quad \nabla_{e_2^+} e_3^+ = 0.$$

Since H_3 is unimodular, we have that ∇ is geodesically complete. If H_{2n+1} denotes the Heisenberg group of dimension $2n+1$ for $n \in \mathbb{N}$, then H_{2n+1} is a flat pseudo-Riemannian Lie group if and only if $n = 1$ (see [1]).

6 The classical pseudo-Riemannian cotangent Lie groups of connected flat affine Lie groups

A simple construction that allows us to obtain flat pseudo-Riemannian Lie groups starting out with connected flat affine Lie groups is the following (see [1]).

Let (G, ∇) be a connected affine flat Lie group of dimension n, $\mathfrak{g}$ its Lie algebra, and $\widetilde{G}$ its universal covering Lie group. Since ∇ is a left invariant flat affine connection, the map $L\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $x \mapsto L_x$, where $L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$, is a Lie algebra homomorphism. The dual representation associated to L is the Lie algebra homomorphism $L^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$, defined by $x \mapsto L_x^* = -{}^tL_x$, where $L_x^*(\alpha) = -\alpha \circ L_x$ for all $\alpha \in \mathfrak{g}^*$. Using the exponential map of G, we obtain a Lie group homomorphism $\Phi\colon \widetilde{G} \to \mathrm{GL}(\mathfrak{g}^*)$ via $\exp_G(x) \mapsto \Phi(\exp_G(x)) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!}(L_x^*)^k$, namely

$$\Phi_{*,\widetilde{\epsilon}}(x) = \left.\frac{d}{dt}\right|_{t=0} (\Phi(\exp(tx))) = L_x^*,$$

for all $x \in \mathrm{Lie}(\widetilde{G}) = \mathfrak{g}$. Therefore, the product manifold $T^*\widetilde{G} = \widetilde{G} \times \mathfrak{g}^*$ is endowed with the structure of a Lie group given by the semidirect product of $\widetilde{G}$ with the Abelian Lie group $(\mathfrak{g}^*, +)$ through Φ; more precisely we have

$$(\sigma, \alpha) \cdot (\tau, \beta) = (\sigma\tau, \Phi(\sigma)(\beta) + \alpha),$$

for all $\sigma, \tau \in G$ and $\alpha, \beta \in \mathfrak{g}^*$. The Lie group $T^*\widetilde{G} = \widetilde{G} \ltimes_\Phi \mathfrak{g}^*$ is called the **classical pseudo-Riemannian cotangent Lie group** associated to the flat affine connected Lie group (G, ∇).

Here the term 'classical' stands in contrast to the more general construction of twisted cotangent Lie groups as used by A. Aubert and A. Medina (see [1]). The Lie group $T^*\widetilde{G}$ is then characterized by the following result.

Proposition 6.1 (Aubert-Medina)**.** *The Lie algebra of $T^*\widetilde{G} = \widetilde{G} \ltimes_\Phi \mathfrak{g}^*$ is the product vector space $\mathfrak{g} \ltimes_L \mathfrak{g}^*$ with Lie bracket*

$$[(x, \alpha), (y, \beta)] = ([x, y], L_x^*(\beta) - L_y^*(\alpha)), \tag{6.1}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$. Moreover,

$$\widetilde{\omega}((x, \alpha), (y, \alpha)) = \alpha(y) + \beta(x), \tag{6.2}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$, is a scalar product over $\mathfrak{g} \ltimes_L \mathfrak{g}^*$ with signature (n, n) which, by formula (5.1)*, defines a left invariant flat pseudo-metric $T^*\widetilde{G}$ whose Levi-Civita connection is determined by*

$$\nabla_{(x,\alpha)^+}(y, \alpha)^+ = (x \cdot y, L_x^*(\beta))^+,$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.

Proof. As the Lie group structure of $T^*\widetilde{G}$ is given by a semidirect product, it is simple to check that its Lie algebra is the vector space $\mathfrak{g} \ltimes_L \mathfrak{g}^*$ with Lie bracket given by (6.1). On the other hand, as the Levi-Civita connection is unique, the proof of the last statement is an immediate consequence of Proposition 5.2. □

A more general construction appears in the study of the twisted pseudo-Riemannian cotangent Lie group of a connected flat affine Lie group (see [1, Proposition 2.1]).

Remark 6.2. If G is simply connected flat affine Lie group, then $T^*\widetilde{G}$ is a trivial vector bundle isomorphic to the cotangent bundle T^*G of G. It is well known that there is a natural way to associate a structure of Lie group to T^*G given that it is isomorphic to the trivial bundle $G\times\mathfrak{g}^*$ through the vector bundle isomorphism

$$\begin{aligned} T^*G &\to G\times\mathfrak{g}^* \\ (\sigma,\alpha_\sigma) &\longmapsto (\sigma,\alpha_\sigma\circ(L_\sigma)_{*,\epsilon}). \end{aligned}$$

If $\mathrm{Ad}^*\colon G\to \mathrm{GL}(\mathfrak{g}^*)$ denotes the co-adjoint representation of G, then the product manifold $G\times\mathfrak{g}^*$ has a Lie group structure given by the semi-direct product of G with the Abelian Lie group $\mathfrak{g}^*$ through Ad^*; consequently we have

$$(\sigma,\alpha)\cdot(\tau,\beta)=(\sigma\tau,\mathrm{Ad}^*_\sigma(\beta)+\alpha), \tag{6.3}$$

for all $\sigma,\tau\in G$ and $\alpha,\beta\in\mathfrak{g}^*$. Therefore, T^*G has the structure of a Lie group induced by (6.3). If $\mathrm{ad}^*\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}^*)$ denotes the co-adjoint representation of $\mathfrak{g}$, the Lie algebra of T^*G is the product vector space $\mathfrak{g}\ltimes_{ad^*}\mathfrak{g}^*$ with Lie bracket

$$[(x,\alpha),(y,\beta)]=([x,y]_\mathfrak{g},\mathrm{ad}^*_x(\beta)-\mathrm{ad}^*_y(\alpha)), \tag{6.4}$$

for all $x,y\in\mathfrak{g}$ and $\alpha,\beta\in\mathfrak{g}^*$. As G is a simply connected Lie group, then it is elementary to verify that $T^*\widetilde{G}$ is locally isomorphic to the cotangent bundle T^*G as Lie groups if and only if the maps $L^*\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}^*)$ and $\mathrm{ad}^*\colon\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}^*)$ are isomorphic representations, that is, there exists a linear isomorphism $\psi:\mathfrak{g}^*\to\mathfrak{g}^*$ such that $\mathrm{ad}^*_x\circ\psi=\psi\circ L^*_x$ for all $x\in\mathfrak{g}$. In this case, the linear map $l:\mathfrak{g}\ltimes_L\mathfrak{g}^*\to\mathfrak{g}\ltimes_{\mathrm{ad}^*}\mathfrak{g}^*$ defined by $(x,\alpha)\mapsto(x,\psi(\alpha))$ is a Lie algebra isomorphism.

7 Orthogonal Lie groups

In this section we will study some elementary properties of those Lie groups which have bi-invariant pseudo-metrics. These will be called

orthogonal Lie groups (see the definition a few lines down). To study the main characteristic of orthogonal Lie groups we introduce the notion of **orthogonal Lie algebra** which we will be used in the method of **double orthogonal extension** described by A. Medina and Ph. Revoy (see [17]).

We describe how to construct the oscillator Lie algebra of the oscillator Lie group which appears in various branches of Physics and Mathematical Physics and give rise to particular solutions of the Einstein-Yang-Mills equations (see [13]). Finally, we will provide another characterization of flat Riemannian Lie groups due to J. Milnor (compare [19]).

For each $\sigma \in G$, we denote by $R_\sigma \colon G \to G$ the right multiplications by σ in G, which is defined by $R_\sigma(\tau) = \tau\sigma$ for all $\tau \in G$.

A pseudo-metric μ on G is **right invariant** if $R_\sigma^*\mu = \mu$ for all $\sigma \in G$. In other words, μ is right invariant if R_σ is an isometry of (G, μ) for all $\sigma \in G$. A pseudo-metric μ on G is called **bi-invariant** if it is left invariant and right invariant. The pair (G, μ), where μ is a bi-invariant pseudo-metric over G, is called an **orthogonal Lie group**.

If μ is a left invariant pseudo-metric on G, it is easy to show that μ is right invariant if and only if

$$\mu_\epsilon(\mathrm{Ad}_\sigma(x), \mathrm{Ad}_\sigma(y)) = \mu_\epsilon(x, y) \tag{7.1}$$

holds for all $\sigma \in G$ and $x, y \in \mathfrak{g}$. This implies in the context that μ is right invariant if and only if the adjoint representation of $\mathfrak{g}$ is antisymmetric with respect to μ_ϵ. The latter implies $\mathrm{ad}_x \in \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ for all $x \in \mathfrak{g}$, namely

$$\mu_\epsilon([x, y], z) + \mu_\epsilon(y, [x, z]) = 0, \tag{7.2}$$

for all $x, y, z \in \mathfrak{g}$.

A scalar product over $\mathfrak{g}$ which satisfies identity (7.2) is called an **invariant scalar product**. A pair $(\mathfrak{g}, \mu_0)$, where $\mathfrak{g}$ is a finite dimensional real Lie algebra and μ_0 is an invariant scalar product over $\mathfrak{g}$ is named an **orthogonal Lie algebra**.

If μ_0 is an invariant scalar product over $\mathfrak{g}$, the left invariant pseudo-metric defined by the formula (5.1) is also right invariant. On the other hand, if (G, μ) is an orthogonal Lie group, the Koszul formula reduced at the identity (5.4) and expression (7.2) imply that the Levi-Civita connection ∇ associated to μ is determined by

$$\nabla_{x^+} y^+ = \frac{1}{2}[x, y]^+, \tag{7.3}$$

for $x, y \in \mathfrak{g}$. Moreover, as a consequence of the Jacobi identity in $\mathfrak{g}$, it follows that the curvature tensor of ∇ is given by the expression

$$R_\nabla(x^+, y^+)z^+ = -\frac{1}{4}[[x, y], z]^+, \tag{7.4}$$

with $x, y, z \in \mathfrak{g}$.

Remark 7.1. A left invariant linear connection on G is called a **Cartan 0-connection** if for all $x \in \mathfrak{g}$, the 1-parameter subgroups of G and the geodesic curves of ∇ determined by the initial condition $(\epsilon, x) \in G \times \mathfrak{g}$ coincide. It is easy to see that every Cartan 0-connection is geodesically complete. Moreover, there exists a unique Cartan 0-connection on G with vanishing torsion, as it is completely determined by Equation (7.3) (see [21, p. 72]).

As an immediate consequence of Identity (7.4) we have the following result (see for instance [1]).

Proposition 7.2 (Aubert-Medina)**.** *Let (G, μ) be an orthogonal Lie group. The bi-invariant pseudo-metric μ is flat if and only if G is a 2-nilpotent Lie group.* $\square$

Example 7.3. *Semisimple Lie groups.* Let G be a semisimple Lie group. It is well known that G is a semisimple if and only if the *Killing form* of $\mathfrak{g}$, which we denote by $k\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, defined by $(x, y) \mapsto k(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$ for $x, y \in \mathfrak{g}$, is non-degenerate. Direct computation shows

$$k([x, y], z) = -k(y, [x, z]),$$

for $x, y, z \in \mathfrak{g}$. Therefore $(\mathfrak{g}, k)$ is an orthogonal Lie algebra.

Example 7.4. *The cotangent bundle of a Lie group.* Let G be a real connected n-dimensional Lie group, $\mathfrak{g}$ its Lie algebra, T^*G the cotangent bundle of G, and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. Recall that T^*G is isomorphic to the trivial bundle $G \times \mathfrak{g}^*$ and this is endowed with a natural Lie group structure given by

$$(\sigma, \alpha) \cdot (\tau, \beta) = (\sigma\tau, \mathrm{Ad}^*_\sigma(\beta) + \alpha), \tag{7.5}$$

for all $\sigma, \tau \in G$ and $\alpha, \beta \in \mathfrak{g}^*$. Therefore, the Lie algebra of T^*G is the product vector space $\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*$ with Lie bracket

$$[(x, \alpha), (y, \beta)] = ([x, y]_\mathfrak{g}, \mathrm{ad}^*_x(\beta) - \mathrm{ad}^*_y(\alpha)), \tag{7.6}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.

We define over $\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*$ the function

$$\mu_0((x, \alpha), (y, \beta)) = \alpha(y) + \beta(x), \tag{7.7}$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$. It is easy to see that μ_0 defines an invariant scalar product over $\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*$, of signature (n, n), so that $(\mathfrak{g} \ltimes_{ad^*} \mathfrak{g}^*, \mu_0)$ is an orthogonal Lie algebra.

Example 7.5. *Oscillator Lie group.* For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with $0 < \lambda_1 \leq \cdots \leq \lambda_n$, the λ-oscillator Lie group, denoted by G_λ, is determined by the product manifold $\mathbb{R}^{2n+2} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n$ endowed with the product

$$(t, s, z_1, \cdots, z_n) \cdot (t', s', z'_1, \cdots, z'_n) =$$
$$= \left(t + t', s + s' + \frac{1}{2} \sum_{j=1}^{n} \mathrm{Im}(\overline{z_j} z'_j e^{i\lambda_j t}), z_1 + z'_1 e^{i\lambda_1 t}, \cdots, z_n + z'_n e^{i\lambda_n t} \right),$$

where $t, t', s, s' \in \mathbb{R}$ y $z_j, z'_j \in \mathbb{C}$ for all $j = 1, \cdots, n$. The Lie algebra of G_λ, denoted by $\mathfrak{g}_\lambda$, is isomorphic to the vector space $\mathbb{R}e \times \mathbb{R}^{2n} \times \mathbb{R}\hat{e} = \mathrm{Vect}_\mathbb{R}\{e, e_j, \hat{e_j}, \hat{e}\}_{j=1,\cdots,n}$, with Lie bracket

$$[e, e_j] = \lambda_j \hat{e_j}, \quad [e, \hat{e_j}] = -\lambda_j e_j, \quad [e_j, \hat{e_j}] = \hat{e},$$

for all $j = 1, \cdots n$.

If $x = \alpha e + \sum_{j=1}^{n} x_j e_j + \sum_{j=1}^{n} y_j \hat{e_j} + \beta \hat{e}$ denotes an element of $\mathfrak{g}_\lambda$, the function μ_0 defined over $\mathfrak{g}_\lambda \times \mathfrak{g}_\lambda$ by

$$\mu_0(x, y) = \sum_{j=1}^{n} \frac{1}{\lambda_j}(x_j x'_j + y_j y'_j) + \alpha\beta' + \alpha'\beta$$

is an invariant scalar product over $\mathfrak{g}_\lambda$. This allow us to conclude that G_λ is an orthogonal Lie group. The signature of the scalar product μ_0 is $(1, 2n+1)$ so that it determines, by means of Formula (5.1), a bi-invariant Lorentzian metric over G_λ.

Remark 7.6. The λ-oscillators Lie groups are the only solvable simply connected non-Abelian Lie groups that admit a bi-invariant Lorentzian metric (see [16]). The oscillator 4-dimensional Lie group has its origin in the study of the harmonic oscillator which is one of the simplest non-relativistic systems where the Schrödinger equation can be completely solved. Moreover, oscillator Lie groups are particular solutions to the Einstein-Yang-Mills equations (see [13]). Over oscillator Lie groups there exist infinitely many solutions to the Yang-Baxter equations (see [3]).

Example 7.7. *A non-orthogonal Lie group: Aff*($\mathbb{R}$). The Lie group of affine transformations of the line Aff($\mathbb{R}$) is a classical example of a non-orthogonal Lie Group. If there were an invariant scalar product μ_0 over $\mathfrak{aff}(\mathbb{R})$, we will get

$$\mu_0([x, y], z) + \mu_0(y, [x, z]) = 0,$$

for all $x, y, z \in \mathfrak{g}$. If we replace here $x = e_1$, $y = e_2$ and $z = e_1$, we obtain $\mu_0(e_1, e_2) = 0$. On the other hand, if we replace $x = e_1$, $y = e_2$ and $z = e_2$ we get $\mu_0(e_2, e_2) = 0$. Therefore, the element e_2 is orthogonal with respect to μ_0 to all elements of $\mathfrak{aff}(\mathbb{R})$, which contradicts the fact that μ_0 is non-degenerate. The bottom line is that there does not exist an invariant scalar product over $\mathfrak{aff}(\mathbb{R})$.

If G is a compact Lie group, using the Haar measure on G we can construct a bi-invariant Riemannian metric over G (see [21, p. 340]). In further generality, a connected Lie group G admits a bi-invariant metric if and only if it is isomorphic to the Cartesian product of a compact group and an additive vector group (see [19]). On the other hand, if $(\mathfrak{g}, \mu_0)$ is an orthogonal Lie algebra with an invariant inner product μ_0 and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ for each y in the orthogonal complement $\mathfrak{h}^{\perp_{\mu_0}}$ of $\mathfrak{h}$ with respect to μ_0, we have

$$\mu_0([x, y], h) = -\mu_0(y, [x, h]) = 0,$$

for $x \in \mathfrak{g}$ and $h \in \mathfrak{h}$. This implies that $\mathfrak{h}^{\perp_{\mu_0}}$ is also an ideal of $\mathfrak{g}$. Therefore, by induction, have shown that $\mathfrak{g}$ can be expressed as an orthogonal direct sum of simple ideals (see [19]).

Remark 7.8. K. Iwasawa showed in [9] that if G is a connected Lie group, then every compact subgroup is contained in a maximal compact subgroup H, which is also connected. Moreover, topologically G is isomorphic to the Cartesian product of H with an Euclidean space $\mathbb{R}^k$. For (G, μ) a flat Riemannian Lie group, if we ignore for a moment the group structure of G and think of it just as a Riemannian manifold, we have that G is isometric to Euclidean space. Therefore, as a consequence of Iwasawa's theorem, every compact subgroup of (G, μ) is commutative (see [19]).

The following characterization of flat Riemannian Lie groups is due to J. Milnor (see [19]).

Theorem 7.9 (Milnor)**.** *Let (G, μ) be a Riemannian Lie group. The metric μ is flat if and only if the Lie algebra $\mathfrak{g}$ decomposes as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is an Abelian subalgebra and $\mathfrak{u}$ is an Abelian ideal such that the linear maps ad_b are antisymmetric with respect to μ_ϵ for all $b \in \mathfrak{b}$.*

Proof. Suppose that (G, μ) is a flat Riemannian Lie group. If ∇ is the Levi-Civita connection associated to μ, then by Proposition 5.2, we know that the linear map $L\colon \mathfrak{g} \to \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ defined by $x \mapsto L_x$, where $L_x(y) = x \cdot y = (\nabla_{x^+} y^+)(\epsilon)$ for all $x, y \in \mathfrak{g}$, is a well defined Lie algebra homomorphism. We denote by $\mathfrak{u}$ the kernel of L. Clearly $\mathfrak{u}$ is an ideal of $\mathfrak{g}$. Since the torsion tensor of ∇ vanishes, we have $[x, y] = L_x(y) - L_y(x)$ for all $x, y \in \mathfrak{g}$. In particular, for $u, v \in \mathfrak{u}$ we have $[u, v] = 0$ and it follows that $\mathfrak{u}$ is an Abelian ideal. Let $\mathfrak{b}$ be the orthogonal complement of $\mathfrak{u}$ with respect to μ_ϵ. For each $b \in \mathfrak{b}$ we have the identity

$$\mathrm{ad}_b(u) = [b, u] = L_b(u) - L_u(b) = L_b(u),$$

for $u \in \mathfrak{u}$. Given that $\mathfrak{u}$ is an ideal of $\mathfrak{g}$, the linear map L_b takes $\mathfrak{u}$ onto itself. Therefore, L_b takes the orthogonal complement $\mathfrak{b}$ to itself, and since this is true for all $b \in \mathfrak{b}$, we conclude that $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$. On the other hand, since L is a Lie algebra homomorphism and $\mathfrak{u} = \mathrm{Ker}(L)$, we have that $\mathfrak{b}$ is sent isomorphically to a Lie subalgebra $L(\mathfrak{b})$ of $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$. For simplicity, we denote $L(\mathfrak{b})$ also by $\mathfrak{b}$. Given that $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ is the Lie algebra of the compact Lie group $O(\mathfrak{g}, \mu_\epsilon)$, which admits a bi-invariant Riemannian metric, we deduce the existence of an invariant inner product μ_0 over $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$. Since $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$, it is easy to verify that μ_0 restricts naturally to an invariant inner product over $\mathfrak{b}$. Therefore, $\mathfrak{b}$ can be written as an orthogonal direct sum $\mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_k$ of simple ideals. If any of these simple ideal, say $\mathfrak{b}_j$, were non Abelian, then the corresponding simple Lie group G_j must be compact (see [19, Thm 2.2]) and the inclusion $\mathfrak{b}_j \subset \mathfrak{b} \subset \mathfrak{g}$ would imply the existence of a nontrivial Lie group homomorphism $G_j \to G$. Hence, G must contain a non-trivial compact subgroup, which is a contradiction. Therefore, each $\mathfrak{b}_j$ must be Abelian and accordingly $\mathfrak{b}$ is an Abelian Lie subalgebra. Finally, since for each $b \in \mathfrak{b}$ the restriction of ad_b to $\mathfrak{b}$ is the trivial map, whereas we have $\mathrm{ad}_b = L_b$ when restricting ad_b to $\mathfrak{u}$, we obtain $\mathrm{ad}_b \in \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ for all $b \in \mathfrak{b}$.

Reciprocally, suppose that the Lie algebra $\mathfrak{g}$ decomposes as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is an Abelian subalgebra and $\mathfrak{u}$ is an Abelian ideal such that $\mathrm{ad}_b \in \mathfrak{o}(\mathfrak{g}, \mu_\epsilon)$ for all $b \in \mathfrak{b}$. As μ_ϵ is non-degenerate, the Koszul formula reduced to the identity (5.4) and both formulas (5.2) and (5.2) imply that the Levi-Civita product associated to μ_ϵ satisfies the identities

$$L_u = 0, \qquad L_b = \mathrm{ad}_b,$$

for all $u \in \mathfrak{u}$ and $b \in \mathfrak{b}$. It is easy to verify that this implies $L_{[x,y]} = [L_x, L_y]_{\mathfrak{gl}(\mathfrak{g})}$ for all $x, y \in \mathfrak{g}$. Therefore, by Proposition 5.2 we have that μ is a left-invariant flat Riemannian metric. $\square$

Example 7.10. *Aff($\mathbb{R}$) does not admit a left-invariant flat Riemannian metric.* Recall that the Lie Algebra of Aff($\mathbb{R}$) is $\mathfrak{aff}(\mathbb{R}) = \mathrm{Vect}_{\mathbb{R}}\{e_1, e_2\}$ with Lie bracket $[e_1, e_2] = e_2$. Suppose that Aff($\mathbb{R}$) admits a left-invariant flat Riemannian metric μ. Let ∇ be the Levi-Civita connection associated to μ and $L_x(y) = (\nabla_{x^+} y^+)(\epsilon)$ the Levi-Civita product determined by ∇. By Theorem 7.9 $\mathfrak{aff}(\mathbb{R})$ decomposes as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$ where $\mathfrak{u} = \mathrm{Ker}(L)$ is an Abelian ideal of $\mathfrak{aff}(\mathbb{R})$ and $\mathfrak{b}$ is an Abelian subalgebra of $\mathfrak{aff}(\mathbb{R})$ such that $\mathrm{ad}_b \in \mathfrak{o}(\mathfrak{aff}(\mathbb{R}), \mu_\epsilon)$ for all $b \in \mathfrak{b}$. Given these conditions, it is clear that we have $\mathfrak{u} = \mathbb{R}e_2$ and $\mathfrak{b} = \mathbb{R}e_1$. Therefore, from $\mathrm{ad}_{e_1} \in \mathfrak{o}(\mathfrak{aff}(\mathbb{R}), \mu_\epsilon)$ we get $\mu_\epsilon(e_1, e_2) = 0$, and since

$$e_2 = [e_1, e_2] = L_{e_1}(e_2) - L_{e_2}(e_1) = L_{e_1}(e_2),$$

the condition $L_{e_1} \in \mathfrak{o}(\mathfrak{aff}(\mathbb{R}), \mu_\epsilon)$ implies $\mu_\epsilon(e_2, e_2) = 0$. Consequently, e_2 is orthogonal to every element of $\mathfrak{aff}(\mathbb{R})$ with respect to μ_ϵ, which contradicts the fact that μ_ϵ is non-degenerate.

8 The double orthogonal extension

In what follows we describe a construction method known by the name of **double orthogonal extension** which is due to A. Medina and Ph. Revoy (compare [17]). This method provides, among other things, a way to construct all finite-dimensional orthogonal Lie algebras. As an application of the double orthogonal extension we indicate how to construct the Lie algebra of the λ-oscillator Lie group.

Given an orthogonal Lie algebra $(\mathfrak{g}, \mu_0)$, the space of **skew - symmetric derivations** of $\mathfrak{g}$ with respect to μ_0, denoted by $\mathrm{Der}_a(\mathfrak{g}, \mu)$, is defined as the set of derivations $D\colon \mathfrak{g} \to \mathfrak{g}$ that verify $\mu_0(D(x), y) = -\mu_0(x, D(y))$ for all $x, y \in \mathfrak{g}$. It is easy to check that $\mathrm{Der}_a(\mathfrak{g}, \mu)$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$. Suppose that there exists a Lie algebra homomorphism $\psi\colon \mathfrak{h} \to \mathrm{Der}_a(\mathfrak{g}, \mu)$ for some Lie algebra $\mathfrak{h}$. Define the map $\Phi\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}^*$ by $\Phi(x, y)(z) = \mu_0(\psi_z(x), y)$ for $x, y \in \mathfrak{g}$ and $z \in \mathfrak{h}$. Such a map is clearly bilinear. Moreover, since $\psi_z \in \mathrm{Der}_a(\mathfrak{g}, \mu)$ for all $z \in \mathfrak{h}$, we have that Φ is skew-symmetric and satisfies

$$\Phi([x, y], w) + \Phi([y, w], x) + \Phi([w, x], y) = 0, \tag{8.1}$$

for all $x, y, w \in \mathfrak{g}$.

The properties of Φ together with identity (8.1) tell us that Φ defines a 2-cocycle of the Lie algebra $\mathfrak{g}$ with values in the vector space $\mathfrak{h}^*$ with respect to the trivial representation of $\mathfrak{g}$ by $\mathfrak{h}^*$ (see [6]). Therefore, the product vector space $\mathfrak{g}_\Phi = \mathfrak{g} \times_\Phi \mathfrak{h}^*$ is a Lie algebra with Lie bracket given by

$$[(x, \alpha), (y, \beta)]_c = ([x, y]_\mathfrak{g}, \Phi(x, y)),$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{h}^*$. In what follows we denote by $\pi^*\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h}^*)$ the co-adjoint representation of $\mathfrak{h}$. For each $z \in \mathfrak{h}$, we define the map $\Theta_z\colon \mathfrak{g} \times_\Phi \mathfrak{h}^* \to \mathfrak{g} \times_\Phi \mathfrak{h}^*$ by $(x, \alpha) \mapsto (\psi_z(x), \pi_z^*(\alpha))$ for all $x \in \mathfrak{g}$ and $\alpha \in \mathfrak{h}^*$. Since $\psi\colon \mathfrak{h} \to \mathrm{Der}_a(\mathfrak{g}, \mu)$ is a Lie algebra homomorphism, it is easy to verify they satisfy

$$\pi_z^*(\Phi(x, y)) = \Phi(\psi_z(x), y) + \Phi(x, \psi_z(y)), \tag{8.2}$$

for all $x, y \in \mathfrak{g}$ and $z \in \mathfrak{h}$. Formula (8.2) aids us to show that Θ_z is a derivation of the Lie algebra $(\mathfrak{g} \times_\Phi \mathfrak{h}^*, [\cdot, \cdot]_c)$ for each $z \in \mathfrak{h}$, namely, we get

$$\Theta_z([(x, \alpha), (y, \beta)]_c) = [\Theta_z(x, \alpha), (y, \beta)]_c + [(x, \alpha), \Theta_z(y, \beta)]_c,$$

for all $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{h}^*$. Therefore, the map $\Theta\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{g} \times_\Phi \mathfrak{h}^*, [\cdot, \cdot]_c)$, defined by $z \mapsto \Theta_z$, is a well behaved Lie algebra homomorphism. The vector space $\widetilde{\mathfrak{g}}\colon = \mathfrak{h} \ltimes_\Theta (\mathfrak{g} \times_\Phi \mathfrak{h}^*)$ has the structure of a Lie algebra given by the semidirect product of $\mathfrak{h}$ with $\mathfrak{g} \oplus_\Phi \mathfrak{h}^*$ through the Lie algebra homomorphism Θ; in other words, the Lie bracket on $\widetilde{\mathfrak{g}}$ is given explicitely by

$$[(z,x,\alpha),(z',y,\beta)] = ([z,z']_{\mathfrak{h}}, \psi_z(y) - \psi_{z'}(x) + [x,y]_{\mathfrak{g}},$$
$$\pi_z^*(\beta) - \pi_{z'}^*(\alpha) + \Phi(x,y)), \qquad (8.3)$$

for all $x, y \in \mathfrak{g}$, $z, z' \in \mathfrak{h}$ and $\alpha, \beta \in \mathfrak{h}^*$. Finally, over $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ we define the function $\widetilde{\mu_0}$ as

$$\widetilde{\mu_0}((z,x,\alpha),(z',y,\beta)) = \mu_0(x,y) + \alpha(z') + \beta(z), \qquad (8.4)$$

for all $x, y \in \mathfrak{g}$, $z, z' \in \mathfrak{h}$ and $\alpha, \beta \in \mathfrak{h}^*$. Since μ_0 is an invariant scalar product over $\mathfrak{g}$, a direct calculation shows that $\widetilde{\mu_0}$ is an invariant scalar product on $\widetilde{\mathfrak{g}}$ so that $(\widetilde{\mathfrak{g}}, \widetilde{\mu_0})$ is an orthogonal Lie algebra called the **double orthogonal extension** of $(\mathfrak{g}, \mu_0)$ by $\mathfrak{h}$ via ψ.

Remark 8.1. If the signature of the invariant scalar product μ_0 is (p,q), then the signature of $\widetilde{\mu_0}$ is $(p + \dim(\mathfrak{h}), q + \dim(\mathfrak{h}))$.

Example 8.2. *The cotangent bundle of a Lie group.* If in the method of double orthogonal extension we set $\mathfrak{g} = \{0\}$, it is easy to see that we get $\widetilde{\mathfrak{g}} = \mathfrak{h} \ltimes_{\pi^*} \mathfrak{h}^*$ with Lie bracket given by (7.6) and $\widetilde{\mu_0}((x,\alpha),(y,\beta)) = \alpha(y) + \beta(x)$ for all $x, y \in \mathfrak{h}$ and $\alpha, \beta \in \mathfrak{h}^*$. Therefore, the orthogonal Lie algebra obtained is the Lie algebra of the cotangent bundle of the connected and simply connected Lie group H with Lie algebra $\mathfrak{h}$.

Example 8.3. *The Lie algebra of the λ-oscillator Lie group.* Let $\mathfrak{g} = \mathbb{R}^{2n}$ be considered as an Abelian Lie algebra and $\mu_0 = \langle \cdot, \cdot \rangle$ the usual inner product on $\mathbb{R}^{2n}$. Clearly $(\mathbb{R}^{2n}, \mu_0)$ is an orthogonal Lie algebra. We define the linear map $\delta \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$x = (x^1, \cdots, x^{2n}) \mapsto (-x^{n+1}, \cdots, -x^{2n}, x_1, \cdots, x_n)$$

which satisfies

$$\mu_0(\delta(x), y) = -\sum_{j=1}^{n} x_{j+n} y_j + \sum_{j=1}^{n} x_j y_{j+n} = -\mu_0(x, \delta(y)),$$

for all $x, y \in \mathbb{R}^{2n}$. If $\mathfrak{h} = \mathbb{R}e$ is a unidimensional Lie algebra, then the map $\psi\colon \mathbb{R}e \to \mathrm{Der}_a(\mathbb{R}^{2n}, \mu_0)$ defined by $te \mapsto \psi(te) = t\delta$, is a well defined Lie algebra homomorphism. Direct calculation shows that $\mathbb{R}^{2n}_{\Phi} = (\mathbb{R}^{2n} \times_{\Phi} \mathbb{R}e^*, [\cdot,\cdot]_c)$ is isomorphic to the Heisenberg Lie algebra of dimension $2n+1$ and that $\widetilde{\mathbb{R}^{2n}} = \mathbb{R}e \ltimes_{\Theta} (\mathbb{R}^{2n} \times_{\Phi} \mathbb{R}e^*)$ is the Lie algebra with bracket

$$[e, e_j] = \delta(e_j) = \hat{e_j}, \qquad [e, \hat{e_j}] = \delta(\hat{e_j}) = -e_j, \qquad [e_j, \hat{e_j}] = e^*,$$

for all $j = 1, \cdots, n$. The invariant product $\widetilde{\mu_0}$ defined by $\widetilde{\mathbb{R}^{2n}}$ is given by

$$\widetilde{\mu_0}(\gamma e + x + \alpha e^*, \gamma' e + y + \beta e^*) = \mu_0(x, y) + \alpha\gamma' + \beta\gamma,$$

for all $x, y \in \mathbb{R}^{2n}$ and $\alpha, \beta, \gamma, \gamma' \in \mathbb{R}$. The orthogonal Lie algebra $(\widetilde{\mathbb{R}^{2n}}, \widetilde{\mu_0})$ is isomorphic to the λ-oscillator Lie algebra with $\lambda_j = 1$ for all $j = 1, \cdots, n$.

A slight modification of this construction allows us to obtain the Lie algebra $\mathfrak{g}_\lambda$ for $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ with arbitrary $0 < \lambda_1 \leq \cdots \leq \lambda_n$.

Remark 8.4. A. Medina and Ph. Revoy proved in [17] that one can inductively produce all orthogonal Lie algebras starting out with simple and unidimensional ones by taking direct sums and double extensions. More precisely, let $\mathfrak{g}$ be an indecomposable orthogonal Lie algebra, that is, an orthogonal Lie algebra that cannot be written as the direct sum of two non-trivial orthogonal Lie algebras. Then either $\mathfrak{g}$ is simple, or $\mathfrak{g}$ is unidimensional, or else $\mathfrak{g}$ is a double extension of an orthogonal Lie algebra $\widetilde{\mathfrak{g}}$ by a unidimensional or a simple Lie algebra $\mathfrak{h}$. As an application, it is possible to show that any indecomposable non-simple Lie algebra of a Lorentzian Lie group with dimension greater than 1 is the double orthogonal extension of an Abelian Lie algebra with inner product by a unidimensional Lie algebra. This provides a classification of Lorentzian orthogonal Lie algebras up to isomorphism (see [16]).

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Resumen

En estas notas estudiamos algunos conceptos básicos de geometría afín y su relación con la geometría Riemanniana. Proporcionamos una caracterización para variedades afines que posee una contraparte válida para variedades pseudo-Riemannianas cuya conexión de Levi-Civita es plana. Se prueba que ningún grupo de Lie semisimple conexo puede admitir una conexión afín plana invariante a izquierda. Caracterizamos grupos de Lie pseudo-Riemannianos planos y se demuestra que la unimodularidad del grupo de Lie es condición necesaria y suficiente para que la conexión de Levi-Civita asociada a una pseudo-métrica plana invariante a izquierda sobre este resulte geodésicamente completa. Adicionalmente, se describe la forma de cómo obtener métricas hiperbólicas planas invariantes a izquierda sobre el fibrado cotangente de un grupo de Lie afín plano simplemente conexo. Por último, se exponen algunas propiedades de grupos de Lie dotados de pseudo-métricas bi-invariantes, además exhibimos la construcción de álgebras de Lie ortogonales mediante el uso del método de doble extensión ortogonal.

Palabras clave: Estructura afín plana, pseudo-métrica plana, grupo de Lie pseudo-Riemanniano plano, grupo de Lie ortogonal, álgebra de Lie ortogonal.

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