The Jacobian conjecture: Approximate roots and intersection numbers

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Abstract

We compute the intersection number of a Jacobian pair in two different ways following Yansong Xu, but using the language of [5]. We obtain nearly the same formulas, but with an inequality instead of an equality.

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1 Introduction

The Jacobian Conjecture in dimension two stated by Keller in [10] claims that any pair of polynomials $P, Q \in L = K[x, y]$, with $[P, Q] = \partial_x P \partial_y Q - \partial_x Q \partial_y P \in K^{\times}$, defines an invertible automorphism of L. If this conjecture is false, then we can find a counterexample such that the shape of the support of the components $P = f(x)$, $Q = f(y)$ is contained in rectangles $(0,0)$, $m(a,0)$, $m(a,b)$, $m(0,b)$ and $(0,0)$, $n(a,0)$, $n(a,b)$, $n(0,b)$, where $m(a,b)$ is in the support of P and $n(a,b)$ is in the support of Q. In a recent paper [14], Yangsong Xu gives two formulas for the intersection number of possible counterexamples, which we call I_M and I_m. If these formulas were true, we would be able to discard several infinite families of possible counterexamples as described in [7].

When we translated the result and proofs of [14] into the language of [12], we obtained the same formula for I_M (Theorem 6.2), but for I_m we achieved only an inequality (Theorem 7.3). Consequently, we cannot discard the infinite families as desired.

Hence, the main result of the present article is the translation of the concept of approximate roots into our language (see [12], also [5] and [7]), which requires a dictionary from Moh's language into our own. This is interesting by itself, and the modified formulas help understand some features of Moh's methods.

Along this paper we freely use the notation of [12].

2 General lower side corners

For $l \in \mathbb{N}$ let $(P, Q) \in L^{(l)}$ be an (m, n)-pair (see [12, Definition 4.3]). In this section we take $(\rho, \sigma) \in](0, -1), (1, 1)]$ subject to

$$\frac{1}{m} \operatorname{en}_{\rho,\sigma}(P) = \frac{1}{n} \operatorname{en}_{\rho,\sigma}(Q) = (a/l, b) \quad \text{with} \quad a/l > b > 0$$

(assuming that such a direction exists). Note that $\rho > 0$ is true by assumption. Suppose $u_p = v_{\rho,\sigma}(P) > 0$. Then the points $(a/l, b)$ and

$(c/l, d) = \dfrac{1}{m}\operatorname{st}_{\rho,\sigma}(P)$ must satisfy certain conditions. The purpose of this section is to analyse them.

Proposition 2.1. *For P, Q and (ρ,σ) as described above we have $[\ell_{\rho,\sigma}(P), \ell_{\rho,\sigma}(Q)] = 0$.*

Proof. By [12, Proposition 1.13] we need to prove $v_{\rho,\sigma}(P)+v_{\rho,\sigma}(Q) > \rho+\sigma$. If $\rho+\sigma \le 0$, then this is true, since we have $v_{\rho,\sigma}(Q) = \dfrac{n}{m}v_{\rho,\sigma}(P) > 0$; while for $\rho+\sigma > 0$, because of $\dfrac{a}{l} > b \ge 1$ and $\rho > 0$, we have

$$v_{\rho,\sigma}(P) + v_{\rho,\sigma}(Q) = (m+n)\left(\rho\frac{a}{l} + \sigma b\right) > (m+n)b(\rho+\sigma) > \rho+\sigma,$$

as desired. □

Proposition 2.2. *Under the above assumptions, if $\rho+\sigma > 0$, then ρ divides l and there exist $\lambda, \mu \in K^{\times}$ such that $\ell_{\rho,\sigma}(P) = \lambda x^{u_p/\rho}(z-\mu)^{mb}$, here $z = x^{-\sigma/\rho}y$.*

Proof. By [12, Theorem 2.6], there exists a (ρ,σ)-homogeneous element $F \in L^{(l)}$ such that

- $v_{\rho,\sigma}(F) = \rho+\sigma$,
- $[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P)$,
- $\operatorname{st}_{\rho,\sigma}(P) \sim \operatorname{st}_{\rho,\sigma}(F)$ or $\operatorname{st}_{\rho,\sigma}(F) = (1,1)$,
- $\operatorname{en}_{\rho,\sigma}(P) \sim \operatorname{en}_{\rho,\sigma}(F)$ or $\operatorname{en}_{\rho,\sigma}(F) = (1,1)$.

If $\operatorname{en}_{\rho,\sigma}(P) = m(a/l, b) \sim \operatorname{en}_{\rho,\sigma}(F)$, then we can find $\lambda > 0$ such that $\operatorname{en}_{\rho,\sigma}(F) = \lambda(a/l, b)$. Therefore

$$\rho+\sigma = v_{\rho,\sigma}(F) = \rho\lambda\frac{a}{l} + \lambda\sigma b > \lambda b(\rho+\sigma)$$

implies $0 < \lambda b < 1$, which is impossible since $\lambda b = v_{0,1}(\operatorname{en}_{\rho,\sigma}(F)) \in \mathbb{Z}$. Consequently we have $\operatorname{en}_{\rho,\sigma}(F) = (1,1)$, and hence $\operatorname{st}_{\rho,\sigma}(F) = (1+$

$\sigma/\rho, 0)$ by [12, Proposition 2.11(2)]. Thus ρ divides l and we have $\mathrm{st}_{\rho,\sigma}(P) \sim \mathrm{st}_{\rho,\sigma}(F)$, which readily implies $v_{0,1}(\mathrm{st}_{\rho,\sigma}(P)) = 0$. Write

$$F = x^{\frac{u}{l}} y^v f(z) \quad \text{and} \quad \ell_{\rho,\sigma}(P) = x^{\frac{c}{l}} y^d p(z), \quad \text{with } p(0), f(0) \neq 0.$$

Note that here $v = d = 0$, $\rho c/l = u_p$, $v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) = mb$ and $f(z) = \lambda_1(z - \mu)$ for some $\lambda_1, \mu \in K^\times$. By [12, Proposition 2.11(1)] we have then $\ell_{\rho,\sigma}(P) = \lambda x^{u_p/\rho}(z-\mu)^{mb}$ for some $\lambda \in K^\times$, which concludes the proof. □

By [12, Proposition 2.1(2)] (which applies thanks to Proposition 2.1) there exist $\lambda_P, \lambda_Q \in K^\times$ and a (ρ, σ)-homogeneous element $R \in L^{(l)}$ such that

$$\ell_{\rho,\sigma}(P) = \lambda_P R^m \qquad \text{and} \qquad \ell_{\rho,\sigma}(Q) = \lambda_Q R^n.$$

Take $\lambda \in K^\times$ and let $R_0 \in L^{(l)}$ be a (ρ, σ)-homogeneous element such that $\ell_{\rho,\sigma}(P) = \lambda R_0^h$ with h maximal (consequently m divides h and we can assume $R = R_0^{h/m}$ and $\lambda_P = \lambda$). Arguing as in [5, Corollary 2.6] we obtain a certain $i \geq 0$ and a (ρ, σ)-homogeneous element $G \in L^{(l)}$ subject to $[G, R] = R^i$.

Let $(a/l, b), (c/l, d) \in \frac{1}{l}\mathbb{Z} \times \mathbb{Z}$ be such that $a/l > b > d \geq 0$ and $a > c > 0$. Assume also $b - d < a/l - c/l$ (we do not claim the existence of P and Q at this point). It is well known that for each $(r/l, s) \in \frac{1}{l}\mathbb{Z} \times \mathbb{Z} \setminus \mathbb{Z}(1,1)$ there exists a unique $(\varrho, \varsigma) \in \mathfrak{V}_{>0}$ (see (3.2) at page 29 of [12]), which we denote by $\mathrm{dir}(r/l, s)$, such that $v_{\varrho,\varsigma}(r/l, s) = 0$. Set $(\rho, \sigma) = -\mathrm{dir}((a/l, b) - (c/l, d))$ and note the inequality $(0, -1) < (\rho, \sigma) < (1, -1)$. We will analyse the existence of $i \in \mathbb{N}$ and (ρ, σ)-homogeneous elements $R, G \in L^{(l)}$, such that

$$v_{\rho,\sigma}(R) > 0, \quad [G, R] = R^i, \quad (a/l, b) = \mathrm{en}_{\rho,\sigma}(R) \quad \text{and} \quad (c/l, d) = \mathrm{st}_{\rho,\sigma}(R). \tag{2.1}$$

Let $\ell \in \mathbb{N}$ be minimal with $\ell v_{\rho,\sigma}(R) + \rho + \sigma > 0$. By [5, Proposition 3.12], if there exist $i \in \mathbb{N}$ and $R, G \in L^{(l)}$ satisfying (2.1), and such that

$$R \neq \lambda x^{\frac{u}{\rho}} h^j(z) \quad \text{for all } \lambda \in K^{\times},\ j \in \mathbb{N} \text{ and all linear polynomials } h, \tag{2.2}$$

where $z = x^{-\frac{\sigma}{\rho}} y$, then either there exist $\vartheta, t' \in \mathbb{N}$ subject to

$$\vartheta \leq N_1, \qquad t' < \ell\vartheta \qquad \text{and} \qquad (\rho, \sigma) = -\operatorname{dir}\Big(t'\Big(\frac{c}{l}, d\Big) + \vartheta(1,1)\Big), \tag{2.3}$$

where $N_1 = \gcd(a-c, b-d)$, or

$$d > 0,\ \ \vartheta \text{ divides } N_2,\ \ t' < \ell\vartheta \quad \text{and} \quad (\rho, \sigma) = -\operatorname{dir}\Big(t'\Big(\frac{c}{l}, d\Big) + \vartheta(1,1)\Big), \tag{2.4}$$

where $N_2 = \gcd(c, d)$. By [5, Remark 3.13] we have then

$$\frac{\vartheta}{t'} = -\frac{\rho a/l + \sigma b}{\rho + \sigma}.$$

Therefore defining

$$s = \frac{\rho a + \sigma b l}{\gcd(\rho l + \sigma l, \rho a + \sigma b l)} \Big| \vartheta,$$

we can take (and we do take it) $\vartheta = s$ in (2.3) and (2.4).

We suspect that the existence of ϑ and t' satisfying the conditions in (2.3) or in (2.4) is enough for the existence of $i \in \mathbb{N}$ and two (ρ, σ)-homogeneous elements $R, G \in L^{(l)}$ such that the conditions in (2.1) and (2.2) are fullfiled (with $(c/l, d) = \operatorname{st}_{\rho,\sigma}(R)$), but at the moment we have no proof of this claim.

Remark 2.3. Since $N_2 < b$, if $s = b$, then necessarily $b \leq N_1$. So, by [5, Proposition 3.12(2)], there exists a linear factor with multiplicity b. As this contradicts (2.2), we have consequently $s < b$.

Remark 2.4. By [5, Theorem 3.4], in (2.1) we can assume that i is the minimal element subject to

$$v_{\rho,\sigma}(R)(i-1) + \rho + \sigma \geq 0,$$

or, equivalently, $i = \left\lceil 1 - \frac{\rho+\sigma}{v_{\rho,\sigma}(R)} \right\rceil$.

For the case $b=2$, we can establish necessary and sufficient conditions on a, l for the existence of $c \in \mathbb{N}$, $d \in \{0,1\}$ and two (ρ,σ)-homogeneous elements $R, G \in L^{(l)}$ satisfying the conditions of (2.1) as soon as we impose that R satisfies (2.2). This additional requirement corresponds to the existence of split roots (see Definition 3.5). Before we establish the result we note that $(0,-1) < (\rho,\sigma) < (1,-1)$ and $(\rho,\sigma) = -\operatorname{dir}\left(\frac{a-c}{l}, b-d\right) \sim (lb-ld, c-a)$ imply $c < a$ and $b-d < a/l - c/l$.

Proposition 2.5. *Let $a, l \in \mathbb{N}$ be such that $a/l > 2$. Set $b=2$. Let $(\rho,\sigma) \in](0,-1),(1,-1)[$ be a direction, and define the number*

$$\vartheta = \frac{\rho a + \sigma b l}{\gcd(\rho l + \sigma l, \rho a + \sigma b l)}.$$

Then the following assertions are equivalent.

(1) *There exist $c \in \mathbb{N}$, $d \in \{0,1\}$ and two (ρ,σ)-homogeneous elements $R, G \in L^{(l)}$ satisfying conditions (2.1) and (2.2).*

(2) *There exist $c \in \mathbb{N}$ and two (ρ,σ)-homogeneous elements $R, G \in L^{(l)}$ satisfying conditions (2.1) and (2.2) with $d=1$.*

(3) *We have $\vartheta = 1$, $v_{\rho,\sigma}(a/l,2) > 0$ and there exist $c \in \mathbb{N}$ such that*

$$(\rho,\sigma) = -\operatorname{dir}\left(\frac{a-c}{l}, 1\right) = -\operatorname{dir}\left(t'\left(\frac{c}{l}, 1\right) + (1,1)\right), \tag{2.5}$$

for some $0 < t' < \ell$, here $\ell \in \mathbb{N}$ is minimal with the property $\ell v_{\rho,\sigma}(a/l,2) + \rho + \sigma > 0$.

(4) *There exists $\Delta \in \mathbb{N}$ with $l < \Delta < a/2$ such that $a - 2\Delta \mid \Delta - l$. Moreover, $(\rho,\sigma) \sim (l, -\Delta)$.*

Proof. We first prove that 1) implies 2). Suppose $d=0$ and write

$$R = \lambda x^{\frac{u}{l}}(z-\alpha_1)(z-\alpha_2) \quad \text{with } z = x^{-\frac{\sigma}{\rho}} y.$$

Note that by (2.2) we have $\alpha_1 \neq \alpha_2$. Also $\rho u/l = 2\sigma + \rho a/l$ implies $u = (2l\sigma + \rho a)/\rho$. Moreover, since $b - d = 2$, we have

$$(2l, c-a) \cdot \Big(\frac{a}{l} - \frac{c}{l}, b-d\Big) = 2(a-c) - (c-a)(b-d) = 0,$$

and consequently $(\rho, \sigma) \sim (2l, c-a)$. Also, since $d = 0$, necessarily (2.3) is satisfied. We claim that 2 divides $a - c$. In fact, this follows from

$$0 = (2l, c-a) \cdot \Big(t'\Big(\frac{c}{l}, 0\Big) + \vartheta(1,1)\Big) = 2ct' + (c-a)\vartheta,$$

for otherwise 2 divides $\vartheta \leq N_1 = \gcd(a-c, 2) = 1$. Set $\Delta = (a-c)/2$ and consider the automorphism φ of $L^{(l)}$ defined by $\varphi(x^{1/l}) = x^{1/l}$ and $\varphi(y) = y + \alpha_1 x^{-\Delta/l}$. Using $(\rho, \sigma) \sim (l, -\Delta)$ it is easy to conclude

$$\varphi(R) = \lambda x^{\frac{u}{l}} z(z - (\alpha_2 - \alpha_1)).$$

By [12, Proposition 3.10], we have $[\varphi(G), \varphi(R)] = \varphi(R)^i$. An easy computation gives $\mathrm{en}_{\rho,\sigma}(\varphi(R)) = (a/l, b)$ and $\mathrm{st}_{\rho,\sigma}(\varphi(R)) = ((a-\Delta)/l, 1)$. So, replacing R by $\varphi(R)$ yields $d = 1$.

That 2) implies 1) is a trivial fact.

Now we prove that 2) implies 3). Since $d = 1$, we get $N_1 = N_2 = 1$. Hence we have $\vartheta = 1$, and Equality (2.5) is satisfied for some $0 < t' < \ell$. Moreover, it is clear that

$$v_{\rho,\sigma}\Big(\frac{a}{l}, 2\Big) = v_{\rho,\sigma}(R) > 0$$

is verified and we also have

$$(\rho, \sigma) = -\operatorname{dir}\big(\mathrm{en}_{\rho,\sigma}(R) - \mathrm{en}_{\rho,\sigma}(R)\big) = -\operatorname{dir}\Big(\frac{a-c}{l}, 1\Big).$$

For 3) implies 4), since

$$(l, c-a) \cdot \Big(\frac{a}{l} - \frac{c}{l}, 1\Big) = 0,$$

we have $(\rho,\sigma)\sim(l,-\Delta)$, with $\Delta=a-c$. Thus, by (2.5), we obtain

$$0=(l,-\Delta)\cdot\Bigl(t'\Bigl(\frac{a-\Delta}{l},1\Bigr)+(1,1)\Bigr)=t'a-2t'\Delta+l-\Delta,$$

which implies $a-2\Delta|l-\Delta$, as desired. But $(\rho,\sigma)\sim(l,-\Delta)$ and $v_{\rho,\sigma}(a/l,2)>0$ yield

$$a-2\Delta=(l,-\Delta)\cdot\Bigl(\frac{a}{l},2\Bigr)=\frac{l}{\rho}(\rho,\sigma)\cdot\Bigl(\frac{a}{l},2\Bigr)>0,$$

and so $\Delta<a/2$. Finally, the relation $l-\Delta=\dfrac{l}{\rho}(\rho+\sigma)<0$ forces $\Delta>l$.

To show that 4) implies 2) we set $c=a-\Delta$, $z=x^{\Delta/l}y$ and $(\rho,\sigma)=-\operatorname{dir}((a/l,2)-(c/l,1))$. Since $0<l<\Delta$, the inequalities $(0,-1)<(\rho,\sigma)<(1,-1)$ hold. Let $k_1\in\mathbb{N}$ be such that $k_1(a-2\Delta)=\Delta-l$ and let $g(z)$ be a polynomial with derivative $g'(z)=z^{k_1}(1+z)^{k_1}$. A straightforward computation shows that

$$R=x^{\frac{a-2\Delta}{l}}z(1+z)=x^{\frac{c}{l}}y(1+z)\qquad\text{and}\qquad G=\frac{l}{2\Delta-a}g(z),$$

satisfy $\Bigl(\dfrac{a}{l},2\Bigr)=\mathrm{en}_{\rho,\sigma}(R)$, $\Bigl(\dfrac{c}{l},1\Bigr)=\mathrm{st}_{\rho,\sigma}(R)$, $v_{\rho,\sigma}(R)>0$ and $[G,R]=R^{k_1+1}$, as desired. □

3 Approximate π-roots

Recall that the **intersection number** of two bivariate polynomials P and Q is defined by $I(P,Q)=\deg_x(\mathrm{Res}_y(P,Q))$, where $\mathrm{Res}_y(P,Q)$ denotes the resultant of P and Q as polynomials in y. In [14], the author defines for a Jacobian pair (P,Q) the polynomial $P_\xi=P(x,y)-\xi$, where ξ is a generic element of the field K, and proposes two different formulas for $I(P_\xi,Q)$: one in terms of the major roots (see [14, Theorem 5.1]), and the other in terms of the minor roots (as in [14, Theorem 4.7]). We will prove the first formula using our language (see Theorem 6.2), however, instead of equality in the formula for I_m we recover an inequality in Theorem 7.3. In order to achieve these results, it will be convenient to provide a proof of the preparatory results of [14] in the language of [12].

We first define approximate roots, final major roots and final minor roots using our language.

In this section we consider a polynomial $P \in L$, monic in y. For $l \in \mathbb{N}$ we take the following algebras:

$$L = K[x, y] \subsetneq K[x^{\pm\frac{1}{l}}, y] \subsetneq K((x^{-1/l}))[y] \subsetneq K[\pi]((x^{-1/l}))[y],$$

where π is a variable (a "symbol" in [14]). We also will use the subring $L_\pi^{(l)} = K[\pi][x^{\pm 1/l}, y]$ of $K[\pi]((x^{-1/l}))[y]$. Note that $\deg_x = v_{1,0}$ is well defined in $K[\pi]((x^{-1/l}))[y]$.

Unless otherwise indicated, the elements P of the above mentioned algebras are polynomials in y with coefficients in one of the algebras $K[x]$, $K[x^{\pm\frac{1}{l}}, y]$, $K[\pi]((x^{-1/l}))$,.... Consequently, expressions like $P(\tau)$, $P(\alpha)$,..., will mean P with y replaced by τ, by α, and so on.

By the Newton-Puiseux theorem (see [4, Corollary 13.15, page 295]) there exist $l \in \mathbb{N}$ and $\alpha_i, \beta_i \in K((x^{-1/l}))$ such that

$$P = \prod_{i=1}^{M} (y - \alpha_i).$$

We set $\mathcal{R}(P) = \{\alpha_i : i = 1, \dots, M\}$. Let $\alpha \in \mathcal{R}(P)$ and write $\alpha = \sum_j a_j x^j$ with $j \in \dfrac{1}{l}\mathbb{Z}$. The **$\pi$-approximation of α up to x^{j_0}** is the element

$$\tau = \sum_{j > j_0} a_j x^j + \pi x^{j_0} \in K[\pi, x^{\pm\frac{1}{l}}].$$

Note the equality $\deg_x(\tau - \alpha) = j_0$.

Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0} \in K[\pi, x^{\pm\frac{1}{l}}]$. Set

$$D_\tau^P = \{\alpha \in \mathcal{R}(P) : \tau \text{ is the } \pi\text{-approximation of } \alpha \text{ up to } x^{j_0}\}.$$

If $\alpha \in D_\tau^P$, we say that τ **approximates α up to x^{j_0}**.

Note that the element $\alpha_i = \sum_j b_j x^j \in \mathcal{R}(P)$ belongs to D_τ^P if and only if $\deg_x(\hat{\alpha}_i) \le j_0$, where $\hat{\alpha}_i = \alpha_i - \sum_{j>j_0} a_j x^j$, that is, if and only if we have $a_j = b_j$ for all $j > j_0$.

We say that $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0} \in K[\pi, x^{\pm\frac{1}{l}}]$ is a **π-root of** P if there exists $\alpha \in \mathcal{R}(P)$ such that τ approximates α up to x^{j_0}. In that case we say that j_0 is the **order** of τ. When we want to underline the dependence of j_0 on τ we will write $\delta_\tau = j_0$.

Notation 3.1. Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P. In what follows denote by φ_τ the automorphism of $L^{(l)}$ determined by $\varphi_\tau(x^{1/l}) = x^{1/l}$ and $\varphi_\tau(y) = y + \sum_{j>j_0} a_j x^j$.

Remark 3.2. Let $\alpha \in \mathcal{R}(P)$. Assume that τ approximates α up to j_0 and τ_1 approximates α up to j_1. If $j_0 > j_1$, then we have $D_{\tau_1}^P \subseteq D_\tau^P$.

In the sequel, for each $j \in \dfrac{1}{l}\mathbb{Z}$, we let $\operatorname{dir}(j)$ denote the unique direction (ρ, σ) such that $\rho > 0$ and $j = \dfrac{\sigma}{\rho}$. Moreover, given a polynomial $\tau = \sum_{i>j_0} a_i x^i + \pi x^{j_0}$, we set $z = x^{-\sigma/\rho} y$, where $(\rho, \sigma) = \operatorname{dir}(j_0)$.

The following proposition shows that our definition of π-root coincides with that given in [11, Definition 1.3] with x^{-1} replaced by t.

Proposition 3.3. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ and $f_{P,\tau}(\pi) \in K[\pi]$ be the polynomial determined by the equality*

$$P(\tau) = f_{P,\tau}(\pi) x^{\lambda_\tau} + \textit{terms with lower order in } x, \tag{3.1}$$

where $\lambda_\tau = \deg_x(P(\tau)) \in \dfrac{1}{l}\mathbb{Z}$. Set $\varphi = \varphi_\tau$ and $(\rho, \sigma) = \operatorname{dir}(j_0)$. Then we have

$$|D_\tau^P| = \deg(f_{P,\tau}) = v_{0,1}(\operatorname{en}_{\rho,\sigma}(\varphi(P))) \tag{3.2}$$

and

$$\ell_{\rho,\sigma}(\varphi(P)) = x^{\lambda_\tau} f_{P,\tau}(z). \tag{3.3}$$

Consequently τ is a π-root of P if and only if we have $\deg(f_{P,\tau}) > 0$.

Proof. Let $\operatorname{ev}_{\pi x^{j_0}} : L_\pi^{(l)} \to L_\pi^{(l)}$ be the evaluation of y at πx^{j_0}. For example we have, $\operatorname{ev}_{\pi x^{j_0}}(y) = \pi x^{j_0} = \pi x^{\sigma/\rho}$ and $\operatorname{ev}_{\pi x^{j_0}}(x^{1/l}) = x^{1/l}$. Note the relation $P(\tau) = \operatorname{ev}_{\pi x^{j_0}}(\varphi(P))$. Since $\operatorname{ev}_{\pi x^{j_0}}$ is (ρ,σ)-homogeneous, we get

$$\ell_{\rho,\sigma}(\operatorname{ev}_{\pi x^{j_0}}(\varphi(P))) = \operatorname{ev}_{\pi x^{j_0}}(\ell_{\rho,\sigma}(\varphi(P))).$$

On the other hand, since ρ divides l, we get

$$\ell_{\rho,\sigma}(\varphi(P)) = x^{r/l} g(z) \quad \text{for some } r \in \mathbb{Z} \text{ and } g(z) \in K[z]. \tag{3.4}$$

Using $\operatorname{ev}_{\pi x^{j_0}}(z) = \pi$ we obtain

$$\ell_{\rho,\sigma}(\operatorname{ev}_{\pi x^{j_0}}(\varphi(P))) = x^{r/l} g(\pi).$$

Therefore we have

$$P(\tau) = \operatorname{ev}_{\pi x^{j_0}}(\varphi(P)) = x^{r/l} g(\pi) + \text{terms with lower order in } x,$$

because $v_{\rho,\sigma}(x^j) = j\rho < \rho r/l = v_{\rho,\sigma}(x^{r/l})$ if and only if $j < r/l$. So we have $f_{P,\tau}(\pi) = g(\pi)$, $\lambda_\tau = r/l$, and Equality (3.4) becomes Equality (3.3). Since $\deg_z(\ell_{\rho,\sigma}(\varphi(P))) = \deg_y(\ell_{\rho,\sigma}(\varphi(P)))$, we also have $\deg(f_{P,\tau}) = v_{0,1}(\operatorname{en}_{\rho,\sigma}(\varphi(P)))$. Consequently, in order to conclude the proof, it suffices to prove $|D_\tau^P| = v_{0,1}(\operatorname{en}_{\rho,\sigma}(\varphi(P)))$. In the chain of equalities

$$v_{0,1}(\operatorname{en}_{\rho,\sigma}(\varphi(P))) = \sum_{i=1}^{M} v_{0,1}(\operatorname{en}_{\rho,\sigma}(\varphi(y-\alpha_i))) = \sum_{i=1}^{M} v_{0,1}(\operatorname{en}_{\rho,\sigma}(y-\hat{\alpha}_i))),$$

where $\hat{\alpha}_i = \alpha_i - \sum_{j>j_0} a_j x^j$, when we replace

$$\operatorname{en}_{\rho,\sigma}(y-\hat{\alpha}_i)) = \begin{cases} (0,1) & \text{if } \deg_x(\hat{\alpha}_i) \le \sigma/\rho = j_0, \\ (\deg_x(\hat{\alpha}_i), 0) & \text{if } \deg_x(\hat{\alpha}_i) > \sigma/\rho = j_0, \end{cases}$$

we finish up with

$$\sum_{i=1}^{M} v_{0,1}(\operatorname{en}_{\rho,\sigma}(y-\hat{\alpha}_i))) = \#\{\alpha_i \in \mathcal{R}(P) : \deg_x(\hat{\alpha}_i) \le j_0\} = |D_\tau^P|,$$

as desired. $\square$

Remark 3.4. Note that if $|D_\tau^P| > 0$, then $|D_\tau^{P_y}| = |D_\tau^P| - 1 \geq 0$. In fact, we get $\varphi(P_y) = (\varphi(P))_y$, and it is straightforward to check that if $\rho > 0$ and $v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) > 0$, then $v_{0,1}(\mathrm{en}_{\rho,\sigma}(P_y)) = v_{0,1}(\mathrm{en}_{\rho,\sigma}(P)) - 1$. The assertion then follows from (3.2).

Definition 3.5. We say that a π-root τ of P is a **final π-root of** P if $f_{P,\tau}(\pi)$ has no multiple roots and $\deg_\pi(f_{P,\tau}(\pi)) > 1$, here $f_{P,\tau}(\pi)$ is defined by Equality (3.1).

Remark 3.6. Let τ be a final π-root of P. Since the support of $f_{P,\tau}$ has more than one point, from Equality (3.3) we conclude $(\rho,\sigma) \in \mathrm{Dir}(\varphi_\tau(P))$.

Proposition 3.7. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P and let $\lambda \in K$. Consider the automorphism $\varphi_1\colon L^{(l)} \to L^{(l)}$ given by $\varphi_1(x^{1/l}) = x^{1/l}$ and $\varphi_1(y) = y + \sum_{j>j_0} a_j x^j + \lambda x^{j_0}$. Assume that $\varphi_1(P)$ is not a monomial and set $(\rho',\sigma') = \mathrm{Pred}_{\varphi_1(P)}(\rho,\sigma)$ (see [12, Definition 3.4]), where $(\rho,\sigma) = \mathrm{dir}(j_0)$. If $\rho' > 0$, then set $j_1 = \dfrac{\sigma'}{\rho'}$, else take any $j_1 \in \dfrac{1}{l}\mathbb{Z}$ with $j_1 < j_0$. In both cases set $(\rho_1,\sigma_1) = \mathrm{dir}(j_1)$. If $\pi - \lambda$ has multiplicity $r > 0$ in $f_{P,\tau}(\pi)$, then*

$$\tau_1 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_1}$$

is a π-root of P and we get $|D_{\tau_1}^P| = r$ (remember that $j_1 < j_0$). Moreover, we have

$$(\rho_1,\sigma_1) \in [\mathrm{Pred}_{\varphi_1(P)}(\rho,\sigma), (\rho,\sigma)[. \tag{3.5}$$

Proof. Write $\varphi_1 = \widetilde{\varphi}\circ\varphi$, where φ is as in Proposition 3.3, $\widetilde{\varphi}(y) = y + \lambda x^{j_0}$, and $\widetilde{\varphi}(x) = x$. By Equality (3.3), the fact that $\widetilde{\varphi}$ is (ρ,σ)-homogeneous together with $\widetilde{\varphi}(z) = z + \lambda$ lead us to

$$\begin{aligned}\ell_{\rho,\sigma}(\varphi_1(P)) &= \widetilde{\varphi}(\ell_{\rho,\sigma}(\varphi(P))) = \widetilde{\varphi}(x^{\lambda_\tau} f_{P,\tau}(z)) \\ &= x^{\lambda_\tau}\widetilde{\varphi}(f_{P,\tau}(z)) = x^{\lambda_\tau} z^r g_1(z),\end{aligned}$$

for some $g_1(z) \in K[z]$ with $g_1(0) \neq 0$. By construction we have then $(\rho_1, \sigma_1) \in]\mathrm{Pred}_{\varphi_1(P)}(\rho, \sigma), (\rho, \sigma)[$, and so, by Proposition 3.3, we get

$$r = v_{0,1}(\mathrm{st}_{\rho,\sigma}(\varphi_1(P))) = v_{0,1}(\mathrm{en}_{\rho_1,\sigma_1}(\varphi_1(P))) = |D^P_{\tau_1}|,$$

as desired. □

Corollary 3.8. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P and let $\lambda \in K$. If $\pi - \lambda$ does not divide $f_{P,\tau}(\pi)$, then there exists no root $\alpha \in \mathcal{R}(P)$ such that $\deg_x(\alpha - (\lambda x^{j_0} + \sum_{j>j_0} a_j x^j)) < j_0$.*

Proof. Let $f_{P,\tau}(\pi) = \prod_{i=1}^{k}(\pi - \lambda_i)^{m_i}$. By Proposition 3.7, for each i there exist $\tau_1(i)$ and m_i roots in $D^P_{\tau_1(i)} \subset D^P_\tau$, for which $\mathrm{Coeff}_{x^{j_0}} = \lambda_i$. Since we have

$$|D^P_\tau| = \deg(f_{P,\tau}(\pi)) = \sum_{i=1}^{k} m_i = \sum_{i=1}^{k} |D^P_{\tau_1(i)}|$$

and the sets $D^P_{\tau_1(i)}$ are pairwise disjoint, we obtain $D^P_\tau = \bigcup_{i=1}^{k} D^P_{\tau_1(i)}$. Consequently, the coefficient of x^{j_0} in each element of D^P_τ is a root of $f_{P,\tau}$. Since λ is not a root of $f_{P,\tau}$, this finishes the proof. □

Remark 3.9. The proof of the corollary shows that if the multiplicity of $\pi - \lambda$ in $f_{P,\tau}(\pi)$ is r, then any π-root τ_2 of P which begins with $\lambda x^{j_0} + \sum_{j>j_0} a_j x^j$ satisfies $|D^P_{\tau_2}| \leq r$.

Remark 3.10. Let $\alpha = \sum_j a_j x^j \in K((x^{-1/l}))$, $j_0 \in \frac{1}{l}\mathbb{Z}$, $(\rho, \sigma) = \mathrm{dir}(j_0)$ and $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. Define $T = \sum_{j \leq j_0} a_j x^j$. Since

$$P(\alpha) = \mathrm{ev}_{y=T}(\varphi_\tau(P)),$$

we have $\ell_{\rho,\sigma}(P(\alpha)) = \ell_{\rho,\sigma}(\mathrm{ev}_{y=\lambda x^{j_0}}(\varphi_\tau(P)))$ whenever the right hand side of the equality is nonzero.

Proposition 3.11. *Let $\alpha = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \sum_{j<j_0} a_j x^j$ and set $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. If $f_{P,\tau}(\lambda) \neq 0$, then $\lambda^P_\tau = \deg_x(P(\tau)) = \deg_x(P(\alpha))$.*

Proof. By Remark 3.10, Equality (3.3) and the fact that $\mathrm{ev}_{y=\lambda x^{j_0}}$ is (ρ,σ)-homogeneous we get

$$\ell_{\rho,\sigma}(P(\alpha)) = \ell_{\rho,\sigma}(\mathrm{ev}_{y=\lambda x^{j_0}}(\varphi(P))) = \mathrm{ev}_{y=\lambda x^{j_0}}(\ell_{\rho,\sigma}(\varphi(P))) = x^{\lambda_\tau^P} f_{P,\tau}(\lambda).$$

Thus, we achieve

$$\deg_x(P(\alpha) = \deg_x(\ell_{\rho,\sigma}(P(\alpha))) = \lambda_\tau^P = \deg_x(P(\tau)),$$

as needed. □

4 Approximate roots for Jacobian pairs

For the rest of the section (P_0,Q_0) will be a Jacobian pair in L satisfying the conditions required in [12, Corollary 5.21]. This in particular means that (P_0,Q_0) is a minimal pair and a standard (m,n)-pair for some coprime integers $m,n>1$. By [12, Proposition 4.6(3)], there exist $a<b$ in $\mathbb{N}$ such that $\mathrm{en}_{1,0}(P_0)=m(a,b)$ and $\mathrm{en}_{1,0}(Q_0)=n(a,b)$. So, by [12, Corollary 5.21(4)], we know that $\ell_{1,1}(P_0)=\lambda x^{am}y^{bm}$ and $\ell_{1,1}(Q_0)=\lambda' x^{an}y^{bn}$ hold for some $\lambda,\lambda'\in K^\times$. Replacing P_0 by $\frac{1}{\lambda}P_0$ and Q_0 by $\frac{1}{\lambda'}Q_0$, we can further assume $\lambda=\lambda'=1$. Let ψ be the automorphism of L characterized by $\psi(y)=y$ and $\psi(x)=x+y$. Set $P=\psi(P_0)$ and $Q=\psi(Q_0)$ (see Figure 1). Since ψ is $(1,1)$-homogeneous we have

$$\ell_{1,1}(P)=\psi(\ell_{1,1}(P_0))=(x+y)^{ma}y^{mb} \quad \text{and} \quad \ell_{1,1}(Q)=(x+y)^{na}y^{nb}. \tag{4.1}$$

Hence, P and Q are monic polynomials in y and, moreover, a straightforward computation yields

$$\mathrm{en}_{1,0}(P)=m(a,b) \quad \text{and} \quad \mathrm{en}_{1,0}(Q)=n(a,b). \tag{4.2}$$

Remark 4.1. We will establish several results about P, but, since by [12, Proposition 4.6] we know that (Q,P) is an (n,m)-pair, the same results remain valid, *mutatis mutandis*, for Q.

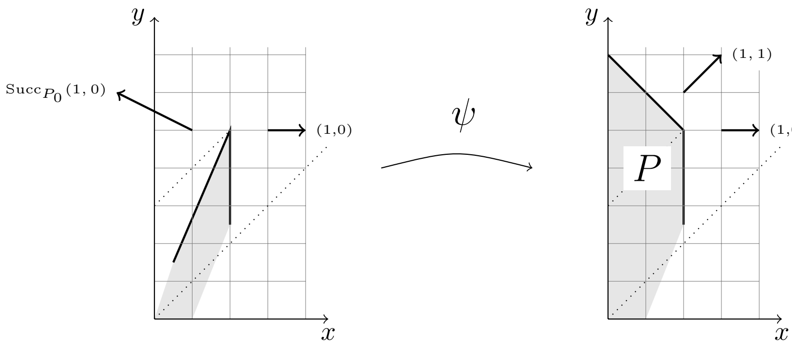

Figure 1: The shapes of P_0 according to [12, Corollary 5.21(4)] and of P according to (4.1) and (4.2).

Proposition 4.2. *Let* $\alpha \in \mathcal{R}(P)$ *and let* τ *be the* π*-approximation of* α *up to* x^{j_0}*. Assume* $\lambda_\tau = \deg_x(P(\tau)) > 0$*, and take* φ *and* (ρ, σ) *as in Proposition 3.3. Then the following facts hold.*

(1) *If* $f_{P,\tau}$ *has multiple roots, then* $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = 0$.

(2) *If* $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = 0$*, then there exists* $\beta \in \mathcal{R}(Q)$ *such that* $\deg_x(\alpha - \beta) < j_0$.

Proof. Write $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. By Proposition 3.7 there exists $j_1 < j_0$ such that

$$\tau_1 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_1}$$

is a π-root of P.

(1) Since we have $\ell_{\rho,\sigma}(\varphi(P)) = x^{\lambda_\tau} f_{P,\tau}(z)$ (see Equality (3.3)), by hypothesis there exist $k > 1$ and $\lambda \in K$ such that $(z-\lambda)^k$ divides $\ell_{\rho,\sigma}(\varphi(P))$. Consequently $(z-\lambda)^{k-1}$ divides $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))]$. As we have $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] \in K$, this yields $[\ell_{\rho,\sigma}(\varphi(P)), \ell_{\rho,\sigma}(\varphi(Q))] = 0$.

(2) Let $(z-\lambda)$ be a linear factor of $\ell_{\rho,\sigma}(\varphi(P))$. Since we have $v_{\rho,\sigma}(P)=\rho\lambda_\tau>0$, from [12, Proposition 2.1(2)b)] it follows that $(z-\lambda)$ divides $\ell_{\rho,\sigma}(\varphi(Q))$. Hence, by Proposition 3.3 we know that τ is a π-root of Q and so, by Proposition 3.7, there exists $j_2<j_0$ such that

$$\tau_2=\sum_{j>j_0}a_jx^j+\lambda x^{j_0}+\pi x^{j_2}$$

is a π-root of Q. We conclude that for any $\alpha\in D^P_{\tau_1}$ and $\beta\in D^Q_{\tau_2}$ the inequality $\deg_x(\alpha-\beta)<j_0$ holds, as claimed. □

Remark 4.3. Let $\alpha=\sum a_jx^j\in\mathcal{R}(P)$. Assume $j_0>j_1$ and that τ approximates α up to x^{j_0} and τ_1 approximates α up to x^{j_1}. Then we must have $\lambda_\tau>\lambda_{\tau_1}$. In fact, setting $(\rho,\sigma)=\operatorname{dir}(j_0)$ and $(\rho_1,\sigma_1)=\operatorname{dir}(j_1)$, Equality (3.3) and [12, Proposition 3.9] show

$$v_{\rho,\sigma}(x^{\lambda_\tau})=v_{\rho,\sigma}(\varphi(P))=v_{\rho,\sigma}(\varphi_1(P))\geq v_{\rho,\sigma}(\operatorname{en}_{\rho_1,\sigma_1}\varphi_1(P)),$$

with $\varphi=\varphi_\tau$ and $\varphi_1=\varphi_{\tau_1}$. A direct computation using $(\rho,\sigma)>(\rho_1,\sigma_1)$, $v_{\rho_1,\sigma_1}(\varphi_1(P))=v_{\rho_1,\sigma_1}(x^{\lambda_{\tau_1}})$ and $v_{0,1}(\operatorname{en}_{\rho_1,\sigma_1}(\varphi_1(P)))>v_{0,1}(x^{\lambda_{\tau_1}})$ derives in

$$v_{\rho,\sigma}(\operatorname{en}_{\rho_1,\sigma_1}(\varphi_1(P)))>v_{\rho,\sigma}(x^{\lambda_{\tau_1}}).$$

Since $\rho>0$, the result follows.

Proposition 4.4. *Let $\alpha\in\mathcal{R}(P)$. Then there exists j_0 such that $\lambda_\tau=0$ for the π-approximation τ of α up to x^{j_0}.*

Proof. Let $\varphi_0\in\operatorname{Aut}(K((x^{-1/l}))[y])$ be given by $\varphi_0(x^{1/l})=x^{1/l}$ and $\varphi_0(y)=y+\alpha$. We will construct a direction $(\rho_0,\sigma_0)\in]（0,-1),(0,1)[$ such that $v_{\rho_0,\sigma_0}(\varphi_0(P))=0$. In order to achieve this, for each point of $\operatorname{Supp}(\varphi_0(P))$ we consider the direction $(\rho,\sigma)\in](0,-1),(0,1)[$ orthogonal to the segment that joins that point to the origin. The minimum (ρ_0,σ_0) of these directions satisfies $v_{\rho_0,\sigma_0}(\varphi_0(P))=0$. Set $j_0=\dfrac{\sigma_0}{\rho_0}$. We assert that the π-approximation

$$\tau=\sum_{j>j_0}a_jx^j+\pi t^{j_0}$$

of α up to x^{j_0} satisfies $\lambda_\tau = 0$. In fact, we have

$$0 = v_{\rho_0,\sigma_0}(\varphi_0(P)) = v_{\rho_0,\sigma_0}(\varphi_\tau(P)) = v_{\rho_0,\sigma_0}(x^{\lambda_\tau}) = \rho_0\lambda_\tau,$$

where the second equality follows from [12, Proposition 3.9], and the third, from (3.3). □

Proposition 4.5. *Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ be a π-root of P, and let (ρ,σ), λ_τ and φ be as in Proposition 3.3. If τ is also a π-root of Q and we have $\lambda_\tau \geq 0$, then we get*

$$\mathrm{en}_{\rho,\sigma}(\varphi(Q)) = \frac{n}{m}\,\mathrm{en}_{\rho,\sigma}(\varphi(P)) \quad \textit{and} \quad \frac{|D^Q_\tau|}{|D^P_\tau|} = \frac{n}{m}.$$

Proof. Write $\mathrm{Dir}(\varphi(P)) \cap [(\rho,\sigma),(1,1)] = \{(\rho,\sigma) = (\rho_0,\sigma_0) < (\rho_1,\sigma_1) < \cdots < (\rho_k,\sigma_k) = (1,1)\}$. Take $\alpha \in D^P_\tau$ and $0 \leq i \leq k$. Let $j_i = \dfrac{\sigma_i}{\rho_i}$ and let τ_i be the π-approximation of α up to x^{j_i}. Set $\lambda_{\tau_i} = \deg_x(P(\tau_i))$ and $\varphi_i = \varphi_{\tau_i}$. Since we have

$$[\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))] \in K,$$

if $[\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))] \neq 0$, then $v_{0,-1}([\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))]) = 0$, and next, by [12, Proposition 1.13], we obtain

$$\begin{aligned} 0 =& v_{0,-1}([\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))]) \\ \leq& v_{0,-1}(\ell_{\rho_i,\sigma_i}(\varphi(P))) + v_{0,-1}(\ell_{\rho_i,\sigma_i}(\varphi(Q))) - (-1+0). \end{aligned}$$

This in turn implies

$$v_{0,-1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(P)) + v_{0,-1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(Q)) \geq -1,$$

or, equivalently,

$$v_{0,1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(P)) + v_{0,1}(\mathrm{st}_{\rho_i,\sigma_i}(\varphi(Q)) \leq 1;$$

hence in this case we obtain $i = 0$. The bottom line is that, if we want $i > 0$, we must have $[\ell_{\rho_i,\sigma_i}(\varphi(P)), \ell_{\rho_i,\sigma_i}(\varphi(Q))] = 0$, where we must take $\lambda_{\tau_i} > 0$ by Remark 4.3. So we can conclude

$$v_{\rho_i,\sigma_i}(\varphi(P)) = v_{\rho_i,\sigma_i}(\varphi_i(P)) = \rho_i\lambda_{\tau_i} > 0;$$

here the first equality follows from [12, Proposition 3.9] and the second from (3.3). Now, an inductive argument using (4.1), [12, Remark 3.1] and $\mathrm{en}_{\rho_i,\sigma_i}(\varphi(P))) = \mathrm{st}_{\rho_{i+1},\sigma_{i+1}}(\varphi(P)))$, for $i = k, \dots, 1$, proves

$$v_{\rho_i,\sigma_i}(\varphi(Q)) > 0 \quad\text{and}\quad \mathrm{st}_{\rho_i,\sigma_i}(\varphi(Q)) = \frac{n}{m}\,\mathrm{st}_{\rho_i,\sigma_i}(\varphi(P)), \text{ for } i = k, \dots, 1.$$

So we get

$$\mathrm{en}_{\rho_0,\sigma_0}(\varphi(Q)) = \frac{n}{m}\,\mathrm{en}_{\rho_0,\sigma_0}(\varphi(P)) \quad\text{and}\quad \frac{v_{0,1}(\mathrm{en}_{\rho_0,\sigma_0}(\varphi(Q)))}{v_{0,1}(\mathrm{en}_{\rho_0,\sigma_0}(\varphi(P)))} = \frac{n}{m}.$$

This finishes the proof, as we have $\dfrac{|D^Q_\tau|}{|D^P_\tau|} = \dfrac{v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(Q)))}{v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi(P)))}$, by Proposition 3.3. □

In [14] the author chooses a generic element $\xi \in K$ and analyses the roots of $P_\xi = P + \xi$. Instead of speaking of a generic element ξ, we will assume (summing eventually to P an element $\xi \in K$) that any π-root τ of P with $\lambda_\tau = 0$ is such that

(1) $f_{P,\tau}$ has no multiple roots;

(2) $f_{P,\tau}$ and $f_{Q,\tau}$ have no common roots (thus are coprime).

This is possible, since, by (3.3), in the case $\lambda_\tau = 0$ adding ξ to P is the same as adding ξ to the univariate polynomial $f_{P,\tau}(z)$. We also can, and will, assume $(0, 0) \in \mathrm{Supp}(P) \cap \mathrm{Supp}(Q)$.

Remark 4.6. Suppose that τ is a π-root of P with $\lambda_\tau < 0$. Then, by Proposition 3.7, Remark 3.2 and Item (1) we have $|D^P_\tau| = 1$. Moreover, we also get $|D^Q_\tau| = 0$. In fact, take $\alpha \in D^P_\tau$. By Proposition 4.4 there exists j_1 and a π-approximation τ_1 of α up to x^{j_1} such that $\lambda_{\tau_1} = 0$. By Remark 4.3 necessarily $j_1 > j_0$, where j_0 is the order of τ. Let λ be the coefficient of α at x^{j_1}. Then $\pi - \lambda$ divides f_{P,τ_1} and so, by item (2), $\pi - \lambda$ does not divide f_{Q,τ_1}. If τ_1 is not a π-root of Q, then clearly $|D^Q_\tau| = 0$. Otherwise, by Corollary 3.8 applied to τ_1 and Q, we also reach $|D^Q_\tau| = 0$.

Remark 4.7. From the first assertion in the previous remark it follows that for any final π-root τ of P we must have $\lambda_\tau \geq 0$.

Notation 4.8. Let $\alpha = \sum_j a_j x^j \in \mathcal{R}(P)$ and set

$$\delta_\alpha = \min\{\deg_x(\alpha - \beta) : \beta \in \mathcal{R}(Q)\}.$$

Remark 4.9. Note that the π-approximation of α up to x^{δ_α} is also a π-root of Q.

Proposition 4.10. *(Compare [14, Lemma 4.2]) Set $\tau = \sum_{j>\delta} a_j x^j + \pi x^{\delta_\alpha}$. Then τ is a final π-root of P.*

Proof. Since clearly τ is a π-root of P, we only need to prove that τ is a final π-root of P, *i.e.*, that we have $\deg(f_{P,\tau}) > 1$ and that $f_{P,\tau}$ has no multiple roots. By Remark 4.6 we know $\lambda_\tau \geq 0$. By Item (1) above, we also know that when $\lambda_\tau = 0$, the polynomial $f_{P,\tau}$ has no multiple roots. If $\lambda_\tau > 0$, then $f_{P,\tau}$ also does not have multiple roots: otherwise by Proposition 4.2 there exists $\beta \in \mathcal{R}(Q)$ such that $\deg_x(\alpha - \beta) < \delta_\alpha$, contradicting the definition of δ_α. Finally, by Proposition 4.5 we know that m divides $|D_\tau^P| = \deg(f_{P,\tau})$, and so, we obtain $\deg(f_{P,\tau}) > 1$, which concludes the proof. □

5 Major and minor final π-roots

A final π-root τ of P is called a **minor final π-root of** P if $\lambda_\tau = 0$, and it is called a **major final π-root of** P if $\lambda_\tau > 0$. The set of minor final π-roots of P is denoted by P_m, while the set of final major π-roots of P is denoted by P_M.

Note that we have

$$\mathcal{R}(P) = \bigcup_{\tau \in P_m \cup P_M} D_\tau^P,$$

since, by Proposition 4.10, every root $\alpha \in \mathcal{R}(P)$ is associated with a final π-root of P (that we will call the **final π-root of** P **associated with**

α), here $\lambda_\tau \geq 0$ by Remark 4.7. Note also that if $\tau \neq \tau_1$ are final π-roots, then $D^P_\tau \cap D^P_{\tau_1} = \emptyset$. In fact, assume by contradiction $\alpha \in D^P_\tau \cap D^P_{\tau_1}$, and take for example $\delta_\tau < \delta_{\tau_1}$, which means that τ is a better approximation of α than τ_1. Then, since the multiplicity of any factor of f_{P,τ_1} is one, by Remark 3.9 we get $|D^P_\tau| \leq 1$, which contradicts the fact that τ is a final π-root of P.

Remark 5.1. Given a final π-root τ of P take $\alpha \in D^P_\tau$. Then, by Proposition 4.10, the π-approximation of α up to x^{δ_α} is a final π-root, and, since $D^P_\tau \cap D^P_{\tau_1} = \emptyset$ for any other final π-root τ_1 of P, necessarily τ is the π-approximation of α up to x^{δ_α}. This is equivalent to $\delta_\tau = \delta_\alpha$.

Proposition 5.2. *Let τ be a final π-root of P, let $\varphi = \varphi_\tau$ and set $\lambda^Q_\tau = \deg_x(Q(\tau))$. Then we have the following.*

(1) *If τ is a minor final π-root of P, then we have*

a) $\lambda^Q_\tau = 0$,

b) $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] = 0$,

c) $\delta_\tau < -1$.

(2) *If τ is a major final π-root of P, then we have*

a) $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \neq 0$,

b) τ is a major final π root of Q,

c) $\lambda^Q_\tau = \dfrac{n}{m} deg_x(P(\tau))$,

d) $\delta_\tau > -1$.

Proof. By Remarks 4.9 and 5.1, any final π-root τ of P is also a π-root of Q. We will use this fact for (1)a) and (2)b).

(1) By Proposition 4.5, since $\lambda_\tau \geq 0$, we have

$$m\, \mathrm{en}_{\rho,\sigma}(\varphi(Q)) = n\, \mathrm{en}_{\rho,\sigma}(\varphi(P)),$$

and so we obtain

$$\rho\lambda^Q_\tau = v_{\rho,\sigma}(\varphi(Q)) = \frac{n}{m} v_{\rho,\sigma}(\varphi(P)) = \frac{n}{m}\rho\lambda^P_\tau = 0,$$

where the first and third equality follow from (3.3). This implies $\lambda_\tau^Q = \deg_x(Q(\tau)) = 0$, thus proving a). Moreover, by [12, Proposition 2.1(1)] we know that the vanishing of $v_{\rho,\sigma}(\varphi(Q))$ and $v_{\rho,\sigma}(\varphi(P))$ implies

$$[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] = 0,$$

this proves item b). Now assume by contradiction $\frac{\sigma}{\rho} = \delta_\tau \geq -1$, which implies $\rho + \sigma \geq 0$. Then, by [12, Proposition 1.13], we have

$$0 = v_{\rho,\sigma}([\varphi(P), \varphi(Q)]) \leq v_{\rho,\sigma}(\varphi(Q)) + v_{\rho,\sigma}(\varphi(P)) - (\rho+\sigma) = -(\rho+\sigma) \leq 0,$$

so we have equality and, again by [12, Proposition 1.13], we also obtain $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \neq 0$. But this contradicts item b) and thus proves $\delta_\tau < -1$, that is, part c).

(2) By Remarks 4.9 and 5.1, we know that τ is a π-root of Q and, that for any $\alpha \in D_\tau^P$, we have

$$\delta_\tau = \min\{\deg_x(\alpha - \beta) | \beta \in \mathcal{R}(Q)\}.$$

Hence, by Proposition 4.2(2), we obtain $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \neq 0$, which proves a). Moreover, by Proposition 4.2(1) with Q and P interchanged, $f_{Q,\tau}$ has no multiple roots. On the other hand, by Proposition 4.5, we have

$$|D_\tau^Q| = \frac{n}{m}|D_\tau^P| > 1,$$

and so τ is a final π-root of Q. Again Proposition 4.5 and Equality (3.3) yield

$$\begin{aligned} \rho \deg_x Q(\tau) =& \rho\lambda_\tau^Q = v_{\rho,\sigma}(\varphi(Q)) = \frac{n}{m} v_{\rho,\sigma}(\varphi(P)) \\ =& \frac{n}{m}\rho\lambda_\tau^P = \rho\frac{n}{m}\deg_x(P(\tau)), \end{aligned}$$

and so $\deg_x Q(\tau) = \frac{n}{m}\deg_x P(\tau)) > 0$, which finishes the proof of b) and c). It remains to check the condition $\delta_\tau > -1$. Assume by contradiction $\frac{\sigma}{\rho} = \delta_\tau \leq -1$. Then $\rho + \sigma \leq 0$, and so

$$v_{\rho,\sigma}(\varphi(Q)) + v_{\rho,\sigma}(\varphi(P)) - (\rho+\sigma) \geq \rho\lambda_\tau^P\left(1 + \frac{n}{m}\right) > 0 = v_{\rho,\sigma}[\varphi(P), \varphi(Q)],$$

which, by [12, Proposition 1.13], implies $[\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] = 0$. This contradicts item a) and thus finishes the proof of part d). □

6 Intersection number and major roots

In this section we first obtain in Theorem 6.2 the same formula for I_M as in [14, Theorem 5.1]. Then we explain how to compute I_M for the families found in [7].

Lemma 6.1. *For τ a final π-root of P, we have $\lambda_\tau^Q = \deg_x(Q(\tau)) = \deg_x(Q(\alpha))$ whenever $\alpha \in D_\tau^P$.*

Proof. We assert that $f_{P,\tau}(z)$ and $f_{Q,\tau}(z)$ have no common roots. In fact, assume on the contrary that $z - s$ is a common factor. If τ is a major final root, then

$$z - s \mid [\lambda_\tau f_{P,\tau}(z), \lambda_\tau^Q f_{Q,\tau}(z)] = [\ell_{\rho,\sigma}(\varphi(Q)), \ell_{\rho,\sigma}(\varphi(P))] \in K^\times,$$

a contradiction. Whereas, if τ is a minor root, then the choice of ξ guarantees that $f_{P,\tau}$ and $f_{Q,\tau}$ have no common roots.

Note that if the coefficient of x^{j_0} in α is s, then $f_{P,\tau}(s) = 0$, since otherwise $\pi - s$ does not divide $f_{P,\tau}(\pi)$ and Corollary 3.8 leads to a contradiction. Hence, by the assertion, we get $f_{Q,\tau}(s) \neq 0$, and from Proposition 3.11, we obtain $\deg_x(Q(\tau)) = \deg_x(Q(\alpha))$. □

Theorem 6.2. *For $I_M = \sum_{\tau \in P_M} |D_\tau^{P_\xi}| \lambda_\tau^Q$ we have $I_M = I(P,Q)$.*

Proof. From the well know equality $\operatorname{Res}_y(P,Q) = \prod_{\alpha \in \mathcal{R}(P)} Q(\alpha)$ we pass to

$$I(P,Q) = \deg_x \prod_{\alpha \in \mathcal{R}(P)} Q(\alpha) = \sum_{\alpha \in \mathcal{R}(P)} \deg_x(Q(\alpha)) \tag{6.1}$$

$$= \sum_{\tau \in P_m \cup P_M} \sum_{\alpha \in D_\tau^P} \deg_x(Q(\alpha)). \tag{6.2}$$

Using Lemma 6.1 we arrive to

$$\begin{aligned} I(P,Q) &= \sum_{\tau\in P_m\cup P_M}\ \sum_{\alpha\in D_\tau^P} \deg_x(Q(\alpha)) \\ &= \sum_{\tau\in P_M} |D_\tau^{P_\xi}|\lambda_\tau^Q + \sum_{\tau\in P_m} |D_\tau^{P_\xi}|\lambda_\tau^Q = \sum_{\tau\in P_M} |D_\tau^{P_\xi}|\lambda_\tau^Q, \end{aligned}$$

since $\lambda_\tau^Q = 0$ if $\tau \in P_m$. □

A root $\alpha \in \mathcal{R}(P)$ is called a **minor root** if the associated final π-root τ is a minor final π-root; it is called a **major root** if τ is a major final π-root.

Proposition 6.3. *Let τ be an approximate π-root of P of order $j_0 \le 0$, with $\lambda_\tau \ge 0$, and let $(\rho,\sigma) = \operatorname{dir}(j_0)$. If $v_{1,-1}(\mathrm{en}_{\rho,\sigma}(\varphi_\tau(P))) > 0$, then any root $\alpha \in D_\tau^P$ is a minor root.*

Proof. The hypotheses guarantee that $(\varphi_\tau(P), \varphi_\tau(Q))$ and (ρ,σ) satisfy the hypotheses of Proposition 2.1 (for instance $(\rho,\sigma) \in \,](0,-1),(1,0)]$, because of $j_0 \le 0$). If $v_{\rho,\sigma}(\varphi_\tau(P)) = \rho\lambda_\tau = 0$, then τ is a minor final π-root and the result is true. Else we have $v_{\rho,\sigma}(\varphi_\tau(P)) = \rho\lambda_\tau > 0$, since $\lambda_\tau \ge 0$. Take $\alpha \in D_\tau^P$. By Proposition 5.2 it suffices to prove $\delta_\alpha < -1$. By Propositions 2.1 and 4.2 we have $\delta_\alpha < \delta_\tau = j_0$, so the result is clear when $\delta_\tau \le -1$. Assume $\delta_\tau > -1$. In this case we have $\rho+\sigma > 0$. Using Proposition 2.2 and Equality (3.3) we obtain $f_{P,\tau}(z) = \varsigma(z-\mu)^{mb}$ for some $\varsigma,\mu \in K^\times$, where $b = \dfrac{1}{m}v_{0,1}(\mathrm{en}_{\rho,\sigma}(\varphi_\tau(P))) = \dfrac{|D_\tau^P|}{m}$ (see Proposition 3.3). Hence, by Proposition 3.7, there exists $j_1 < j_0$ such that for the π-root

$$\tau_1 = \sum_{j>j_0} a_j x^j + \mu x^{j_0} + \pi x^{j_1}$$

we have $D_{\tau_1}^P = D_\tau^P$. If $j_1 \le -1$, then we finish the proof immediately after applying the above argument with τ replaced by τ_1 since we must have $\lambda_{\tau_1} \ge 0$ (in fact, if $\lambda_{\tau_1} < 0$, then, by Remark 4.6, we get $|D_{\tau_1}^P| = 1$, which is impossible because of $bm = |D^P_\tau|$). Assume now $j_1 > -1$ and set $(\rho_1, \sigma_1) = \operatorname{dir}(j_1)$. By Proposition 2.2 ρ_1 divides l, and so $j_1 \in \frac{1}{l}\mathbb{Z}$. Hence, if $j_0 = -\frac{k}{l}$ for some $0 \le k \le l$, then $-j_1 \in \{\frac{k+1}{l}, \frac{k+2}{l}, \dots, \frac{l-1}{l}, \frac{l}{l}\}$, so, after repeating the same procedure a finite number of times, say t, we arrive at $\delta_\alpha < j_t \le -1$, as desired. □

Proposition 6.4. *Let a, b satisfy Equalities* (4.1)*. There exist ma minor roots α of P with $\deg_x(\alpha) = 1$ and leading term $-x$, and mb roots β of P with $\deg_x(\beta) \le 0$.*

Proof. Take $\tau_0 = \pi x^0$. Then we have $j_0 = 0$, $\operatorname{dir}(j_0) = (1, 0)$ and $\varphi_{\tau_0} = \mathrm{id}$. From the first equality in (4.2), we get

$$\operatorname{en}_{\rho,\sigma}(\varphi_{\tau_0}(P)) = \operatorname{en}_{1,0}(P) = m(a, b),$$

and by Proposition 3.3, we obtain $|D^P_{\tau_0}| = mb$. Since $\deg_x(\beta) \le j_0 = 0$ for all $\beta \in D^P_{\tau_0}$, this yields mb roots with $\deg_x(\beta) \le 0$. On the other hand, by Proposition 3.7, with $\tau = \pi x$, $\lambda = -1$ and $\varphi_1(y) = y - x$, there exists $j_1 < 1$ such that the π-root $\tau_1 = -x + \pi x^{j_1}$ satisfies $|D^P_{\tau_1}| = ma$ since $f_{P,\tau}(z) = (z+1)^{ma} z^{mb}$; therefore the multiplicity of $\lambda = -1$ is ma. Moreover, by (3.5) and the first equality in (4.1), we have

$$\operatorname{en}_{\rho_1,\sigma_1}(\varphi_1(P)) = \operatorname{st}_{1,1}(\varphi_1(P)) = m(b, a),$$

and so $v_{1,-1}(\operatorname{en}_{\rho_1,\sigma_1}(\varphi_1(P))) > 0$. Then every root $\alpha \in D^P_{\tau_1}$ is a minor root. □

Following [14], the minor roots in Proposition 6.4 are called **top minor roots**.

Proposition 6.5. *Let $\alpha \in \mathcal{R}(P)$ be a major root, let τ be the associated final π-root and let $(\rho, \sigma) = \operatorname{dir}(\delta_\alpha)$. Then $\left(\frac{1}{m}\operatorname{en}_{\rho,\sigma}(\varphi_\tau(P)), (\rho, \sigma)\right)$ is a regular corner of type I of $(\varphi_\tau(P), \varphi_\tau(Q))$ (see [12, Definition 5.5] and the discussion above Remark 5.9 in [12] for the classification of regular corners).*

Proof. Item (3) of [12, Definition 5.5] holds by hypothesis, Item (1) holds by the very definition of π-root, Proposition 6.3 and [12, Theorem 2.6(4)], and Item (2) holds by Remark 3.6. Moreover, Proposition 5.2(2)a) proves that $\left(\frac{1}{m}\operatorname{en}_{\rho,\sigma}(\varphi_\tau(P)),(\rho,\sigma)\right)$ is of type I. □

Proposition 6.6. *Let $j_0 < j_1 < \dots < j_k \in \frac{1}{l}\mathbb{Z}$ and let $(\rho,\sigma) = \operatorname{dir}(j_0)$. Consider the automorphism φ of $L^{(l)}$ defined by*

$$\varphi(x^{1/l}) = x^{1/l} \quad \textit{and} \quad \varphi(y) = y + \sum_{i=1}^{k} a_i x^{j_i}.$$

Let $A = ((a/l,b),(\rho,\sigma))$ be a regular corner of $(\varphi(P),\varphi(Q))$. Then the following facts hold.

(1) *$\tau = \sum_{i=1}^{k} a_i x^{j_i} + \pi x^{j_0}$ is a π-root of P and Q.*

(2) *If A is of type Ib, then τ is a final major π-root of P and Q, with*

$$|D_\tau^P| = mb \quad \textit{and} \qquad |D_\tau^Q| = nb; \tag{6.3}$$

moreover, if $\operatorname{st}_{\rho,\sigma}(\varphi(Q)) = (k/l,0)$ for some $1 \le k < l - a/b$, then $\lambda_\tau^Q = \frac{k}{l}$.

Proof. (1) By Items (1) and (3) of [12, Definition 5.5], we have $A = \frac{1}{m}\operatorname{en}_{\rho,\sigma}(\varphi(P))$ and $b \ge 1$. Hence, by Equalities (3.2) and (3.3) we conclude $\deg(f_{P,\tau}) > 0$ and that τ is a π-root of P. Since by [12, Corollary 5.7] and Remark 4.1 the equality $A = \frac{1}{n}\operatorname{en}_{\rho,\sigma}(\varphi(Q))$ holds and (Q,P) is an (n,m)-pair, we infer that τ is also a π-root of Q.

(2) The two expressions for A obtained in the proof of Item (1), combined with Equality (3.2) and the corresponding equality for Q yield the equalities in (6.3). Since A is of type Ib, we have

$$[\ell_{\rho,\sigma}(\varphi(P)),\ell_{\rho,\sigma}(\varphi(Q))] \neq 0,$$

and so, by Proposition 4.2(1), the polynomial $f_{P,\tau}$ has no multiple root. Moreover, using again Equality (3.2) and Equality (6.3) we obtain $\deg(f_{P,\tau}) = mb > 1$. This proves that τ is a major final π-root of P, and then, by Proposition 5.2(2)b), also of Q. Finally, assuming $\mathrm{st}_{\rho,\sigma}(\varphi(Q)) = (k/l, 0)$, Equality (3.3) for Q implies $\rho\lambda_\tau^Q = v_{\rho,\sigma}(\varphi(Q)) = \rho\frac{k}{l}$, from which the last assertion follows, as $\rho \neq 0$. □

Example 6.7. Consider the family F_1 of [7], corresponding to an (m, n)-pair (P_0, Q_0) as in [12, Corollary 5.21]:

$$A_0 = (4, 12),\ A_0' = (1, 0),\ A_1 = (7/4, 3),\ k = 1,\ m = 2j+3,\ n = 3j+4. \tag{6.4}$$

Then (P_0, Q_0) has the shape given in Figure 2 and we get $\ell_{-2,1}(P) = R^{4m}$ for a $(-2, 1)$-homogeneous element $R = \lambda_0(\lambda_1 y - xy^3)$ with $\lambda_0, \lambda_1 \neq 0$.

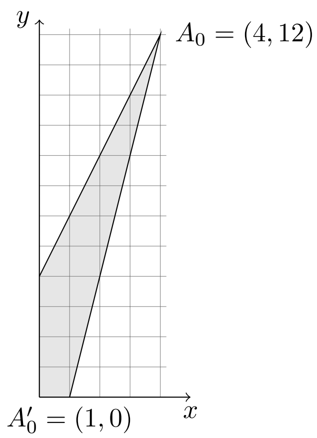

Figure 2: The shape of (P_0, Q_0)

In fact, by (6.4), the edge from A_0 to A_0' is determined. So we only need to prove

$$(\rho, \sigma) = \mathrm{Succ}_{P_0}(1, 0) = \mathrm{Succ}_{Q_0}(1, 0) = (-2, 1)$$

and $\frac{1}{m}\operatorname{en}_{-2,1}(P) = (0,4)$. From [12, Corollary 5.21(4)] we obtain $(-1,1) < (\rho,\sigma) < (-1,0)$. Moreover, by the second equality in [7, (2.13)] we have

$$q_0 = \frac{v_{4,-1}(4,12)}{\gcd(v_{4,-1}(4,12),4-1)} = \frac{4}{\gcd(4,3)} = 4.$$

On the other hand, at the beginning of [5, Subsection 2.4] it is shown the equality

$$\operatorname{en}_{\rho,\sigma}(F_0) = \frac{p_0}{q_0}\frac{1}{m}\operatorname{en}_{\rho_0,\sigma_0}(P_0),$$

and therefore, by [12, Corollary 7.2], there exists a (ρ,σ)-homogeneous element R such that $\ell_{\rho,\sigma}(P) = R^{4m}$. This is only possible if $(\rho,\sigma) = (-k,1)$ for some $k \in \mathbb{N}$, with $k \geq 2$. But $k \geq 3$ leads to $v_{\rho,\sigma}(P_0) \leq 0$ and then to $\deg_y(P_0(0,y)) \leq 0$, which contradicts [13, Proposition 10.2.6]. Therefore we have $k = 2$, $R = \lambda_0(\lambda_1 y - xy^3)$ and hence we get

$$\frac{1}{m}\operatorname{en}_{-2,1}(P) = 4\operatorname{en}_{-2,1}(R) = (0,4),$$

as desired. Since $P = \psi(P_0)$ and $Q = \psi(Q_0)$, where $\psi(y) = y$ and $\psi(x) = x + y$ (see the beginning of Section 4), the shape of P is as in Figure 3, and P is a monic polynomial in y of degree $16m$.

Write $\ell_{4,-1}(P) = x^m g(z)^m$, where $z = x^{1/4}y$. By [5, Theorem 2.20(6)] and the condition $v_{1,-1}(A_0') > 0$, we know that $(A_0,(\rho,\sigma)) = ((4,12),(4,-1))$ is a regular corner of type IIb) of (P,Q). Hence, by item (8) of the same theorem, we get $v_{0,1}(A_1) = \frac{m_\lambda}{m}$, where m_λ is the multiplicity of $z-\lambda$ in $\mathfrak{p}_0(z) = g(z)^m$. Since we have $v_{0,1}(A_1) = 3$, by [5, Remarks 3.8 and 3.9] we get

$$g(z) = \lambda_0(z^4 - \lambda_1^4)^3,$$

for some $\lambda_0, \lambda_1 \in K^\times$. We obtain then

$$\ell_{4,-1}(P_\xi) = \lambda_0 x^m (z-\lambda_1)^{3m}(z-i\lambda_1)^{3m}(z+\lambda_1)^{3m}(z+i\lambda_1)^{3m},$$

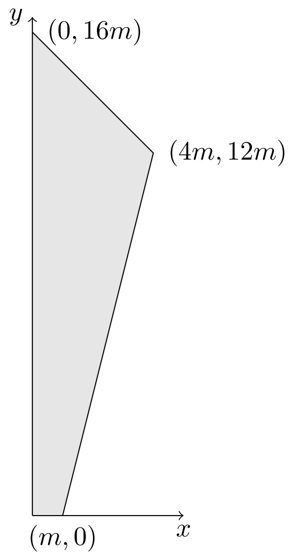

Figure 3: The shape of P_ξ

and so, we have four final major π-roots

$$\tau_0 = \lambda_1 x^{1/4} + \pi x^{\delta}, \qquad \tau_1 = i\lambda_1 x^{1/4} + \pi x^{\delta},$$
$$\tau_2 = -\lambda_1 x^{1/4} + \pi x^{\delta}, \qquad \tau_3 = -i\lambda_1 x^{1/4} + \pi x^{\delta},$$

where $\delta = \sigma/\rho$, with $(\rho,\sigma) = \operatorname{dir}\left(m\left(\frac{7}{4},3\right) - \left(\frac{3}{4},1\right)\right)$. Here $A_1 = \left(\frac{7}{4},3\right)$ is the same final corner (see [5, Definition 2.18]) for all major final roots, corresponding to the regular corner $(A_1,(\rho,\sigma))$ of type Ib) of each of the four (m,n)-pairs $(\varphi_{\tau_j}(P),\varphi_{\tau_j}(Q))$. By the first equality in (6.3), there are $3m$ roots of P associated to each of these major roots, and by Proposition 6.4, the remaining $4m$ roots of P are minor roots. Now we can compute

$$I_M = \sum_{\tau \in P_M} |D_\tau^P| \lambda_\tau^Q = \sum_{j=0}^{3} |D_{\tau_j}^P| \lambda_{\tau_j}^Q$$
$$= 4 \cdot mb \cdot \frac{k}{l} = 4 \cdot m \cdot 3 \cdot \frac{1}{4} = 3m = 3(2j+3).$$

7 Intersection number and minor roots

In this section we obtain an inequality for I_m in Theorem 7.3, as opposed to the equality in [14, Theorem 4.7], whose proof has a serious gap. We also show how to compute I_m for the families of [7]. For the sake of brevity we write P_x, Q_x, P_y and Q_y instead of the partial derivatives $\partial_x P$, $\partial_x Q$, $\partial_y P$ and $\partial_y Q$.

Lemma 7.1. *Let (P,Q) be as above, (ρ,σ) be a direction, with $\rho \neq 0$, and $\alpha \in \mathcal{R}(P)$. Write $\ell_{\rho,\sigma}(P) = x^u g(z)$ with $z = x^{-\sigma/\rho} y$. The following facts hold.*

(1) *If $\deg(g) > 0$, then $\ell_{\rho,\sigma}(P_y) = x^{u-\sigma/\rho} g'(z)$.*

(2) *α is a minor root if and only if $\deg_x(Q(\alpha)) = 0$.*

(3) *Let $\beta \in \mathcal{R}(P_y)$. There exists $\tau \in P_m$ such that $\beta \in D_\tau^{P_y}$ if and only if $\deg_x(P(\beta)) = 0$.*

(4) *If α is a minor root, then $\deg_x(P_y(\alpha)) = -\delta_\alpha$.*

(5) *Let $\tau \in P_m$ and assume that $f_{P_y,\tau}$ and $f_{Q_y,\tau}$ are coprime. Then $\deg_x(Q_y(\beta)) = -\delta_\tau$ for all $\beta \in D_\tau^{P_y}$.*

(6) *Let $\tau \in P_m$ and assume that $f_{P_y,\tau}$ and $f_{Q_y,\tau}$ are not coprime. Then there exists $\beta \in D_\tau^{P_y}$ such that $\deg_x(Q_y(\beta)) < -\delta_\tau$.*

Proof. (1) This follows from the fact that the morphism ∂_y satisfies $\partial_y(x^i y^j) = j x^i y^{j-1}$ for $j > 0$: in fact we have

$$v_{\rho,\sigma}(\partial_y(x^i y^j)) = v_{\rho,\sigma}(x^i y^j) - \sigma.$$

Hence $\ell_{\rho,\sigma}(\partial_y P) = \partial_y \ell_{\rho,\sigma}(P)$ when $\partial_y \ell_{\rho,\sigma}(P) \neq 0$, and so

$$\ell_{\rho,\sigma}(P_y) = \partial_y(x^u g(z)) = x^{u-\sigma/\rho} g'(z),$$

because of $\deg(g) > 0$.

(2) By Items (1)a) and (2)c) of Proposition 5.2, we know that α is a minor root if and only if we have $\lambda_\tau^Q = 0$ for the π-root τ associated to α. This proves (2), since we have $\lambda_\tau^Q = \deg_x(Q(\alpha))$ by Lemma 6.1.

(3) If we define

$$\delta_\beta = \min\{\deg_x(\alpha - \beta) | \alpha \in \mathcal{R}(P)\},$$

and

$$\beta = \sum_{j>\delta_\beta} a_j x^j + \lambda x^{\delta_\beta} + \sum_{j<\delta_\beta} a_j x^j,$$

then $\tau = \sum_{j>\delta_\beta} a_j x^j + \pi x^{\delta_\beta}$ is a π-root of P. From Remark 3.4 we obtain $0 < |D_\tau^{P_y}| = |D_\tau^P| - 1$, and so, by Remark 4.6 we get $\lambda_\tau^P \geq 0$. Take $\alpha \in D_\tau^P$ and let τ_1 be the final π-root of P associated with α. We have $\delta_\alpha \leq \delta_\beta$ (since $\delta_\beta < \delta_\alpha$ implies $|D_\tau^P| = 1$) and hence $\lambda_{\tau_1} \leq \lambda_\tau$, so $\lambda_\tau^P = 0$ if and only if $\tau = \tau_1$ is a final minor π-root of P.

We claim the equality $\lambda_\tau^P = \deg_x(P(\beta))$. In fact, we have $f_{P,\tau}(\lambda) \neq 0$ since otherwise, by Proposition 3.7, there exists $j_1 < \delta_\beta$ such that the π-approximation of β up to j_1 is a π-root of P, contradicting the minimality of δ_β. Hence, by Proposition 3.11, we have $\deg_x(P(\beta)) = \lambda_\tau^P \geq 0$. Therefore, if $\deg_x(P(\beta)) = 0$, then $\beta \in D_\tau^{P_y}$ and $\tau \in P_m$. On the other hand, if $\beta \in D_{\tau_2}^{P_y}$ for some $\tau_2 \in P_m$, then $\delta_\beta \leq \delta_{\tau_2}$, hence $0 \leq \lambda_\tau \leq \lambda_{\tau_2} = 0$, and so $0 = \lambda_\tau^P = \deg_x(P(\beta))$, as desired.

(4) Let $\tau = \sum_{j>\delta_\alpha} a_j x^j + \pi x^{\delta_\alpha}$ be the minor final π-root of P associated with α. Write

$$\alpha = \sum_{j>\delta_\alpha} a_j x^j + \lambda x^{\delta_\alpha} + \sum_{j<\delta_\alpha} a_j x^j.$$

Since $f_{P,\tau}(\lambda) = 0$, and $f_{P,\tau}$ has no multiple roots, we have $f'_{P,\tau}(\lambda) \neq 0$. But by Item (1) we have $f_{P_y,\tau} = f'_{P,\tau}$, and, by Proposition 3.11, we obtain $\lambda_\tau^{P_y} = \deg_x(P_y(\tau)) = \deg_x(P_y(\alpha))$. Using again Item (1) we get $\lambda_\tau^{P_y} = \lambda_\tau - \sigma/\rho$, and the result follows immediately since $\lambda_\tau = 0$.

(5) Let $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$ and take $\beta \in D_\tau^{P_y}$. Set

$$\beta = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \sum_{j<j_0} a_j x^j.$$

By Corollary 3.8, we have $f_{P_y,\tau}(\lambda) = 0$. As $f_{P_y,\tau}$ is coprime with $f_{Q_y,\tau}$, we get $f_{Q_y,\tau}(\lambda) \neq 0$. By Proposition 3.11, we obtain $\lambda_\tau^{Q_y} = \deg_x(Q_y(\tau)) = \deg_x(Q_y(\beta))$ and by Item (1) we have $\lambda_\tau^{Q_y} = \lambda_\tau^Q - \sigma/\rho$. The result follows immediately from the equality $\lambda_\tau^Q = 0$.

(6) Write $\tau = \sum_{j>j_0} a_j x^j + \pi x^{j_0}$. Let $\lambda \in K$ be such that $f_{Q_y,\tau}(\lambda) = 0 = f_{P_y,\tau}(\lambda)$. By Proposition 3.7, there exist $j_1, j_2 < j_0$ such that $\tau_1 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_1}$ is a π-root of P_y and $\tau_2 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_2}$ is a π-root of Q_y. Take $j_3 = \max\{j_1, j_2\}$. Then $\tau_3 = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \pi x^{j_3}$ is a π-root of Q_y and P_y. If $\beta \in D_{\tau_3}^{P_y}$, then

$$\beta = \sum_{j>j_0} a_j x^j + \lambda x^{j_0} + \sum_{j<j_0} a_j x^j.$$

With $T = \lambda x^{j_0} + \sum_{j<j_0} a_j x^j$ we have

$$\begin{aligned} Q_y(\beta) &= \mathrm{ev}_{y=T}(\varphi_\tau(Q_y)) = \mathrm{ev}_{y=\lambda x^{j_0}}(\ell_{\rho,\sigma}(\varphi_\tau(Q_y))) + R \\ &= x^{\lambda_\tau^{Q_y}} f_{Q_y,\tau}(\lambda) + R \end{aligned}$$

for some R with $v_{\rho,\sigma}(R) < v_{\rho,\sigma}(\varphi_\tau(Q_y))) = \rho\lambda_\tau^{Q_y}$. Since $f_{Q_y,\tau}(\lambda) = 0$, we obtain

$$\rho \deg_x(Q_y(\beta)) = v_{\rho,\sigma}(Q_y(\beta)) < \rho\lambda_\tau^{Q_y}.$$

However, from Item (1) we have $\lambda_\tau^{Q_y} = \lambda_\tau^Q - \sigma/\rho$, and since $\lambda_\tau^Q = 0$ we conclude

$$\deg_x(Q_y(\beta)) < -\sigma/\rho,$$

as desired. □

Lemma 7.2. *For any $\alpha \in K((x^{-1/l}))$ we have*

$$Q_y(\alpha)\frac{d}{dx}P(\alpha) - P_y(\alpha)\frac{d}{dx}Q(\alpha) \in K^{\times}.$$

Proof. This follows directly from $\frac{d}{dx}P(\alpha) = P_x(\alpha)+P_y(\alpha)\frac{d\alpha}{dx}$, $\frac{d}{dx}Q(\alpha) = Q_x(\alpha) + Q_y(\alpha)\frac{d\alpha}{dx}$ and the Jacobian condition. □

Theorem 7.3. *For $I_m = 1 - \sum_{\tau \in P_m}(\delta_\tau + 1)$ we get $I_m \le I(P,Q)$. We also have*

$$I(P, P_yQ) = \deg(P) - \sum_{\tau \in P_m} |D_\tau^P|(1 + \delta_\tau). \qquad (7.1)$$

Proof. It suffices to prove (7.1) and

$$I(P, P_y) \le \deg(P) - 1 - \sum_{\tau \in P_m} (|D_\tau^P| - 1)(\delta_\tau + 1). \qquad (7.2)$$

In fact, Equality (7.1) and Inequality (7.2) yield

$$I(P,Q) = I(P, P_yQ) - I(P, P_y) \ge 1 - \sum_{\tau \in P_m} (\delta_\tau + 1),$$

as desired.

Proof of Equality (7.1). By Lemma 7.2, for each $\alpha \in \mathcal{R}(P)$ we have $P_y(\alpha)\frac{d}{dx}Q(\alpha) \in K^{\times}$. Moreover, by Lemma 7.1(2), if α is a major root, then $\deg_X(Q(\alpha)) = \lambda_\tau^Q > 0$, and so $\deg_x(P_y(\alpha)Q(\alpha)) = 1$. On the other hand, if α is a minor root, then by Proposition 5.2(1)a), Lemma 6.1 and Lemma 7.1(4) we obtain

$$\deg_x(P_y(\alpha)Q(\alpha)) = \deg_x(P_y(\alpha)) = -\delta_\alpha = -\delta_\tau,$$

where τ is the minor final π-root associated with α. Using these facts we obtain

$$\begin{aligned}
I(P,P_yQ) &= \sum_{\alpha\in\mathcal{R}(P)} \deg_x(P_y(\alpha)Q(\alpha))\\
&= \sum_{\tau\in P_m}\sum_{\alpha\in D_\tau^P} \deg_x(P_y(\alpha)Q(\alpha)) + \sum_{\tau\in P_M}\sum_{\alpha\in D_\tau^P} \deg_x(P_y(\alpha)Q(\alpha))\\
&= \sum_{\tau\in P_m} |D_\tau^P|(-\delta_\tau) + \sum_{\tau\in P_M} |D_\tau^P| + \sum_{\tau\in P_m} |D_\tau^P| - \sum_{\tau\in P_m} |D_\tau^P|\\
&= \deg(P) - \sum_{\tau\in P_m} |D_\tau^P|(1+\delta_\tau),
\end{aligned}$$

where the first equality is justified as in the proof of Theorem 6.2.

Proof of Inequality (7.2). By Lemma 7.2, for each $\beta\in\mathcal{R}(P_y)$, we have $Q_y(\beta)\dfrac{d}{dx}P(\beta)\in K^\times$. Define

$$P_{y,m} = \{\beta\in\mathcal{R}(P_y):\ \text{there is a minor final } \pi\text{-root } \tau \text{ with } \beta\in D_\tau^{P_y}\}.$$

Then, by Lemma 7.1(3), if β is not in $P_{y,m}$, we have $\deg_x(Q_y(\beta)P(\beta)) = 1$. On the other hand, if β is in $P_{y,m}$, then by Items (3), (5) and (6) of Lemma 7.1 we obtain

$$\deg_x(P(\beta)Q_y(\beta)) = \deg_x(Q_y(\beta)) \le -\delta_\tau,$$

where τ is the minor final π-root associated with β. Using these facts we obtain

$$
\begin{aligned}
I(P_y, PQ_y) &= \sum_{\beta\in\mathcal{R}(P_y)} \deg_x(P(\beta)Q_y(\beta))\\
&= \sum_{\tau\in P_m}\sum_{\beta\in D_\tau^{P_y}} \deg_x(P(\beta)Q_y(\beta)) + \sum_{\beta\notin P_{y,m}} \deg_x(P(\beta)Q_y(\beta))\\
&\le \sum_{\tau\in P_m} |D_\tau^{P_y}|(-\delta_\tau) + \deg(P_y) - \sum_{\tau\in P_m} |D_\tau^{P_y}|\\
&= \deg(P) - 1 - \sum_{\tau\in P_m} |D_\tau^{P_y}|(1+\delta_\tau)\\
&= \deg(P) - 1 - \sum_{\tau\in P_m} (|D_\tau^{P}|-1)(1+\delta_\tau),
\end{aligned}
\tag{7.3}
$$

where the last equality follows from Remark 3.4.

Since the Jacobian condition implies

$$
\operatorname{Res}_y(P_y, Q_y)\operatorname{Res}_y(P_y, P_x) = \operatorname{Res}_y(P_y, Q_yP_x) = \prod_{\beta\in\mathcal{R}(P_y)} Q_y(\beta)P_x(\beta) = 1,
$$

we have $I(P_y, Q_y) = 0$, and thus (7.3) yields Inequality (7.2). □

Example 6.7 (continuation). In the case of family F_1 of [7] there is only one minor root τ corresponding to the remaining $4m$ roots of P_ξ, and $\delta_\tau = -3$. Hence we have $I_m = 1-(-3+1) = 3$. Since $I_M = 3(2j+3)$, we have $I_m < I_M$ (which is compatible with Theorem 7.3 and doesn't allow us to disregard this family).

In fact, consider the diagram in Figure 4, where flip : $K[x,y] \to K[x,y]$ is given by $\operatorname{flip}(x) = y$ and $\operatorname{flip}(y) = -x$, and the three morphisms $\varphi, \varphi_1, \widetilde{\varphi} : K[x,x^{-1},y] \to K[x,x^{-1},y]$ are characterized by

$$
\begin{aligned}
&\varphi(x) = x+y, &\quad &\varphi_1(x) = x, &\quad &\widetilde{\varphi}(x) = x,\\
&\varphi(y) = y, & &\varphi_1(y) = y-x, & &\widetilde{\varphi}(y) = y+\lambda_1 x^{-2}.
\end{aligned}
$$

In order to define G set $u = x + y + \lambda_1 x^{-2}$. The morphism $G : K[y]((x^{-1})) \to K[y]((x^{-1}))$ is given by $G(x) = u$ and $G(y) = y +$

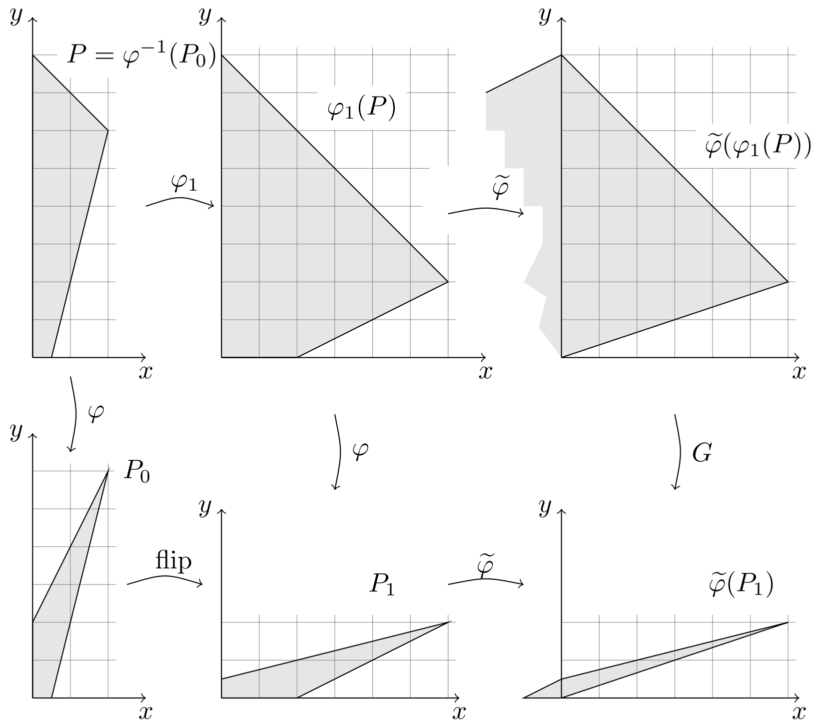

Figure 4: Finding minor roots: $\ell_{1,-3}(\widetilde{\varphi}(\varphi_1(P))) = \ell_{1,-3}(\widetilde{\varphi}(P_1))$

$\lambda_1(x^{-2} - u^{-2})$. Note that u is invertible in $K[y]((x^{-1}))$ with

$$u^{-1} = x^{-1}\left(1 - x^{-1}(y + \lambda_1 x^{-2}) + x^{-2}(y + \lambda_1 x^{-2})^2 - \dots\right).$$

We also have $G \circ \widetilde{\varphi} = \widetilde{\varphi} \circ \varphi$ and $\ell_{1,-3}(G(S)) = \ell_{1,-3}(S)$ for all $S \in K[y]((x^{-1}))$, which implies $\ell_{1,-3}(\widetilde{\varphi}(S)) = \ell_{1,-3}(\widetilde{\varphi}(\varphi(S)))$.

Now, the previous computations yield $\ell_{1,-2}(P_1) = R_1^{4m}$ where

$$R_1 = \text{flip}(R) = \lambda_0 x^3 (y - \lambda_1 x^{-2}).$$

Moreover, we also have $\ell_{-1,4}(\widetilde{\varphi}(P_1)) = \ell_{-1,4}(P_1)$ and the element F

of [12, Corollary 7.4] satisfies

$$\mathrm{st}_{-1,4}(F) = (9,3) = \frac{3}{4}\frac{1}{m}\,\mathrm{st}_{-1,4}(\widetilde{\varphi}(P_1)),$$

and so we must have $q = 4$ in [12, Corollary 7.4]. Hence, if we assume by contradiction $(\rho_1, \sigma_1) = \mathrm{Pred}_{\widetilde{\varphi}(P_1)}(1,-2) > (1,-3)$, from [12, Corollary 7.4] we obtain a (ρ_1,σ_1)-homogeneous element $R_2 \in K[x, x^{-1}, y]$ such that $\ell_{\rho_1,\sigma_1}(\widetilde{\varphi}(P_1)) = R_2^{4m}$, which is impossible.

It follows that

$$\mathrm{Pred}_{\widetilde{\varphi}(P_1)}(1,-2) = (1,-3) = \mathrm{Pred}_{\widetilde{\varphi}(Q_1)}(1,-2)$$

holds, as we also know that $(0,0)$ belongs to $\mathrm{Supp}(\widetilde{\varphi}(P_1)) \cap \mathrm{Supp}(\widetilde{\varphi}(Q_1))$. But we have $\ell_{1,-3}(\widetilde{\varphi}(P_1)) = \ell_{1,-3}(\widetilde{\varphi}(\varphi_1(P)))$ and so, from Proposition 3.3 we know that $\tau = -x + \lambda_1 x^{-2} + \pi x^{-3}$ is a π-root of P, since $\varphi_\tau = \widetilde{\varphi} \circ \varphi_1$. Moreover we get $\lambda_\tau = 0$; and $f_{\pi,P}(z) = \ell_{1,-3}(\varphi_\tau(P))$ has no multiple roots (eventually replace P by $P + \xi$ for an adequate constant ξ). Consequently τ is a final minor root of P and $|D_\tau^P| = v_{0,1}(\mathrm{en}_{1,-3}(\varphi_\tau(P))) = 4m$, where $4m$ is the number of remaining minor roots of P. So we conclude that τ is the only minor root of P and we also obtain $\delta_\tau = \dfrac{\sigma}{\rho} = \dfrac{-3}{1} = -3$, as claimed.

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Resumen

En [14] Yansong Xu calcula el número de intersección de un par jacobiano usandos dos igualdades diferentes. Probamos la primera de estas desigualdades usando el lenguaje de [12], pero en lugar de la segunda solamente obtenemos una desigualdad.

Palabras clave: Conjetura del jacobiano, teorema de Newton-Puisseux

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