A simplified proof of the Granja-Merle factorization theorem

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Abstract

In [G] Granja proved a factorization theorem for power series in two variables which generalized the Merle theorem [M] on polar curves. Our aim is to reprove his result without resorting to Hamburger-Noether expansions and Apéry sequences. We base our proof on the method developed by us in [GB-Pł].

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1 Introduction

Let $\mathbf{K}$ be an algebraically closed field of characteristic $p \geq 0$. Let $f \in \mathbf{K}[[x,y]]$ be a non-zero power series with no constant term. An **algebroid curve** $\{f = 0\}$ is the ideal generated by $f \in \mathbf{K}[[x,y]]$. We say that $\{f = 0\}$ is **irreducible** (respectively reduced) if f is irreducible (respectively f has no multiple factors). The irreducible curves are also called **branches**. The **order** ord f of the power series f is the **multiplicity** of the curve $\{f = 0\}$.

For any power series $f, g \in \mathbf{K}[[x,y]]$ we define the **intersection multiplicity** (also called **intersection number**) by

$$i_0(f,g) = \dim_{\mathbf{K}} \mathbf{K}[[x,y]]/(f,g),$$

where (f,g) is the ideal of $\mathbf{K}[[x,y]]$ generated by f and g. If f, g are non-zero power series with no constant term we have $i_0(f,g) < +\infty$ if and only if $\{f = 0\}$ and $\{g = 0\}$ have no common branch.

For any irreducible power series $f \in \mathbf{K}[[x,y]]$ we set

$$\Gamma(f) = \{i_0(f,g) : g \text{ runs over all power series such that } g \not\equiv 0 \pmod{f}\}.$$

A subset of $\mathbb{N}$ is a **semigroup** if it is closed under addition and it contains 0. Since we have $i_0(f,gh) = i_0(f,g) + i_0(f,h)$, then $\Gamma(f)$ should be a semigroup. We call $\Gamma(f)$ the **semigroup associated with the branch** $\{f = 0\}$.

Suppose that the branch $\{f = 0\}$ is different from $\{x = 0\}$. Write $n = i_0(f,x)$. Let $(\overline{b_0}, \ldots, \overline{b_h})$, where $\overline{b_0} = n$, be a n-minimal system of generators of $\Gamma(f)$ defined by the conditions

- $\overline{b_0} = n$,
- $\overline{b_k} = \min\left(\Gamma(f) - (\mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}})\right)$ for $1 \leq k \leq h$ and,
- $\Gamma(f) = \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_h}$

(cf. [4, Preliminaries]). Then we write $\overline{\mathrm{char}_x} f = (\overline{b_0}, \ldots, \overline{b_h})$. Note that the semigroup $\Gamma(f)$ is a numerical semigroup, i.e., it satisfies $\gcd(\Gamma(f)) =$

1. Let $c(f)$ be the smallest element of $\Gamma(f)$ such that $c(f) + N \in \Gamma(f)$ for any integer $N \geq 0$ (compare [1, p. 136]). The number $c(f)$ is called the **conductor** of $\Gamma(f)$.

Put $e_i = \gcd(\overline{b_0}, \dots, \overline{b_i})$ for $0 \leq i \leq h$ and $n_i = \dfrac{e_{i-1}}{e_i}$ for $1 \leq i \leq h$. We have $n_k > 1$ for $1 \leq k < h$ and $e_h = 1$. The following theorem is well-known, its proof employs Puiseux series when they are available, that is, when the characteristic of the field $\mathbf{K}$ is zero or n is not a multiple of it. We present a formulation of the theorem in use of *key polynomials* instead of Puiseux series, since they are available without any restriction.

Theorem 1.1 (Semigroup theorem)**.** *With the assumptions and notations introduced above, there exists a sequence of monic polynomials* $f_0, f_1, \dots, f_{h-1} \in \mathbf{K}[[x]][y]$ *such that for* $1 \leq k \leq h$ *we have*

(a_k) $\deg_y(f_{k-1}) = \dfrac{n}{e_{k-1}}$,

(b_k) $i_0(f_{k-1}, f) = \overline{b_k}$,

(c_k) *if* $k > 1$ *then* $n_{k-1}\overline{b_{k-1}} < \overline{b_k}$.

Proof. See [4, Theorem 3.2]. □

The polynomials $f_0, f_1, \dots, f_{h-1} \in \mathbf{K}[[x]][y]$ are called the **key polynomials** of f. They are not uniquely determined by f.

Theorem 1.2 (Granja factorization theorem)**.** *Let* $\{f = 0\}$ *be a branch different from* $\{x = 0\}$*. Let* $\overline{\mathrm{char}_x} f = (\overline{b_0}, \dots, \overline{b_h})$*, where* $\overline{b_0} = n = \operatorname{ord} f(0, y) > 1$*. Fix* k *between 1 and* h*. Let* $g = g(x, y)$ *be a power series with no constant term, subject to* $i_0(g, x) < \dfrac{n}{e_k}$ *and* $i_0(f, g) = \sum_{i=1}^{k} \lambda_i \overline{b_i}$*, with* $0 \leq \lambda_i < n_i$*, for* $1 \leq i \leq k$*. Then there is a factorization* $g = g_1 \cdots g_k \in \mathbf{K}[[x, y]]$ *such that*

- $i_0(g_i, x) = \lambda_i \dfrac{n}{e_{i-1}}$ *for* $1 \leq i \leq k$,
- *if* $\lambda_i > 0$ *and* $\phi \in \mathbf{K}[[x, y]]$ *is an irreducible factor of* g_i*, then*

(a) $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{i-1}\overline{b_i}}{n}$,

(b) $i_0(\phi, x) \equiv 0 \pmod{n/e_{i-1}}$.

We postpone the proof of Theorem 1.2 until Section 3.

A generalization of the Granja factorization theorem is due to Delgado [2, Section 2], who employs, like Granja, Hamburger-Noether expansions and Apéry sequences.

Example 1.3. Let $f_0, \dots, f_{h-1}$ be a sequence of key polynomials of f. Fix k between 1 and h. Take $g = f_0^{\lambda_1} \cdots f_{k-1}^{\lambda_k}$. Then g satisfies the assumptions of Theorem 1.2; here $g_i = f_{i-1}^{\lambda_i}$, for $1 \le i \le k$. Also, if ϕ is an irreducible factor of g_i then $\phi = f_{i-1}.unit$. Clearly in this case we should have $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{i-1}\overline{b_i}}{n}$ and $i_0(\phi, x) = \dfrac{n}{e_{i-1}}$.

Example 1.4. Suppose $\mathbf{K}$ is a field of characteristic different from 2. The power series $f(x,y) = (y^2 - x^3)^2 - 4x^5y - x^7$ is irreducible and we have $\overline{\mathrm{char}_x} f = (4, 6, 13)$ (see [4, page 246]). Let $k = 2$. Obviously we have $\lambda_1 = \lambda_2 = 1$. Take $g(x,y) = y^3 - x^3y$. Then we get $i_0(g,x) = 3 < \dfrac{n}{e_2} = 4$ and $i_0(f,g) = \lambda_1\overline{b}_1 + \lambda_2\overline{b}_2$. For $g_1 = y$ and $g_2 = y^2 - x^3$ we get $g = g_1g_2$ and $i_0(g_1,x) = 1$, $i_0(g_2,x) = 2$. Now, for $\tilde{g} = y^3 - x^2y - x^5$ we get $i_0(\tilde{g},x) = 3$ and $i_0(f,\tilde{g}) = 19$. We have $\tilde{g} = \tilde{g}_1\tilde{g}_2 = (y + \cdots)(y^2 - x^3 + \cdots)$, where the dots denote higher order terms.

As a corollary of Theorem 1.2 we present the following result.

Theorem 1.5 (Merle factorization theorem)**.** *(See [7],[8],[3] for the case $\mathbf{K} = \mathbb{C}$) Suppose $\overline{\mathrm{char}_x} f = (\overline{b_0}, \dots, \overline{b_h})$, $\overline{b_0} = \mathrm{ord}\, f(0,y) = n > 1$ and $n \not\equiv 0 \pmod{\mathrm{char}\, \mathbf{K}}$. Then we have $\dfrac{\partial f}{\partial y} = g_1 \cdots g_h$ in $\mathbf{K}[[x,y]]$, where*

1. *$i_0(g_i, x) = \dfrac{n}{e_i} - \dfrac{n}{e_{i-1}}$ for $1 \le i \le h$;*
2. *if $\phi \in \mathbf{K}[[x,y]]$ is an irreducible factor of g_i, then $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{i-1}\overline{b_i}}{n}$ and $i_0(\phi,x) \equiv 0 \pmod{n/e_{i-1}}$.*

Proof. Since $n \not\equiv 0 \pmod{\mathrm{char}\, \mathbf{K}}$ we have $i_0\left(\frac{\partial f}{\partial y}, x\right) = n-1$. Let $\mathcal{O} = \mathbf{K}[[x,y]]/(f)$ be the local ring of the branch $\{f=0\}$; write $\overline{\mathcal{O}}$ for its integral closure and set $\mathcal{C} = \overline{\mathcal{O}}$ for its conductor.

It is well-known (see [5, Lemma 3.1]) that we have $i_0\left(f, \frac{\partial f}{\partial y}\right) = \dim_{\mathbf{K}} \overline{\mathcal{O}}/\mathcal{C} + n - 1$ and $\dim_{\mathbf{K}} \overline{\mathcal{O}}/\mathcal{C} = c(f)$ (see [1, pp. 136-139]).

On the other hand, as the conductor satisfies $c(f) = \sum_{k=1}^{h}(n_k - 1)\overline{b_k} - \overline{b_0} + 1$ (see [4, Corollary 3.5]), we obtain $i_0\left(f, \frac{\partial f}{\partial y}\right) = \sum_{k=1}^{h}(n_k-1)\overline{b_k}$. We apply Theorem 1.2 to the series f and $g = \frac{\partial f}{\partial y}$ for $\lambda_k = n_k - 1$ to get the result. □

The first result on factorization of the derivative was proved by Henry J.S. Smith in [8]. Nevertheless his work fell into oblivion for a long time. Merle proved the factorization theorem in the case where the vertical axis is tranverse to the branch and Ephraim proved the theorem in any coordinates [3]. The Granja theorem is a natural generalization of the result due to Merle. That is the reason for the title of this note.

The Granja theorem was originally formulated in terms of Apéry sequences and its proof employs Hamburger-Noether expansions (see [6]). In [4] we developed a new approach to the theory of plane branches. We used the logarithmic distance on the set of branches without resorting to Hamburger-Noether expansions or a resolution process. Our proof of the Granja factorization theorem follows the spirit of [4].

2 The contact coefficient

For any pair of branches $\{f=0\}$ and $\{g=0\}$ different from $\{x=0\}$ set

$$d_x(f,g) = \frac{i_0(f,g)}{i_0(f,x)i_0(g,x)}.$$

The function d_x satisfies the **strong triangle inequality (STI)**: for any branches $\{f=0\}$, $\{g=0\}$ and $\{h=0\}$ different from $\{x=0\}$ we get

$$d_x(f,g) \geq \inf\{d_x(f,h), d_x(g,h)\},$$

that is, at least two of the numbers $d_x(f,g), d_x(f,h)$, $d_x(g,h)$ are equal among them and the third is not smaller than the equal two (see [4, Section 2, Theorem 2.8]).

Let $\{f=0\}$ and $\{g=0\}$ be two branches. Then we write

$$h_x(f,g) = \frac{i_0(f,g)}{i_0(g,x)}$$

and call the number $h_x(f,g)$ the **contact coefficient** of the branches $\{f=0\}$ and $\{g=0\}$. For f_{k-1} a $(k-1)$-th key polynomial of f with $\overline{\mathrm{char}}_x f = (\overline{b_0}, \dots, \overline{b_h})$ we have

$$h_x(f, f_{k-1}) = \frac{e_{k-1}\overline{b_k}}{n}.$$

Proposition 2.1. *With the assumptions and notations introduced above for the contact coefficient, we have the following possibilities.*

(a) If $h_x(f,\phi) > \dfrac{e_{k-1}\overline{b_k}}{n}$ *then* $i_0(\phi,x) \equiv 0 \pmod{n/e_k}$.

(b) If $h_x(f,\phi) < \dfrac{e_{k-1}\overline{b_k}}{n}$ *then* $i_0(f,\phi) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$.

(c) If $h_x(f,\phi) = \dfrac{e_{k-1}\overline{b_k}}{n}$ *then* $i_0(\phi,x) \equiv 0 \pmod{n/e_{k-1}}$ *and* $i_0(f,\phi) \equiv 0$ $\big(\mathrm{mod}\ \overline{b_k}\big)$.

Proof. For a) see [4, Lemma 5.6]. By assumption we have $h_x(f,\phi) < h_x(f, f_{k-1})$, and this lead us to $d_x(f,\phi) < d_x(f, f_{k-1})$. Using the STI with the power series ϕ, f_{k-1} and f we get $d_x(f,\phi) = d_x(f_{k-1},\phi)$, which implies $i_0(f,\phi) = \dfrac{i_0(f,x)}{i_0(f_{k-1},x)} i_0(f_{k-1},\phi) = e_{k-1} i_0(f_{k-1},\phi) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$ since $\overline{\mathrm{char}}_x f_{k-1} = \left(\dfrac{\overline{b_0}}{e_{k-1}}, \dots, \dfrac{\overline{b_{k-1}}}{e_{k-1}}\right)$ (see [4, Proposition 4.2]). This proves b). For c), first we check $i_0(\phi,x) \equiv 0 \pmod{n/e_{k-1}}$. If

$k = 1$ then it is obvious, so assume $k > 1$. We have $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{k-1}\overline{b_k}}{n} > \dfrac{e_{k-2}\overline{b_{k-1}}}{n}$, and hence $i_0(\phi,x) \equiv 0 \pmod{n/e_{k-1}}$ by the first statement of this same proposition. All these facts yield $i_0(f,\phi) = \dfrac{e_{k-1}\overline{b_k}}{n} i_0(\phi,x) \equiv 0$ $(\mathrm{mod}\ \overline{b_k})$. □

3 Proof of Theorem 1.2

Fix k and choose $g \in \mathbf{K}[[x,y]]$ so that the assumptions of Theorem 1.2 hold. Let $g = \phi_1 \cdots \phi_s$ with irreducible $\phi_j \in \mathbf{K}[[x,y]]$. First we check

$$\frac{i_0(f,\phi)}{i_0(\phi,x)} \leq \frac{e_{k-1}\overline{b_k}}{n} \tag{3.1}$$

if ϕ is an irreducible factor of g. Indeed, if not, we have an irreducible factor ϕ of g such that $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} > \dfrac{e_{k-1}\overline{b_k}}{n}$. Proposition 2.1 lead us then to $i_0(\phi,x) \equiv 0$ (mod $\dfrac{n}{e_k}$), a contradiction since $i_0(\phi,x) \leq i_0(g,x) < \dfrac{n}{e_k}$. Now if $\lambda_k \neq 0$, then there exists at least one irreducible factor ϕ of g such that $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} = \dfrac{e_{k-1}\overline{b_k}}{n}$. If we have $\dfrac{i_0(f,\phi)}{i_0(\phi,x)} \neq \dfrac{e_{k-1}\overline{b_k}}{n}$ for all irreducible factors of g, Equation 3.1 take us to $\dfrac{i_0(f,\phi_j)}{i_0(\phi_j,x)} < \dfrac{e_{k-1}\overline{b_k}}{n}$ for $1 \leq j \leq s$. In use of Proposition 2.1 we have $i_0(f,\phi_j) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$ and consequently $i_0(f,g) = \sum_{j=1}^{s} i_0(f,\phi_j) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$. This is impossible because of $i_0(f,g) = \sum_{i=1}^{k} \lambda_i \overline{b_i} \not\equiv 0 \pmod{e_{k-1}}$ (if we have $\sum_{i=1}^{k} \lambda_i \overline{b_i} \equiv 0 \pmod{e_{k-1}}$, then we would get $\lambda_k \overline{b_k} \equiv 0 \pmod{e_{k-1}}$ and $\lambda_k \dfrac{\overline{b_k}}{e_k} \equiv 0 \pmod{n_{k-1}}$, which is impossible since $0 < \lambda_k < n_k$).

Now, we prove Theorem 1.2 by induction on $k > 0$. If $\lambda_k = 0$, then the theorem reduces to the case $k-1$. Therefore we assume $\lambda_k \neq 0$. Let g_k be the product of all factors ϕ_j of g satisfying $\dfrac{i_0(f,\phi_j)}{i_0(\phi_j,x)} = \dfrac{e_{k-1}\overline{b_k}}{n}$.

Therefore we have $\frac{i_0(f,g_k)}{i_0(g_k,x)} = \frac{e_{k-1}\overline{b_k}}{n}$ and $g = \tilde{g}g_k$ in $\mathbf{K}[[x,y]]$. By Proposition 2.1 (b) we obtain $i_0(f,\tilde{g}) \in \mathbb{N}\overline{b_0} + \cdots + \mathbb{N}\overline{b_{k-1}}$. Similarly, by Proposition 2.1 (c), we get $i_0(f,g_k) \equiv 0 \pmod{\overline{b_k}}$ since we have $i_0(f,\phi) \equiv 0 \pmod{\overline{b_k}}$ for any irreducible factor ϕ of g_k. From $\frac{i_0(f,g_k)}{i_0(g_k,x)} = \frac{e_{k-1}\overline{b_k}}{n}$ we get

$$i_0(f,g_k) = \frac{e_{k-1}\overline{b_k}}{n} i_0(g_k,x) \leq \frac{e_{k-1}\overline{b_k}}{n} i_0(g,x) < \frac{e_{k-1}\overline{b_k}}{n}\frac{n}{e_k} = n_k\overline{b_k}.$$

With $i_0(f,g_k) = a_k\overline{b_k}$, we have $a_k < n_k$. Any element of the semigroup $\mathbb{N}\overline{b_0}+\cdots+\mathbb{N}\overline{b_{k-1}}$ has a representation $a_0\overline{b_0}+\cdots+a_{k-1}\overline{b_{k-1}}$, where $a_0 \geq 0$ and $0 \leq a_i < n_i$. Therefore we have $i_0(f,g) = i_0(f,\tilde{g}) + i_0(f,g_k) = a_0\overline{b_0} + a_1\overline{b_1} + \cdots + a_{k-1}\overline{b_{k-1}} + a_k\overline{b_k}$, where $0 \leq a_i < n_i$ for $1 \leq i \leq k$ and $a_0 \geq 0$. On the other hand, by assumption we have $i_0(f,g) = \lambda_1\overline{b_1} + \cdots + \lambda_k\overline{b_k}$. The unicity of the representation of $i_0(f,g)$ implies $a_0 = 0$ and $a_i = \lambda_i$. In particular $a_k = \lambda_k$ and $i_0(f,g_k) = \lambda_k\overline{b_k}$ hold. Since $\frac{i_0(f,g_k)}{i_0(g_k,x)} = \frac{e_{k-1}\overline{b_k}}{n}$ we have $i_0(g_k,x) = \frac{n}{e_k}\lambda_k$. If $k = 1$ we are done (in that case we conclude $i_0(f,\tilde{g}) = 0$, so $\tilde{g}$ is a unit and we work with $\tilde{g}g_1$ instead of g_1). If $k > 1$ then $\tilde{g} = \frac{g}{g_k}$ satisfies the assumptions of Theorem 1.2 with $k-1$. By induction we are done. □

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Resumen

En [6] Granja provó una generalización del teorema de Merle [7] para curvas polares de ramas planas. El trabajo presenta una prueba de este resultado sin recurrir a la expansión de Hamburger-Noether o secuencias de Apéry (presentes en la prueba original), sino basándonos en el método desarrollado en [4].

Keywords: Curvas algebroides planas, semigrupo asociado a una rama, polinomios llave, coeficientes de contacto.

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