About the stability between a foliation of degree two and the pencil of conics that defines it

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Abstract

In this paper, we study foliations on the projective plane of degree two which have a first integral with degree two. Such first integrals define a pencil of conics.

The Hilbert-Mumford criterion is a powerful tool of the Geometric Invariant Theory. An application of this theory is the characterizarion of the instability of the space of foliations of degree two, with respect to the action by a change of coordinates, and the characterization of the stability of pencils of conics, given by Alcántara.

The aim of the paper is to give another proof of the fact that a foliation of degree two defined by a pencil of conics is unstable if, and only if, the pencil is unstable.

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1 Introduction

The problem of classification of holomorphic foliations in complex manifolds has been the object of intense study in the last decades, see [5, 6, 8] and the references therein.

The set of holomorphic foliations on the complex projective plane of degree d, denoted by $\mathbb{F}ol(d)$, is a projective space and accepts a linear action of $\mathrm{Aut}(\mathbb{P}^2)$ under a change of coordinates. In this paper, we study foliations on $\mathbb{P}^2$ of degree two which have a first integral of degree two. Our main tool would be Geometric Invariant Theory (GIT), as developed mainly by Hilbert and Mumford [11]. We also deal with the pencil of conics defined by said first integral. In [10], Miranda found a characterization of the stability of pencils of cubic curves. Based on this result, Alcántara and Sánchez-Argáez [13] (see also [4]) proved the following characterization on the stability of a pencil of conics.

Theorem 1.1 ([13, Teorema 8])**.** *Let $A(x, y, z)$ and $B(x, y, z)$ be degree two homogeneous polynomials in $\mathbb{P}^2$ defining conics without common components. Let $\mathcal{L}_{A,B}$ be the pencil generated by such conics, and let $B(\mathcal{L}_{A,B})$ the set of common zeros of A and B. Then $P(\mathcal{L})$ is unstable if and only if $B(\mathcal{L}_{A,B})$ contains at most three different points.*

In [2], Alcántara obtained maximal sets of generators for unstable foliations, namely

$$CN_1 = \mathbb{P}\left\langle xy\frac{\partial}{\partial x}, xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial z}\right\rangle$$
$$CN_2 = \mathbb{P}\left\langle xz\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial x}, yz\frac{\partial}{\partial x}, z^2\frac{\partial}{\partial x}, xz\frac{\partial}{\partial y}, yz\frac{\partial}{\partial y}, z^2\frac{\partial}{\partial y}\right\rangle.$$

Moreover, Alcántara also achieved a characterization of the instability of a degree two foliation.

Theorem 1.2 ([2])**.** *Let $X \in \mathbb{F}ol(2)$ be a foliation with isolated singularities. Then X is unstable if and only if it has one of the following properties:*

1. *there exists a singular point* p *of multiplicity* 2*, or*

2. *it has an invariant line with a unique singular point with multiplicity* 1 *and Milnor number* 5*.*

Moreover, a foliation X *satisfies (1) if and only there exists* $g \in SL(3,\mathbb{C})$ *such that* $gX \in CN_1$*, and satisfies (2) if and only if there exist* $g \in SL(3,\mathbb{C})$ *such that* $gX \in CN_2 \setminus CN_1$*.*

In [7], Cerveau *et al.* proved that there exist, up to the action, three foliations of degree two on $\mathbb{P}^2$ defined by a pencil of conics. On the other hand, in [3], Alcántara constructs a stratification (based on GIT, see [11]) of the space of foliations with respect to the action by a change of coordinates and presents the following corollary: *a foliation of degree two defined by a pencil of conics is unstable if and only if the pencil is unstable.* In this work, we present a new proof of this corollary, where, instead of using the aforementioned stratification, we rely on Proposition 3.2, due to Darboux, by studying the singular fibers of the first integral defined by the pencil. We also use the characterization given in Theorem 1.2 for the instability of a foliation and the characterization given in Theorem 1.1 for the instability of a pencil of conics. In other words, we will establish the folllowing result.

Theorem 1.3. *Let* $\mathcal{F}$ *be a foliation of degree two in* $\mathbb{P}^2$ *with fist integral* $H = \dfrac{F}{G}$*, here* F, G *are polynomials of degree two. Then* $\mathcal{F}$ *is unstable if and only if* $\mathcal{L}_{F,G}$ *is unstable.*

2 Preliminaries

Let k be an algebraically closed field. An **algebraic group** G over k is a group that by own right is a variety over k and is such that the multiplication and inversion operations are morphisms of the variety.

Classical examples of algebraic groups include

$$GL(n,k) = \{A \in M_n(k) \ : \ \det(A) \neq 0\},$$
$$SL(n,k) = \{A \in M_n(k) \ : \ \det(A) = 1\},$$

respectively called the **general** and **special linear** groups. Given G, H algebraic groups, a **morphism** $\varphi : G \to H$ should be both a group homomorphism and a morphism of varieties.

A **linear algebraic group** is an algebraic group that is isomorphic to an algebraic subgroup of $GL(n,k)$. Note that both $GL(n,k)$ and $SL(n,k)$ are linear algebraic groups.

From now on, π will denote the projection from $\mathbb{A}_k^{n+1} \setminus \{0\}$ to $\mathbb{P}^n$. Let G be an algebraic group acting on an affine variety $X \subset \mathbb{A}_k^{n+1}$ (respectively, a projective variety $X \subset \mathbb{P}^n$). We say that the action is **linear** if there exists a group homomorphism

$$\rho : G \to GL(n+1,k)$$

with $g \cdot x = \rho(g)(x)$ (respectively, $g \cdot x = \pi(\rho(g)(\bar{x}))$, where $\bar{x} \in \pi^{-1}(x)$).

Remark 2.1. Note that if an action over a projective variety X is linear, then ρ induces an action in the affine cone

$$\check{X} = \{0\} \cup \bigcup_{x \in X} \pi^{-1}(x).$$

of X.

Denote by $\mathbb{C}[x,y,z]_d$ the space of homogeneous polynomials of degree d. Given $F \in \mathbb{C}[x,y,z]_d$, the set

$$V(F) = \{[x:y:z] \in \mathbb{P}^2 \ : \ F(x,y,z) = 0\}$$

is called the **locus** of F. Recall that $F, G \in \mathbb{C}[x,y,z]_d$ define the same curve (*i.e.,* locus) if and only if $F = \lambda G$, for some $\lambda \in \mathbb{C}^*$. Then

$$X = \{V(F) : F(x,y,z) \in \mathbb{C}[x,y,z]_d\}$$

is a proyective variety, because of the identification

$$X \simeq \frac{\mathbb{C}[x, y, z]_d \setminus \{0\}}{\mathbb{C}^*} \simeq \mathbb{P}^N,$$

where $N = \binom{d+2}{2} - 1$.

Example 2.2. Let X be the set of cubic planar curves, that is

$$X = \{V(F) : F(x, y, z) \in \mathbb{C}[x, y, z]_3\} \simeq \mathbb{P}^9.$$

The action in X given by

$$\begin{aligned} \rho : SL(3, \mathbb{C}) \times X &\longrightarrow X \\ (g, F(x, y, x)) &\longmapsto F(g^{-1}(x, y, z)). \end{aligned}$$

is linear.

Let G be an algebraic group acting on X. In general the quotient space X/G is not a variety.

Example 2.3. Let $G = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $X = \mathbb{C}^2$. Consider the action $\rho : G \times \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\rho(\lambda, (z_1, z_2)) = \lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2).$$

The orbits of X are given by

$$\begin{aligned} O(0, 0) &= \{(0, 0)\}, \\ O(z_1, 0) &= \{(\lambda z_1, 0) : \lambda \in G\}, \\ O(0, z_2) &= \{(0, \lambda^{-1} z_2) : \lambda \in G\}, \\ O(z_1, z_2) &= \{(x, y) : xy = z_1 z_2\}, \end{aligned}$$

where $z_1, z_2 \neq 0$. We conclude that X/G is not an algebraic variety.

Let G be an algebraic group acting on a variety X. Denote by $A(X)$ the algebra of morphisms $\varphi : X \to k$. The action of G on X induces an action of G on $A(X)$. We define

$$A(X)^G = \{f \in A(X) : f(g \cdot x) = f(x), \text{ for all } g \in G\},$$

called the **invariant ring** of X.

Example 2.4. For the action ρ of Example 2.3 we have

$$A(X) = \{f : \mathbb{C}^2 \to \mathbb{C} : f \text{ is a polynomial}\}.$$

Here we obtain

$$A(X)^G = k[xy] = A(\mathbb{C})$$

and, so, the ring is finitely generated.

The set $V(A(X)^G) = V(A(\mathbb{C})) = \mathbb{C}$ is called the **GIT-quotient** and is denoted by $X//G$. The question of whether or not $A(X)^G$ is finitely generated is a variation on Hilbert's 14th problem. The most general answer was given by Nagata [12] in 1963. See also [1].

Theorem 2.5 (Nagata)**.** *If G is a reductive group then $A(X)^G$ is finitely generated.* □

We say that a linear algebraic group G is **reductive** if for every linear action of G in k^n, and every invariant point $v \in k^n \setminus \{0\}$ there exists a G-invariant homogeneous polynomial f of degree ≥ 1 such that $f(v) \neq 0$.

Example 2.6 ([1, Ejemplo 10])**.** The groups $G = GL(3, \mathbb{C})$ and $G = SL(3, \mathbb{C})$ are known to be reductive.

One of the main goals of Geometric Invariant Theory (GIT) is to classify objects in Algebraic Geometry. The case of interest here is when X is the space of foliations of degree d, see Section 3.

Definition 2.7 ([11, Proposition 2.2], see also [1, Proposición 4])**.** Let X be a projective variety in $\mathbb{P}^n$ and G a reductive group acting linearly on X. Let $x \in X$ and $\overline{x} \in \pi^{-1}(x) \subset \check{X} \subset \mathbb{C}^{n+1}$. Then x is **semistable** when $0 \notin \overline{O(\overline{x})}$. We write X^{ss} for the set of semistable points of X.

Also, we say that x is **stable** whenever $0 \notin \overline{O(\overline{x})}$, the orbit of x is closed in X^{ss}, and $\dim O(x) = \dim G$.

In general, it is difficult to determine when a point in a projective variety is stable. We now describe a usable criterion, originally given by Hilbert for $G = SL(n)$, and later extended by Mumford for arbitrary G.

A 1**-parameter subgroup of** G is a non-trivial homomorphism of algebraic groups $\lambda : \mathbb{C}^* \to G$.

Given a 1-parameter subgroup λ and a linear action G on $X \subset \mathbb{P}^n$, we introduce the representation

$$\begin{aligned} \mathbb{C}^* &\to GL_{n+1}(\mathbb{C}) \\ t &\mapsto \lambda_t : \lambda_t(v) = \lambda(t) \cdot v. \end{aligned}$$

This representation is diagonalizable.

Proposition 2.8 ([1, Proposición 5])**.** *There is a basis* $\{e_0, e_1, \ldots, e_n\}$ *of* $\mathbb{C}^{n+1}$ *such that* $\lambda(t)e_i = t^{r_i} e_i$*, with* $r_i \in \mathbb{Z}$*.*

By Proposition 2.8, given a point $\overline{x} \in \check{X}$, with $\overline{x} = \sum_{i=0}^{n} \overline{x}_i e_i$, we have

$$\lambda(t)\overline{x} = \sum t^{r_i} \overline{x}_i e_i.$$

Mumford used this 1-parameter subgroup to calculate the stability of elements of X by the action of G. In that way he introduced the now called **Mumford's function**: for $x \in X$ and λ in a given 1-parameter subgroup of G, we set

$$\mu(x, \lambda) = \min\{r_i : \overline{x}_i \neq 0\}.$$

Theorem 2.9 ([11, Theorem 2.1], see also [1, Teorema 12])**.** *Let X be a projective variety in $\mathbb{P}^n$ and G a reductive group acting linearly on X. Then, for any $x \in X$ we have that*

- *x is semistable if and only if $\mu(x, \lambda) \leq 0$ for all 1-parameter subgroups λ of G.*
- *x is stable if and only if $\mu(x, \lambda) < 0$ for all 1-parameter subgroups λ of G.*

Remark 2.10. Almost directly from the definition we get

$$\mu(g \cdot x, \lambda) = \mu(x, g^{-1}\lambda g),$$

for any $g \in G$.

Proposition 2.11 ([1, Proposición 6])**.** *Every 1-parameter subgroup $\lambda : \mathbb{C}^* \to SL(3, \mathbb{C})$ is conjugated to a diagonal one. In other words, we have*

$$\lambda(t) = g \begin{pmatrix} t^{n_0} & 0 & 0 \\ 0 & t^{n_1} & 0 \\ 0 & 0 & t^{n_2} \end{pmatrix} g^{-1},$$

for some $g \in SL(3, \mathbb{C})$, where $n_0 \geq n_1 \geq n_2$, $n_0 + n_1 + n_2 = 0$.

Example 2.12. Let $X = \{V(F) : F \in \mathbb{C}[x, y, z]_3\}$ and consider the linear action

$$\begin{aligned} \rho : SL(3, \mathbb{C}) \times X &\longrightarrow X \\ (g, F(x, y, x)) &\longmapsto F(g^{-1}(x, y, z)). \end{aligned}$$

We now calculate the Mumford's function associated to X and ρ. For that purpose let

$$B = \{x^3, y^3, z^3, x^2y, x^2z, xy^2, xz^2, yz^2, y^2z, xyz\}$$

be a base of X. By Remark 2.10, in order to analyze the stability of $F \in X$, it is enough to consider the 1-parameter subgroups given by

$$\begin{aligned} \lambda : \mathbb{C}^* &\to SL(3, \mathbb{C}) \\ t &\mapsto \operatorname{diag}(t^{n_0}, t^{n_1}, t^{n_2}), \end{aligned}$$

where $n_0 \geq n_1 \geq n_2$ and $n_0 + n_1 + n_2 = 0$. A few calculations yield $\mu(x^{3-i-j}y^i z^j, \lambda) = (i + j - 3)n_0 - in_1 - jn_2$ and

$$\mu(F, \lambda) = \min\{(i + j - 3)n_0 - in_1 - jn_2 : a_{ij} \neq 0\}.$$

where

$$F(x, y, x) = \sum_{i,j=0}^{3} a_{ij} x^{3-i-j} y^i z^j.$$

3 Foliations in the complex projective plane

An holomorphic foliation $\mathcal{F}$ on a complex compact surface X is a family of holomorphic 1-forms $\{\omega_i\}_{i \in I}$ defined on an open covering $\{U_i\}_{i \in I}$ of X, such that $\omega_i = g_{ij}\omega_j$, for some holomorphic functión g_{ij} without zeroes on $U_i \cap U_j$. A **foliation** of degree d in $\mathbb{P}^2$ is determined by either

- a projective 1-form $\Omega = Pdx + Qdy + Rdz$, with P, Q, R homogeneous polynomials of degree $d + 1$ subject to $xP + yQ + zR = 0$ (called **Euler's condition**), that satisfies $\Omega \wedge d\Omega = 0$, the **integrabitility condition** or
- a vector field $X = A\dfrac{\partial}{\partial x} + B\dfrac{\partial}{\partial y} + C\dfrac{\partial}{\partial z}$, with A, B, C homogeneous polynomials of degree d, modulo a product GX_R of the **radial vector field** $X_R = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}$ and a homogeneous polynomial G.

The **singular set** of a foliation $\mathcal{F}$, denoted by $\mathrm{Sing}(\mathcal{F})$, is given by the set

$$\mathrm{Sing}(\mathcal{F}) = \{p \in \mathbb{P}^2 \ : \ P(p) = Q(p) = R(p) = 0\}$$

when $\mathcal{F}$ is defined via a 1-form, and by

$$\mathrm{Sing}(\mathcal{F}) = \{p = [x : y : z] \in \mathbb{P}^2 \ : \ X(p) = \lambda X_R(p), \ \text{for some } \lambda \in \mathbb{C}\}$$

when $\mathcal{F}$ is defined via a vector field.

Example 3.1. Let P, Q be homogeneous polynomials of degree k in $\mathbb{P}^2$ without common factors. Then the 1-form $\Omega = PdQ - QdP$ satisfies $i_{X_R}(PdQ - QdP) = 0$, where i_{X_R} is the contraction of the 1-form, and defines a foliation $\mathcal{F}_\Omega$ of degree $2k - 2$.

Let P, Q be homogeneous polynomials of degree k in $\mathbb{P}^2$. For $\alpha = [a : b] \in \mathbb{P}^1$, let $L_\alpha = aP + bQ$ be a fiber of P/Q, whose decomposition in irreducible factors is $L_\alpha = f_{\alpha,1}^{n_1} \cdots f_{\alpha,j}^{n_j}$, with $n_1, \ldots, n_j \in \mathbb{N}$ and $j \in \mathbb{N}$, all depending on α. In this setting, we write $G_\alpha = f_{\alpha,1}^{n_1-1} \cdots f_{\alpha,j}^{n_j-1}$. We have the following proposition due to Darboux [9].

Proposition 3.2 (Darboux)**.** *Let $\Omega = PdQ - QdP$. Then there exist $\Delta = G_{\beta_1} \cdots G_{\beta_n}$, with $\beta_1, \ldots, \beta_n \in \mathbb{P}^1$, and a 1-form ω such that*

$$\Omega = \Delta \cdot \omega.$$

Moreover, ω defines a foliation $\mathcal{F}(P, Q)$ of degree $2k - 2 - \deg(\Delta)$ with isolated singularities, where $k = \deg(P) = \deg(Q)$.

Remark 3.3. In Proposition 3.2, the quotient $H = \dfrac{P}{Q}$ is called a **first integral** of the foliation $\mathcal{F} = \mathcal{F}(P, Q)$. When this is the case, the quotient $\dfrac{P + \lambda Q}{P + \mu Q}$, whenever $\lambda \neq \mu$, is also a first integral of $\mathcal{F}$.

Remark 3.4. By Proposition 3.2, if P and Q are polynomials of degree two, we have $PdQ - QdP = \Delta\omega$, $\Delta = G_{\beta_1} \cdots G_{\beta_n}$, with $\beta_1, \ldots, \beta_n \in \mathbb{P}^1$, ω defines $\mathcal{F}$ and

$$\deg(\mathcal{F}) = 2 = 2 \cdot 2 - 2 - \deg(\Delta),$$

which implies $\deg(\Delta) = 0$. Then, if $G_\beta = f_{\beta,1}^{n_1-1} \ldots f_{\beta,k}^{n_k-1}$ is a factor of Δ then $n_j = 1$, for $j = 1 \ldots n$, and $L_\alpha = f_{\alpha,1}^{n_1} \cdots f_{\alpha,k}^{n_k}$ is a singular fiber of H. In particular, every singular fiber of H is a product of two lines.

Lemma 3.5. *Let $\mathcal{F}$ be a foliation of degree two in $\mathbb{P}^2$ with isolated singularities and with $H = \dfrac{F}{G}$ as first integral, where F and G are polynomials of degree 2. Then, under an automorphism of $\mathbb{P}^2$ we can reach one of the following three forms*

- $H = \dfrac{z(ax + by + cz)}{xy}$, *with at most one of a, b, c equal to zero,*
- $H = \dfrac{(x - y)(ax + cz)}{xy}$, *where $a \cdot c \neq 0$,*
- $H = \dfrac{xy}{Q(x, y, z)}$, *with $Q = ax^2 + cy^2 + mxz + nyz + pz^2$ irreducible.*

Proof. For $\alpha = [a : b] \in \mathbb{P}^1$, the quadratic form $H_\alpha = aP + bQ$ is a fiber of H. Let A_α be the associated matrix of H_α. Given $\beta, \gamma \in \mathbb{P}^1$ we have the identity

$$\det(A_\beta + tA_\gamma) = \det(A_\gamma)\det(A_\gamma^{-1}A_\beta - (-t)Id) = \det(A_\gamma)q(-t),$$

where $q(x)$ is the characteristic polynomial of $A_\gamma^{-1}A_\beta$. It follows that there is at least one singular fiber of H, a product of two lines by Remark 3.4. Let us call L_α this singular fiber.

By Remark 3.3, any two fibers of H determine a first integral of $\mathcal{F}$. Therefore, we have two possibilities:

- if H has two singular fibers, then $H = \dfrac{\ell_1\ell_2}{\ell_3\ell_4}$, where $\ell_1\ell_2$ and $\ell_3\ell_4$ are the singular fibers of H;

- if H has one singular fiber, then $H = \dfrac{\ell_1\ell_2}{Q}$, with Q irreducible.

In the first case, after a change of coordinates H assumes one of the following shapes

- $H = \dfrac{z\ell}{xy} = \dfrac{z(ax+by+cz)}{xy}$, where at most one of a, b, c is zero, since $\mathcal{F}$ has isolated singularities.
- $H = \dfrac{(x-y)\ell}{xy} = \dfrac{(x-y)(ax+by+cz)}{xy}$. By Remark 3.3, we can further assume $b = 0$. Note that $a \neq 0$ returns us to the first case, so we can asume $a \neq 0$. Finally, we should have $c \neq 0$ if we wish $\mathcal{F}$ to have only isolated singularities.

In the second case, it is clear that under a linear change of coordinates H takes the form

$$H = \frac{xy}{Q} = \frac{xy}{ax^2 + cy^2 + mxz + nyz + qxy + pz^2},$$

and, by Remark 3.3, we can assume that $q = 0$. $\square$

4 Pencils of curves of degree d

To $\mathbb{C}[x,y,z]_d$, the space of homogeneous polynomials of degree d, we associate the corresponding projective space $\mathbb{P}(\mathbb{C}[x,y,z]_d) \simeq \mathbb{P}^{n-1}$, here $n = \dbinom{d+2}{2} = \dfrac{(d+1)(d+2)}{2}$.

Given $F, G \in \mathbb{C}[x,y,z]_d$, with $F \neq G$, the **pencil of plane curves of degree d generated by F and G** is defined by

$$\mathcal{L} = \mathcal{L}_{F,G} = \{aF + bG \,:\, [a:b] \in \mathbb{P}^1\},$$

to which we associate the **base locus**

$$B(\mathcal{L}_{F,G}) = V(F) \cap V(G).$$

The set of pencils of plane curves of degree d is denoted by $\mathbb{G}_d$.

On the other hand, writing

$$F = \sum_{i=0}^{d}\sum_{j=0}^{d-i} a_{(i,j)} x^i y^j z^{d-i-j}, \qquad G = \sum_{i=0}^{d}\sum_{j=0}^{d-i} b_{(i,j)} x^i y^j z^{d-i-j},$$

we consider the $2 \times n$-matrix

$$\begin{bmatrix} a_{(0,0)} & a_{(0,1)} & \cdots & a_{(0,d)} & \cdots & a_{(d,0)} \\ b_{(0,0)} & b_{(0,1)} & \cdots & b_{(0,d)} & \cdots & b_{(d,0)} \end{bmatrix}.$$

Note that the indices of the columns of the previous matrix are lexicographically ordered. Define the determinants

$$\mathcal{M}_{i,j,k,l} = \det \begin{bmatrix} a_{ij} & a_{kl} \\ b_{ij} & b_{kl} \end{bmatrix}$$

and the set of N-tuples

$$(\mathcal{M}_{i,j,k,l})_{(i,j)<_{\text{lex}}(k,l)},$$

where $N = \binom{n}{2} = \dfrac{n(n-1)}{2}$. The function

$$\begin{aligned} P : \mathbb{G}_d &\longrightarrow \mathbb{P}^{N-1} \\ \mathcal{L} &\longmapsto (\mathcal{M}_{i,j,k,l}) \end{aligned}$$

determines what are called the **Plucker coordinates of $\mathcal{L}$**.

Theorem 4.1. *With the notations above, P is an embedding and $P(G_d)$ is a projective variety.*

Proof. See [13, Teorema 7]. □

In Theorem 1.1, the action of the group $SL(3,\mathbb{C})$ on $\mathbb{G}_d$ is

$$\begin{aligned} \sigma : SL(3,\mathbb{C}) \times \mathbb{G}_d &\longrightarrow \mathbb{G}_d \\ \sigma(g, \{k_1 A + k_2 B\}_{(k_1:k_2)\in\mathbb{P}(\mathbb{C})^1}) &\longmapsto \{k_1 A g + k_2 B g\}_{(k_1:k_2)\in\mathbb{P}(\mathbb{C})^1}. \end{aligned}$$

5 Relating the pencil and the foliation

The group $\mathbb{P}GL(3,\mathbb{C})$ of automorphisms of $\mathbb{CP}^2$ is a reductive group that acts linearly on $\mathbb{F}ol(d)$ by the change of coordinates

$$\begin{aligned}\mathbb{P}GL(3,\mathbb{C}) \times \mathbb{F}ol(d) &\longrightarrow \mathbb{F}ol(d)\\ (g, X) &\longmapsto g\cdot X = DgX \circ (g^{-1}),\end{aligned}$$

or more specifically, on $\mathcal{F}_2$, as

$$\left(g, \begin{pmatrix} P(x,y,z) \\ Q(x,y,z \\ R(x,y,z) \end{pmatrix} \right) \longmapsto g \begin{pmatrix} P(g^{-1}(x,y,z)) \\ Q(g^{-1}(x,y,z)) \\ R(g^{-1}(x,y,z)) \end{pmatrix}.$$

Lemma 5.1. *Let $\mathcal{L}_{F,G}$ be the pencil of conics for $F = z(ax+by+cz)$ and $G = xy$. Then $\mathcal{L}_{F,G}$ is unstable if and only if we have $abc = 0$.*

Proof. The base locus $B(\mathcal{L})$ of F and G is obtained by solving

$$\begin{aligned} xy &= 0\\ z(ax+by+cz) &= 0. \end{aligned}$$

For $c = 0$ we obtain

$$B(\mathcal{L}) = \{[1:0:0], [0:1:0], [0:0:1]\}$$

while for $c \neq 0$ we get

$$B(\mathcal{L}) = \{[1:0:0], [0:1:0], [0:c:-b], [c:0:-a]\}.$$

Thus, $B(\mathcal{L})$ has at most three points provided $a = 0$ or $b = 0$ or $c = 0$ (or $abc = 0$ in short). Therefore, by Theorem 1.1, the pencil $\mathcal{L}$ is unstable if and only if $abc = 0$. □

Lemma 5.2. *Let $\mathcal{L}_{F,G}$ be the pencil of conics for $F = (x-y)(ax+cz)$ and $G = xy$. Then $\mathcal{L}_{F,G}$ is unstable if and only if $c \neq 0$.*

Proof. When $c = 0$ (hence $a \neq 0$), the base locus $B(\mathcal{L})$ of F and G is

$$B(\mathcal{L}) = \{[0 : y : z]\} \simeq \mathbb{P}^1.$$

This in turn implies that the pencil $\mathcal{L}$ is stable. On the other hand, when $c \neq 0$ we obtain

$$B(\mathcal{L}) = \{[0 : 0 : 1], [0 : 1 : 0], [c : 0 : -a]\}.$$

Since $B(\mathcal{L})$ has at most three points, the pencil $\mathcal{L}$ is unstable. Therefore, by Theorem 1.1 the pencil $\mathcal{L}$ is unstable if and only if we have $c \neq 0$. □

Lemma 5.3. *Let $\mathcal{L}_{F,G}$ be the pencil of conics for $F = xy$ and $G = ax^2 + cy^2 + mxz + nyz + pz^2$, irredutible. Then $\mathcal{L}_{F,G}$ is unstable if and only if $p = 0$, or $p \neq 0$ and $\Delta_1\Delta_2 = 0$, here $\Delta_1 = n^2 - 4pc$ and $\Delta_2 = m^2 - 4pa$.*

Proof. A point $[x : y : z]$ of the base locus of the pencil must be either $[0 : y : z]$, where $cy^2+nyz+pz^2 = 0$, or $[x : 0 : z]$, where $ax^2+mxz+pz^2 = 0$. If $p = 0$, these points reduce to $[0 : 0 : 1]$, $[0 : m : -a]$ and $[m : 0 : -a]$, so the base locus has at most three points. When $p \neq 0$, if $\Delta_1 = n^2-4pc = 0$ or $\Delta_2 = m^2-4pa = 0$, then the base locus also has at most three points. On the other hand, if $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$, the base locus has four points. The lemma now follows by Theorem 1.1. □

Theorem 5.4. *Let $\mathcal{F}$ be a foliation of degree two in $\mathbb{P}^2$ with fist integral $H = F/G$, here F, G are polynomials of degree 2. Then $\mathcal{F}$ es unstable if and only if $\mathcal{L}_{F,G}$ is unstable.*

Proof. By Lemma 3.5, the first integral $H = \dfrac{F}{G}$ must be of one of the following three types:

- $H = \dfrac{z(ax + by + cz)}{xy}$, where at most one among a, b, c is zero;
- $H = \dfrac{(x - y)(ax + cz)}{xy}$, with $a \cdot c \neq 0$;

- $H = \dfrac{xy}{Q} = \dfrac{xy}{ax^2 + cy^2 + mxz + nyz + pz^2}$, with Q irreducible.

For the first case set $L = ax+by+cz$. Then the foliation $\mathcal{F}$ induced by

$$\begin{aligned}\Omega_1 &= zLd(xy) - xyd(zL)\\ &= yz(by+cz)dx + xz(ax+cz)dy - xy(ax+by+2cz)dz\end{aligned}$$

has singularities in the solution set of the system

$$\begin{aligned} xy(ax+by+2cz) &= 0,\\ xz(ax+cz) &= 0,\\ xyz(by+cz) &= 0.\end{aligned}$$

Thus, the singular set of $\mathcal{F}$ is

$$\mathrm{Sing}(\mathcal{F}) = \left\{\begin{matrix}[1:0:0],[0:1:0],[0:0:1],[c:0:-a],\\ [-b:a:0],[0:c:-b],[bc:ac:-ab]\end{matrix}\right\}.$$

The foliation $\mathcal{F}$ is induced around the singularities $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ by

$$\Omega_1|_{x=1} = z(a+cz)dy - y(a+by+2cz)dz, \tag{5.1}$$

$$\Omega_1|_{y=1} = z(b+cz)dx - x(ax+b+2cz)dz, \tag{5.2}$$

$$\Omega_1|_{z=1} = y(by+c)dx + x(ax+c)dy, \tag{5.3}$$

respectively. Then, by Equations 5.1, 5.2, and 5.3, if $abc=0$ the foliation has a singularity with multiplicity 2 and $\mathcal{F}$ is unstable. On the other hand, if $abc \neq 0$, the foliation $\mathcal{F}$ has 7 singularities of multiplicity one, so $\mathcal{F}$ is semistable. In conclusion, in the first case $\mathcal{F}$ is unstable if and only if $abc = 0$. Hence, by Lemma 5.1, the foliation $\mathcal{F} = \mathcal{F}(F,G)$ turns out to be unstable if and only if the pencil $\mathcal{L}_{F,G}$ is unstable.

In the second case the foliation $\mathcal{F}$ is induced by

$$\Omega_2 = -y(cyz+ax^2)dx + x^2(ax+cz)dy - cxy(x-y)dz$$

with singular set

$$\mathrm{Sing}(\mathcal{F}) = \{[0:1:0], [0:0:1], [c:0:-a], [c:c:-a]\}.$$

The foliation $\mathcal{F}$ around $p_1 = [0:0:1]$ is induced by

$$\Omega_1|_{z=1} = -y(cy + ax^2)dx + x^2(ax + c)dy,$$

so it is unstable because the multiplicity of p_1 is 2. Therefore again by Lemma 5.2 we conclude that $\mathcal{F} = \mathcal{F}(F, G)$ is unstable and the pencil $\mathcal{L}_{F,G}$ is unstable.

In the third case the foliation $\mathcal{F} = \mathcal{F}(F, G)$ (here $F = xy$ and $G = Q$) is induced by

$$\begin{aligned}\Omega_3 &= (xy)dQ - Qd(xy) = y(xQ_x - Q)dx + x(yQ_y - Q)dy + xyQ_z dz \\ &= y(ax^2 - cy^2 - nyz - pz^2)dx + x(cy^2 - ax^2 - mxz - pz^2)dy + \\ &\quad + xy(mx + ny + 2pz)dz,\end{aligned}$$

and so now we get $\mathrm{Sing}(\mathcal{F}) = \{[0:0:1]\} \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, where

$$\begin{aligned}\mathcal{S}_1 &= \{[0:y:z] \,:\, cy^2 + nyz + pz^2 = 0\},\\ \mathcal{S}_2 &= \{[x:0:z] \,:\, ax^2 + mxz + pz^2 = 0\},\\ \mathcal{S}_3 &= \{[x:y:z] \,:\, Q = xQ_x = yQ_y,\, Q_z = 0,\, xy \neq 0\}.\end{aligned}$$

We now prove that $\mathcal{F}$ is unstable if and only if $p = 0$ or $p \neq 0$ and $\Delta_1\Delta_2 = 0$, where $\Delta_1 = n^2 - 4pc$ and $\Delta_2 = m^2 - 4pa$.

First assume $p = 0$. In this case, we get $m \neq 0$ or $n \neq 0$, otherwise Q becomes $Q = ax^2 + cy^2$, which is not irreducible. The foliation near $[0:0:1]$ is induced by

$$\Omega_3|_{z=1} = y(ax^2 - cy^2 - ny)dx + x(cy^2 - ax^2 - mx)dy,$$

so $[0:0:1]$ has multiplicity two. This implies that $\mathcal{F}$ is unstable.

Now assume $p \neq 0$ and $\Delta_1 = 0$. In this case we have the singular point $[0:1:-\frac{n}{2p}] \in \mathcal{S}_1$ and the foliation is given by

$$\Omega|_{y=1} = (ax^2 - c - nyz - pz^2)dx + x(mx + n + 2pz)dz.$$

After a translation to the origin, the foliation can be rewritten as

$$\omega = (ax^2 - pz^2)dx + x(mx + 2pz)dz.$$

It is clear now that $\mathcal{F}$ is unstable, because the multiplicity at $[0:1:-\frac{n}{2p}]$ is two. The case $p \neq 0$ and $\Delta_2 = 0$ is handled analogously.

Finally, assume $p \neq 0$ and $\Delta_1 \cdot \Delta_2 \neq 0$. In this case $\mathcal{S}_1$ and $\mathcal{S}_2$ have two singularities each. The singularities in $\mathcal{S}_3$ are obtained from the equations

$$Q = xQ_x, \quad Q = yQ_y, \quad Q_z = 0.$$

Note that $Q_z = mx + ny + 2pz = 0$ implies $z = -\dfrac{mx+ny}{2p}$. Replacing this in either $Q = xQ_x$ or $Q = yQ_y$, we obtain

$$\Delta_1 y^2 + \Delta_2 x^2 = 0,$$

so $\mathcal{S}_3$ has two singularities, totalling seven of them. This implies that $\mathcal{F}$ is semistable.

Therefore, if $p \neq 0$, then $\mathcal{F}$ is unstable if and only if $\Delta_1 = 0$ or $\Delta_2 = 0$. By Lemma 5.3 the foliation $\mathcal{F} = \mathcal{F}(F, G)$ is unstable if and only if the pencil $\mathcal{L}_{F,G}$ is unstable. □

Remark 5.5. It follows from the proof of the previous theorem, when $H = \dfrac{(x-y)(ax+cz)}{xy}$, with $a \cdot c \neq 0$, that both $\mathcal{F}$ and $\mathcal{L}_{F,G}$ are unstable (see Lemma 5.2).

References

[1] Alcántara, C. R. Introducción a la teoría de invariantes geométricos, 2010. URL: `https://www.cimat.mx/Eventos/SIGA2012/ClaudiaReynoso.pdf`.

[2] Alcántara, C. R. Geometric invariant theory for holomorphic foliations on $\mathbb{CP}^2$ of degree 2. *Glasg. Math. J. 53*, 1 (2011), 153–168.

[3] Alcántara, C. R. Foliations on $\mathbb{CP}^2$ of degree 2 with degenerate singularities. *Bull. Braz. Math. Soc. (N.S.) 44*, 3 (2013), 421–454.

[4] Alcántara, C. R., and Sánchez-Argáez, D. Instability of pencils of plane conics and their base locus. In *Papers of the Mexican Mathematical Society (Spanish)*, vol. 43 of *Aportaciones Mat. Comun.* Soc. Mat. Mexicana, México, 2011, pp. 3–14.

[5] Bedrouni, S., and Marín, D. Classification of foliations of degree three on $\mathbb{P}^2_{\mathbb{C}}$ with a flat Legendre transform. *arXiv e-prints* (Mar. 2018), arXiv:1803.08526.

[6] Calvo-Andrade, O. Irreducible components of the space of holomorphic foliations. *Math. Ann. 299*, 4 (1994), 751–767.

[7] Cerveau, D., Déserti, J., Garba Belko, D., and Meziani, R. Géométrie classique de certains feuilletages de degré deux. *Bull. Braz. Math. Soc. (N.S.) 41*, 2 (2010), 161–198.

[8] Cerveau, D., and Lins Neto, A. Irreducible components of the space of holomorphic foliations of degree two in $\mathbf{C}\mathrm{P}(n)$, $n \geq 3$. *Ann. of Math. (2) 143*, 3 (1996), 577–612.

[9] Darboux, G. Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré. *Bulletin des Sciences Mathématiques et Astronomiques 2e série, 2*, 1 (1878), 60–96.

[10] Miranda, R. On the stability of pencils of cubic curves. *Amer. J. Math. 102*, 6 (1980), 1177–1202.

[11] Mumford, D., Fogarty, J., and Kirwan, F. *Geometric invariant theory*, third ed., vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)].* Springer-Verlag, Berlin, 1994.

[12] Nagata, M. Invariants of a group in an affine ring. *J. Math. Kyoto Univ. 3* (1963/64), 369–377.

[13] Sánchez-Argáez, D. Inestabilidad de pinceles de cónicas y cúbicas planas y su lugar base. Master's thesis, Universidad de Guanajuato, Mexico, 2008.

Resumen
En este artículo estudiamos foliaciones de grado dos en el plano proyectivo que acepten integral primera, también, de grado dos. Tales integrales primera definen una familia lineal de cónicas. El criterio de Hilbert-Munford es una poderosa herramienta de la teoría de invariantes geométricos. Una aplicación de esta teoría es la caracterización de la inestabilidad en el espacio de foliaciones de grado dos respecto a la acción por un cambio de coordenadas, y asimismo la caracterización de la estabilidad de las familias lineales de cónicas, ambas dadas por Alcántara. El objeto de este artículo es presentar una prueba alternativa del hecho de que una foliación de grado dos definida por una familia lineal de cónicas es inestable si y solo si la correspondiente familia lineal es inestable.

Palabras clave: Foliaciones, pincel de cónicas, inestabilidad.

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