Local dynamics of parabolic skew-products

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Abstract

The local dynamics around a fixed point has been extensively studied for germs of one and several complex variables. In dimension one, there exist a complete picture of the trajectory of the orbits on a full neighbourhood of the fixed point. In greater dimensions some partial results are known. In this paper we analyze a case that lies between one and several variables. We consider skew product maps of the form $F(z, w) = (\lambda(z), f(z, w))$ and deal with the parabolic case, that is, when $DF(0, 0) = Id$. We describe the behaviour of orbits around a neighbourhood of the origin. We establish formulas for conjugacy maps in different regions of these neighbourhoods.

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1 Introduction

The dynamics of skew-product maps $F(z,w) = (\lambda(z), f(z,w))$ has been studied by several authors [11, 13, 16, 17, 18]. In this work we focus on local aspects of the theory, namely, we look at the dynamics of F close to a fixed point. For the sake of simplicity, we take said fixed point at the origin $(0,0)$. We turn our attention to a class of skew-product maps that we call **parabolic**, defined as those subject to $\lambda(z) = z + O(|z|^2)$ and $f(z,w) = w + O(|(z,w)|^2)$.

Skew-product maps are suitable to test general aspects of the dynamics of self-maps on several dimensions. Since the first coordinate depends only on one variable, we can borrow results from one dimensional complex dynamics to gain information. Nonetheless, they provide a richer theory than in dimension one. An instance of this fact can be seen in a recent article by Astorg *et al.* [3] where they describe a polynomial skew-product map in two dimensions that has a wandering Fatou component.

We center our study on maps given by

$$\begin{aligned} F :&(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \\ F(z,w) &= (\lambda(z), f(z,w)), \end{aligned} \tag{1.1}$$

where $\lambda(z) = z + a_2 z^2 + O(z^3)$, with $a_2 \neq 0$, and $f(z,w) = w + b_2 w^2 + O((z,w)^3)$, with $b_2 \neq 0$.

Our goal is to describe the dynamics of such maps in a neighbourhood of the origin. We divide our program into two categories.

A. Describe regions in which F is conjugated to a simpler map.

B. Find formulas for the conjugation map in each region, as in the one dimensional case.

Finding a conjugacy map to a simpler map depends strongly on the type of map we are studying and the dimension of the space.

Consider a holomorphic germ $F : (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ with a fixed point p. A local conjugacy of F to G is a one-to-one map $\phi : U_p \to \mathbb{C}^n$, from an open neighbourhood U_p around p, in such way that the conjugation $G = \phi^{-1} \circ F \circ \phi$ holds. In general, the main goal is to obtain a conjugacy to an easier map to study than F. This is a rich history that goes back to Schroeder. We redirect the reader to [1] or [15] for a list of helpful results.

In the case of a parabolic map F, that is, when $DF(p) = \mathrm{Id}$, only partial results are available when $n \geq 2$. The dynamics of parabolic maps in several dimensions is in general unpredictable [2, 8], and although some results have been proven for generic maps, much less is known in comparison with the one dimensional case.

One common feature of the study of parabolic maps is partial conjugacy to a translation. While usually this conjugacy cannot be realized on a whole neighbourhood around p, it is well defined on certain open sets with p at its boundary. The conjugacy is commonly referred to as a **Fatou coordinate**.

Fatou coordinates are useful for the study of parabolic maps. In dimension one for instance it is a crucial tool in the understanding of parabolic bifurcations.

Let us recall standard facts in dimension one. Consider the map $f(z) = z + a_2 z^2 + O(z^3)$, with $a_2 \neq 0$, where the origin is a parabolic fixed point. The Leau Fatou flower theorem states that there exists a parabolic basin B for the origin, that is, an open set with the origin at its boundary, where every point converges to the origin after iteration by f. There exists in fact a conjugacy of f to the translation map $g(w) = w+1$ in the set B. Similarly, there exists a repelling basin R converging to 0 under backward iteration. Likewise, we can construct a conjugacy to the translation. In this particular case, the union of B and R contains a full pinched neighbourhood of the origin [15].

Our goal is to describe the dynamics of parabolic maps in two dimensions in a similar fashion. That is, we would like to divide an entire neighbourhood of the origin into several open sets, in such a way that we can conjugate our parabolic map to a simpler map.

Our main results are stated as Theorem 5.1 and Theorem 5.2 which can be sumarized as follows.

Theorem. *Let F be as in* (1.1). *Then, after a change of coordinates, the set*

$$U = \{(z, w) \in \mathbb{C}^2, |z| < \epsilon, |w| < \epsilon, |w| < |z|^M\},$$

(where M can be chosen as large as desired) can be divided into regions where F is conjugated to a translation map.

One immediate consequence of the theorem above is that the set U is foliated by invariant curves.

While most of the conjugacy maps for the hyperbolic case can be obtained as a limit of iterates, Fatou coordinates are in general not so easily computed. In this article we provide formulas for Fatou coordinates for the class of skew-product parabolic maps as in (1.1).

This paper is organized as follows. In Section 2 we write down properties of Fatou coordinates, namely the way they are modified after we perform change of coordinates. In Section 3 we recall results from dimension one. Section 4 gathers results from [26], where we find a complete description of the dynamics of a more particular class of parabolic maps on a whole neighbourhood of the origin. In Section 5 we work the main theorem.

2 Fatou coordinates

Since we use Fatou coordinates of different maps throughout our work, we write down here the main definitions and properties. Set $F^1 = F$ and $F^k = F \circ F^{k-1}$ for all $k \geq 2$.

Let $F : (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ be a holomorphic germ with a fixed point p subject to $DF(p) = \mathrm{Id}$ and $F \neq \mathrm{Id}$. We will alternate between our fixed point p being the origin and the point at infinity. The hypotheses on the derivative of F guarantees that there is a local well defined inverse holomorphic germ F^{-1} on a neighbourhood of p.

Given $\zeta \in \mathbb{C}^k$ we write $T_\zeta : \mathbb{C}^k \to \mathbb{C}^k$ for the translation map $T_\zeta(z) = z + \zeta$. Unless otherwise stated, we take $\zeta \neq 0$.

Let $U^{\mathrm{i},F} \subset \mathbb{C}^n$ be an open connected set such that $p \in \partial U^{\mathrm{i},F}$ and such that for any $z \in U^{\mathrm{i},F}$, we have *(i)* $F(z) \in U^{\mathrm{i},F}$ and *(ii)* $\lim_{k\to\infty} F^k(z) = p$. When this is possible, we say $U^{\mathrm{i},F}$ is a **parabolic attracting basin** of F.

Let $U^{\mathrm{i},F} \subset \mathbb{C}^n$ be an attracting basin of F. Assume there is a holomorphic map $\phi^{\mathrm{i},F} : U^{\mathrm{i},F} \to \mathbb{C}^k$ such that the diagram

$$\begin{array}{ccc} U^{\mathrm{i},F} & \xrightarrow{F} & U^{\mathrm{i},F} \\ {\scriptstyle \phi^{\mathrm{i},F}}\big\downarrow & & {\scriptstyle \phi^{\mathrm{i},F}}\big\downarrow \\ \mathbb{C}^k & \xrightarrow{T_\zeta} & \mathbb{C}^k \end{array} \tag{2.1}$$

commutes. Then we say $\phi^{\mathrm{i},F}$ is an **incoming Fatou map** for F and $U^{\mathrm{i},F}$ with translation T_ζ.

Remark 2.1. If $\phi^{\mathrm{i},F}$ is an incoming Fatou map for F and $U^{\mathrm{i},F}$ with translation T_ζ, then $\lambda\phi^{\mathrm{i},F}$ is an incoming Fatou map for F and $U^{\mathrm{i},F}$ with translation $T_{\lambda\zeta}$ once we choose $\lambda \in \mathbb{C}^*$.

Remark 2.2. Note that we do not require $\phi^{\mathrm{i},F}$ to have an inverse map. In fact, in some cases, k, the target dimension of $\phi^{\mathrm{i},F}$, can be strictly smaller than n; in such case $\phi^{\mathrm{i},F}$ cannot be injective. In the literature this is sometimes referred as a **semi-conjugacy**.

Repelling basins as well as repelling Fatou maps are defined by considering the local inverse map F^{-1}. We do this next.

Whenever $U^{\mathrm{o},F} \subset \mathbb{C}^n$ is an open set such that $p \in \partial U^{\mathrm{o},F}$ and $U^{\mathrm{o},F}$ is an open attracting basin of F^{-1}, we will call $U^{\mathrm{o},F}$ a **repelling basin** of F.

Assume there exists an incoming Fatou map ψ for F^{-1} and $U^{\mathrm{o},F}$ with translation $T_{-\zeta}$ so that the diagram

$$\begin{array}{ccc} U^{\mathrm{o},F} & \xrightarrow{F^{-1}} & U^{\mathrm{o},F} \\ {\scriptstyle \psi}\big\downarrow & & {\scriptstyle \psi}\big\downarrow \\ \mathbb{C}^k & \xrightarrow{T_{-\zeta}} & \mathbb{C}^k \end{array}$$

commutes. Assume in addition that ψ has a holomorphic inverse map (this will imply $n = k$). Then we call the inverse $\phi^{\mathrm{o},F} = \psi^{-1}$ defined on $\psi(U^{\mathrm{o},F})$ the **outgoing Fatou map** for F and U^{o} with respect to T_ζ. A closer look at the functional equation satisfied by ψ and F^{-1} yields

$$F(\phi^{\mathrm{o},F}(z)) = \phi^{\mathrm{o},F}(z+\zeta) \text{ or } F \circ \phi^{\mathrm{o},F} = \phi^{\mathrm{o},F} \circ T_\zeta. \tag{2.2}$$

We point out some basic facts related to the concepts above.

Remark 2.3. When $n = 1$, we can assume without loss of generality $\zeta = 1$. The incoming or outgoing change of coordinates are usually referred to as, respectively, incoming or outgoing Fatou coordinates.

Remark 2.4. When $n \geq 2$ and $k = 1$, the incoming change of coordinate has been used in the past to prove the existence of Fatou-Bieberbach maps for automorphisms of $\mathbb{C}^n$ (compare [8], [25]).

Remark 2.5. It is easy to see that Fatou coordinates are not unique. From the functional relations we see that compositions (respectively pre-compositions) of translations with incoming (respectively outgoing) Fatou coordinates are also incoming (respectively outgoing) Fatou coordinates.

When there is no risk of confusion, we simply write ϕ^{i} and ϕ^{o}. For now, though, we stick to the superscript for referring to the maps in question since we want to establish how they behave when changing coordinates.

Proposition 2.6. *Let F be a parabollic germ as above. Assume F and F^{-1} have attracting basins $U^{i,F}$ and $U^{o,F} = U^{i,F^{-1}}$. Then F^{-1} has also a repelling basin, namely we can take $U^{o,F^{-1}} = U^{i,F}$. Let also $\phi^{i,F}$ (respectively $\phi^{o,F}$) be an incoming (respectively outgoing) Fatou coordinate for F and $U^{i,F}$ with respect to T_ζ. Assume further that $\phi^{i,F}$ has a well defined inverse map. Then the following formulas*

$$\phi^{o,F^{-1}}(z) = (\phi^{i,F})^{-1}(-z), \qquad \phi^{i,F^{-1}}(z) = -(\phi^{o,F})^{-1}(z) \tag{2.3}$$

yield outgoing (respectively incoming) Fatou maps for F^{-1} and $U^{o,F^{-1}}$ (respectively $U^{i,F^{-1}}$) with respect to T_ζ.

Proof. The proof follows easily by verifying directly the respective equations and using Remark 2.1. □

Although the following proposition is trivial, we will use the transformation between Fatou coordinates for different maps repeatedly on the following sections.

Proposition 2.7. *Let η be a (local) change of coordinates between F and G as in*

$$\begin{array}{ccc} (\mathbb{C}^n, q) & \xrightarrow{G} & (\mathbb{C}^n, q) \\ \eta \big\downarrow & & \eta \big\downarrow \\ (\mathbb{C}^n, p) & \xrightarrow{F} & (\mathbb{C}^n, p). \end{array} \tag{2.4}$$

Assume $U^{i,F}$ and $U^{o,F}$ are attracting and repelling basins for F along with $\phi^{i,F}$ and $\phi^{o,F}$ injective Fatou coordinates for F. Then the following provides attracting and repelling basins for G as well:

$$\begin{aligned} U^{i,G} &= \eta^{-1}(U^{i,F}), \qquad & U^{o,G} &= \eta^{-1}(U^{o,F}), \\ \phi^{i,G} &= \phi^{i,F} \circ \eta, \qquad & \phi^{o,G} &= \eta^{-1} \circ \phi^{o,F}. \end{aligned} \tag{2.5}$$

here $\phi^{i,G}$ (respectively $\phi^{o,G}$) is defined in $U^{i,G}$ (respectively $U^{o,G}$). □

Remark 2.8. One observation that we will use repeatedly on the next sections is that we do not need η to be defined on a whole neighbourhood of the origin. In fact, it is enough for η to be defined only on $U^{\mathrm{i},G}$.

3 Fatou coordinates in one dimension

Consider a parabolic germ at the origin of the form

$$f(z) = z + az^2 + O(|z|^3),$$

with $a \neq 0$. By the standard change of coordinates $Z = -az$ we reduce to the case $a = -1$. The following is the classic theorem of Leau and Fatou. See [15] for details.

Theorem 3.1. *(Leau-Fatou theorem) Take f as above. Then there exist $U^{i,f}$ and $U^{o,f}$ such that $U^{i,f} \cup U^{o,f}$ forms a punctured neighbourhood of the origin. In each of these open sets we can define incoming and outgoing Fatou coordinates $\phi^{i,f} : U^{i,f} \to \mathbb{C}$, $\phi^{o,f} : \psi(U^{o,f}) \to U^{o,f}$.* □

We can write down an explicit choice for the sets $U^{\mathrm{i},f}$ and $U^{\mathrm{o},f}$. Indeed, for any f there exist $\epsilon > 0$ so that $V_\epsilon = \{\zeta \in \mathbb{C}, |\zeta| < \epsilon, |\mathrm{Arg}(\zeta)| < 3\pi/4\}$ is an attracting basin and $-V_\epsilon$ is a repelling basin.

Next we translate all the action to a neighbourhood of ∞ using the reciprocal involution $S(z) = 1/z$. We then obtain

$$g(w) = w + 1 + \frac{\alpha}{w} + O(1/w^2),$$

where $g = S \circ f \circ S$. Our fixed point is relocated now at infinity. Let $S_R = S(V_\epsilon) = \{|w| > R, |\mathrm{Arg}(w)| < 3\pi/4\}$, where $R = 1/\epsilon$. Then we see that $U^{\mathrm{i},g}$ and $U^{\mathrm{o},g}$ can be chosen as S_R and $-S_R$, respectively.

We first start with a lemma as preparation.

Lemma 3.2. *Let $g(w) = w + 1 + \frac{\alpha}{w} + O(1/w^{1+\delta})$, where $\alpha \in \mathbb{C}$ and $\delta > 0$, be a holomorphic map defined on a neighbourhood of infinity. Take $U^{i,g}$ and $U^{o,g}$ as, respectively, attracting and repelling basins for g. Define $L_\alpha(w) = w + \alpha \log(w)$. Then $\rho = L_{-\alpha} \circ g \circ (L_{-\alpha})^{-1}$ is defined on $W = L_{-\alpha}(U^{i,g})$ and sastifies $\rho(W) \subset W$ and $\rho(w) = w + 1 + O(1/w^{1+\delta})$. Similarly, for $\tau = (L_\alpha)^{-1} \circ g \circ L_\alpha$ defined on $V = (L_\alpha)^{-1}(U^{o,g})$ we have $\tau(V) \supset V$ and $\tau(w) = w + 1 + O(1/w^{1+\delta})$.*

Proof. Note that L_α and $L_{-\alpha}$ are one-to-one maps on $U^{\mathrm{i},g}$ and $U^{\mathrm{o},g}$. The rest of the assertions are immediate. $\square$

Now consider the maps $\phi^{\mathrm{i},\rho}(w) = \lim_{n\to\infty} \rho^n(w) - n$. We see that this map is well defined on all of W (since $\rho(W) \subset W$) and from the estimate

$$|\rho^{n+1}(w) - \rho^n(w) - 1| = |O(1/(\rho^n(w))^{1+\delta}| = O(1/n^{1+\delta})$$

we deduce that $\{\rho^n(w) - n\}$ forms a Cauchy sequence. Hence the convergence is uniform on compact subsets of $U^{\mathrm{i},g}$.

Similarly $\phi^{o,\tau}(w) = \lim_{n\to\infty} \tau^n(w - n)$ is well defined on the open subset of $\mathbb{C}$ where it converges. Using the relations $\phi^{\mathrm{i},g} = \phi^{\mathrm{i},\rho} \circ L_{-\alpha}$ and $\phi^{\mathrm{o},g} = L_\alpha \circ \phi^{o,\tau}$ we can establish the following.

Proposition 3.3. *Let $g(w) = w + 1 + \frac{\alpha}{w} + O\left(\frac{1}{w^2}\right)$ be a germ at infinity. Then we take the incoming Fatou coordinate $\phi^{i,g} : U^{i,g} \to \mathbb{C}$ of g as the limit*

$$\phi^{i,g}(w) = \lim_{n\to\infty} L_{-\alpha}(g^n(w)) - n. \tag{3.1}$$

Similarly, we can find an outgoing Fatou coordinate $\phi^{i,g} : U^{o,g} \to \mathbb{C}$ *by means of the formula*

$$\phi^{o,g}(w) = \lim_{n\to\infty} g^n(L_\alpha(w-n)) \tag{3.2}$$

(recall the definition $L_\alpha(w) = w + \alpha\log(w)$*).*

4 Fatou coordinates in two dimensions

Let us recall our results from [26] for skew parabolic maps of the particular form

$$F(z,w) = \left(\frac{z}{1+z}, f_z(w)\right) = \left(\frac{z}{1+z}, w - w^2 + w^3 + O(w^4, zw^4)\right). \tag{4.1}$$

As usual, set $V_\epsilon = \{\zeta \in \mathbb{C}, |\zeta| < \epsilon, |\mathrm{Arg}(\zeta)| < 3\pi/4\}$. In dimension one, the union $V_\epsilon \cup (-V_\epsilon)$ forms a punctured neighbourhood of the origin. In dimension two, we use the four subsets

$$\begin{aligned} U^{\mathrm{i}} &= V_\epsilon \times V_\epsilon, \\ U^{\mathrm{o}} &= (-V_\epsilon) \times (-V_\epsilon), \\ U^{\mathrm{a}} &= (-V_\epsilon) \times V_\epsilon, \\ U^{\mathrm{b}} &= V_\epsilon \times (-V_\epsilon). \end{aligned} \tag{4.2}$$

Note that their union covers a full neighbourhood of the origin with the exception of the two axis $\{zw = 0\}$. (Anyway, since we have $F(0,w) = (0, f_0(w)))$ and $F(z,0) = (\frac{z}{1+z}, 0)$, the orbits of F on both axis are fully understood.)

As in the one dimensional case, we change variables so that the fixed point is at infinity by using the conjugation map $S(z,w) = (1/z, 1/w)$. In this way $G = S \circ F \circ S$ can be explicitly written as

$$G(u,v) = (u+1, g_u(v)) = \left(u+1, v+1+O\left(\frac{1}{v^2}, \frac{1}{uv^2}\right)\right). \tag{4.3}$$

Let $S_R = \{|\zeta| > R, |\mathrm{Arg}(\zeta)| < 3\pi/4\}$ with $R = 1/\epsilon$ (so that we have $I(\pm V_\epsilon) = \pm S_R$). We focus our attention on the sets

$$\begin{aligned} W^{\mathrm{i}} &= \quad S_R \times S_R, \\ W^{\mathrm{o}} &= -S_R \times -S_R, \\ W^{\mathrm{a}} &= -S_R \times S_R, \\ W^{\mathrm{b}} &= \quad S_R \times -S_R. \end{aligned} \tag{4.4}$$

Let $T_{(a,b)} : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as $T_{(a,b)}(z, w) = (z + a, w + b)$.

Theorem 4.1. *Let G be as in* (4.3).
(a) For any $p \in W^i$, the iterates $G^n(p)$ converge to infinity. We have a Fatou coordinate given by $\Phi^{i,G} = \lim_{n\to\infty} T_{(-n,-n)} \circ G^n$ so that the diagram

$$\begin{array}{ccc} W^i & \xrightarrow{G} & W^i \\ {\scriptstyle \Phi^{i,G}}\big\downarrow & & {\scriptstyle \Phi^{i,G}}\big\downarrow \\ \mathbb{C}^2 & \xrightarrow{T_{(1,1)}} & \mathbb{C}^2 \end{array}$$

commutes.
(b) For any $p \in W^o$, the backward iterates $G^{-n}(p)$ converges to infinity. We have a Fatou coordinate given by $\Phi^{o,G} = \lim_{n\to\infty} G^n \circ T_{(-n,-n)}$ so that the diagram

$$\begin{array}{ccc} G^{-1}(W^o) & \xrightarrow{G} & W^o \\ {\scriptstyle \Phi^{o,G}}\big\uparrow & & {\scriptstyle \Phi^{o,G}}\big\uparrow \\ N \subset \mathbb{C}^2 & \xrightarrow{T_{(1,1)}} & T_{(1,1)}(N) \subset \mathbb{C}^2 \end{array}$$

commutes. Here we have $N = (\Phi^{o,G})^{-1}(G^{-1}(W^o))$ and $T_{(1,1)}(N) = (\Phi^{o,G})^{-1}(W^o)$.

Proof. We prove first the existence of $\Phi^{\mathrm{i},G}$. Then we will apply this same result to prove the analogue for $\Phi^{\mathrm{o},G}$.

Let $(u, v) \in W^{\mathrm{i}}$, that is $u \in S_R$ and $v \in S_R$, and write $(u_n, v_n) = G^n(u, v)$. We have $u + 1 \in S_R$ and $|v_1 - v - 1| < 1/10$ for R large enough. In this way we have $(u_1, v_1) \in W^{\mathrm{i}}$ and therefore, by induction, also $(u_n, v_n) \in W^{\mathrm{i}}$ for positive n. In fact we have $u_n = u_0 + n$ and $|v_n - v_0| = O(n)$.

For $\Phi_n^{\mathrm{i},G} = T_{(-n,-n)} \circ G^n$, a simple computation yields

$$\begin{aligned} |\Phi_{n+1}^{\mathrm{i},G} - \Phi_n^{\mathrm{i},G}| &= |G^{n+1}(u,v) - G^n(u,v) - (1,1)| \\ &= |G(u_n, v_n) - (u_n, v_n) - (1,1)| \\ &= |(0, O(1/v_n^2, 1/(u_n v_n^2)))|. \end{aligned}$$

As u_n and v_n are of growth $O(n)$ when $(u,v) \in W^{\mathrm{i}}$, we conclude that $\Phi_n^{\mathrm{i},G}$ converges uniformly in compact sets of W^{i}.

For the outgoing coordinate we write $\Phi_n^{\mathrm{o},G} = G^n \circ T_{(-n,-n)}$. Then we have the relation

$$\Phi_n^{\mathrm{o},G} \circ \eta \circ \Phi_n^{\mathrm{i},H} \circ \eta = \mathrm{Id}, \tag{4.5}$$

here $H = \eta \circ G^{-1} \circ \eta$ and $\eta(u,v) = (-u,-v)$. Since we have $H(u,v) = (u+1, v+1+O(1/v^2))$, we note that $\Phi_n^{\mathrm{i},H}$ converges, and therefore $\Phi_n^{\mathrm{o},G}$ does also. Finally, as $\eta(W^{\mathrm{i}}) = W^{\mathrm{o}}$ and $H(W^{\mathrm{i}}) \subset W^{\mathrm{i}}$ hold, we conclude $W^{\mathrm{o}} \subset \Phi^{\mathrm{o},G}(W^{\mathrm{o}})$. □

We also have the following.

Theorem 4.2. *Let G be as in (4.3).*
(a) The map $\Psi^{a,G} = \lim_{n\to\infty} T_{(n,-n)} \circ G^n \circ T_{(-2n,0)}$ converges uniformly on compact subsets of W^a and fits into the commutative diagram

$$\begin{array}{ccc} W^a & \xrightarrow{(-1,g_\infty)} & W^a \\ {\scriptstyle \Psi^{a,G}}\big\downarrow & & {\scriptstyle \Psi^{a,G}}\big\downarrow \\ \mathbb{C}^2 & \xrightarrow{T_{(-1,1)}} & \mathbb{C}^2. \end{array}$$

(b) The map $\Psi^{b,G} = \lim_{n\to\infty} T_{(-2n,0)} \circ G^n \circ T_{(n,-n)}$ converges uniformly on compact sets of W^b and we have the diagram

$$\begin{array}{ccc} L^{-1}(W^b) & \xrightarrow{L=(-1,g_\infty)} & W^b \\ {\scriptstyle \Psi^{b,G}}\big\uparrow & & {\scriptstyle \Psi^{b,G}}\big\uparrow \\ E \subset \mathbb{C}^2 & \xrightarrow{T_{(-1,1)}} & T_{(-1,1)}(E) \subset \mathbb{C}^2. \end{array}$$

Proof. Define $\Psi_n^{\mathrm{a},G} = T_{(n,-n)} \circ G^n \circ T_{(-2n,0)}$ and $\Psi_n^{\mathrm{b},G} = T_{(-2n,0)} \circ G^n \circ T_{(n,-n)}$. Unraveling the definition we arrive to

$$\Psi_n^{\mathrm{b},G}(u,v) = (u, g_{u+2n-1} \circ \ldots \circ g_{u+n+1} \circ g_{u+n}(v-n)).$$

In [26] it is proven that the sequence of functions

$$\psi_n^o(v) = g_{-v+\alpha+2n-1} \circ \ldots \circ g_{-v+\alpha+n+1} \circ g_{-v+\alpha+n}(v-n)$$

converges for any $\alpha \in \mathbb{C}$ and $v \in -S_R$ with limit $\psi^o(v+1) = g_\infty(\psi^o(v))$. Applying this result for $\alpha = u - v$, we obtain the convergence of the sequence $\Psi_n^{\mathrm{b},G}(u,v) \to (u, \psi^o(v))$, where $\psi^o(v+1) = g_\infty(\psi^o(v))$.

For the other coordinate we use the identity

$$\Psi_n^{\mathrm{a},G} \circ \eta \circ \Psi_n^{\mathrm{b},H} \circ \eta = \mathrm{Id}, \tag{4.6}$$

where $H = \eta \circ G^{-1} \circ \eta$ and $\eta(u,v) = (-u,-v)$. Since we have $H(u,v) = (u+1, v+1+O_u(1/v^2))$, we conclude that $\Psi_n^{\mathrm{b},H}$ converges, and therefore $\Psi_n^{\mathrm{a},G}$ also does. □

We sumarize the results thus obtained for our map F. We name $(0,w)$ the **invariant fiber** of F.

Theorem 4.3. *Let F be as in* (4.1) *and U^i, U^o, U^a and U^b be defined as in* (4.2).
(a) The union of U^i, U^o, U^a and U^b together with the axes form a neighbourhood of the origin in $\mathbb{C}^2$.
(b) For any $q \in U^i$ we have $F^n(q) \in U^i$. Furthermore F^n converges to the origin uniformly in compact sets of U^i.
(c) For any $q \in U^o$ we have $F^{-n}(q) \in U^o$. Furthermore F^{-n} converges to the origin uniformly in compact sets of U^o.
(d) For any $q \in U^a$ we have that $F^{-n}(q)$ converges to the w-axis, the invariant fiber of the map F.
(e) For any $p \in U^b$ we have that $F^n(p)$ converges to the w-axis, the invariant fiber of the map F.

5 The general case

We are ready to tackle the general case. Consider now the map

$$F(z,w) = (\lambda(z), f_z(w)), \tag{5.1}$$

where $\lambda(z) = z + O(z^2)$ and $f_z(w) = w + O(|(z,w)|^2)$. We focus on the particular case

$$F(z,w) = (z + a_2 z^2 + O(z^3), w + b_2 w^2 + O(|(z,w)|^3)),$$

with $a_2 \neq 0$ and $b_2 \neq 0$.

By a change of coordinates we can even assume $a_2 = -1$ and $b_2 = -1$. Using a shear polynomial as a further change of coordinates, we can increase the power of z on the second term. Similarly by using another polynomial change of variables we can increase the degree of the z term that is multiplied by w. Therefore we can assume F takes the form

$$F(z,w) = (z - z^2 + O(z^3), w - w^2 + O(w^3, zw^2, z^{M+1}w, z^{M+1})), \tag{5.2}$$

with M as large as we wish.

As we still have $F(0,w) = (0, w - w^2 + O(w^3))$, we can again refer to the w-axis as an invariant fiber of F. However, this time we have $F(z,0) = (z - z^2 + O(z^3), O(z^{M+1}))$, so the z-axis is no longer invariant. To work around this issue we use the main result of [8] which states that on V_ϵ there exists an analytic function $\phi_1(z)$ subject to $\phi_1(\lambda(z)) = f_z(\phi_1(z))$. Similarly on $-V_\epsilon$ there exists an analytic function $\phi_2(z)$ such that $\phi_2(\lambda(z)) = f_z(\phi_2(z))$.

We can therefore change coordinates on $V_\epsilon \times \{|w| < \epsilon\}$ by means of $(z,w) \mapsto (z, w - \phi_1(z))$, and on the set $(-V_\epsilon) \times \{|w| < \epsilon\}$ by $(z,w) \mapsto (z, w - \phi_2(z))$. On these new coordinates we read

$$F(z,w) = (z - z^2 + O(z^3), w - w^2 + O(w^3, zw^2, z^{M+1}w)). \tag{5.3}$$

The next step is to linearize the first coordinate. As we know from the one dimensional theory, there exist two maps, ρ_1 on V_ϵ and ρ_2 on $-V_\epsilon$, that conjugate $\lambda(z)$ to the translation $u \to u+1$ on V_ϵ and $-V_\epsilon$. We have for both the estimate $\rho(z) = z + O(z^2 \log(z))$ (see [15] for details).

One more time we use the change of coordinate $(u, w) \mapsto (u, v) = (u, 1/w)$. On the new system we have

$$G(u,v) = \left(u+1, v+1+O\left(\frac{1}{u}, \frac{1}{v}, \frac{\log(u)}{u^2}, \frac{v}{u^{M+1}}, \frac{v\log(u)}{u^{M+2}}\right)\right). \quad (5.4)$$

Now, as before, we divide a punctured neighbourhood of infinity in several sets:

$$\begin{aligned} W^{\mathrm{i}} &= S_R \times S_R, & W^{\mathrm{b}} &= S_R \times (-S_R) \\ W^{\mathrm{a}} &= (-S_R) \times S_R, & W^{\mathrm{o}} &= (-S_R) \times (-S_R). \end{aligned}$$

From now on, when we refer to a region W, we mean one of the four possible $W^{\mathrm{i}}, W^{\mathrm{b}}, W^{\mathrm{a}}$ or W^{o}.

Consider the class of maps $\Theta(u,v) = (u, v + \alpha\log(u) + \beta\log(v))$. It is immediate that after choosing R large enough Θ is injective in each region W. So, by conjugating G_j by Θ and choosing α and β appropriately, we can get rid of the linear terms $O(1/u, 1/v)$.

To emphasize that each of these maps is a different conjugation of G on each set W: we use θ_{i} for the change of coordinates on W^{i}, θ_{o} on W^{o}, θ_{a} on W^{a} and θ_{b} on W^{b}. We write $G_{\mathrm{i}} = (\theta_{\mathrm{i}})^{-1} \circ G_1 \circ \theta_{\mathrm{i}}$, the corresponding map defined on W^{i}, and $G_{\mathrm{o}} = (\theta_{\mathrm{o}})^{-1} \circ G_2 \circ \theta_{\mathrm{o}}$ on W^{o}, and $G_{\mathrm{a}} = (\theta_{\mathrm{a}})^{-1} \circ G_2 \circ \theta_{\mathrm{a}}$ on W^{a}, and $G_{\mathrm{b}} = (\theta_{\mathrm{b}})^{-1} \circ G_1 \circ \theta_{\mathrm{b}}$ on W^{b}. Now, the composition $\Theta^{-1} \circ G_j \circ \Theta(u,v)$ equals

$$\left(u+1, v+1+O\left(\frac{1}{u^2}, \frac{1}{v^2}, \frac{\log(v)}{v^2}, \frac{\log(u)}{v^2}, \frac{v}{u^{M+1}}, \frac{v\log(u)}{u^{M+2}}\right)\right).$$

Note the similarity with the special map on last section where we have

$$G(u,v) = (u+1, g_u(v)) = \left(u+1, v+1+O\left(\frac{1}{v^{1+\delta}}, \frac{1}{uv^{1+\delta}}\right)\right).$$

In order to control the mixed terms in u and v on our maps G we define

$$\begin{aligned} \widetilde{W^{\mathrm{i}}} &= \{(u,v) \in S_R \times S_R, |u|^{M-1} > |v|\}, \quad (5.5) \\ \widetilde{W^{\mathrm{o}}} &= \{(u,v) \in (-S_R) \times (-S_R), |u|^{M-1} > |v|\}. \end{aligned}$$

For $(u,v) \in \widetilde{W^{\mathrm{i}}}$ we have $G_{\mathrm{i}}^n(u,v) = (u+n, v+O(n))$. Therefore, eventually we reach $G_{\mathrm{i}}^n(u,v) \in \widetilde{W^{\mathrm{i}}}$. Using results from the last section, it is possible to conjugate G_{i} to a translation on $\widetilde{W^{\mathrm{i}}}$ by means of the limit $\Phi^{\mathrm{i},G_{\mathrm{i}}} = \lim_{n\to\infty} T_{(-n,-n)} \circ G_{\mathrm{i}}^n$, with $\Phi^{\mathrm{i},G_{\mathrm{i}}} : \widetilde{W^{\mathrm{i}}} \to \mathbb{C}^2$. If we unravel for G, we obtain a formula for the Fatou coordinate on the incoming basin for G as

$$\Phi^{\mathrm{i},G}(u,w) = \lim_{n\to\infty} T_{(-n,-n)} \circ \theta_{\mathrm{i}}^{-1} \circ \Psi_1^{-1} \circ G^n, \tag{5.6}$$

where Ψ_1 is the composition of the change of coordinates from above and $W^{\mathrm{i},G} = \Psi_1(\widetilde{W^{\mathrm{i}}})$.

Similarly, we obtain $G_{\mathrm{o}}(\widetilde{W^{\mathrm{o}}}) \supset G_{\mathrm{o}}$. By our work on the last section we achieve a conjugation $\Phi^{\mathrm{o},G_{\mathrm{o}}} : \widetilde{W^{\mathrm{o}}} \to \mathbb{C}^2$ of G on $W^{\mathrm{o},G}$ to the translation

$$\Phi^{\mathrm{o},G_{\mathrm{o}}} = \lim_{n\to\infty} G_{\mathrm{o}}^n \circ T_{(-n,-n)}.$$

Rewritting for G we obtain the Fatou coordinate $\Phi^{\mathrm{o},G} : W^{\mathrm{o},G} \to \mathbb{C}^2$ on the outgoing basin for G as

$$\Phi^{\mathrm{o},G}(u,w) = \lim_{n\to\infty} G^n \circ \Psi_2 \circ \theta_{\mathrm{o}} \circ T_{(-n,-n)}. \tag{5.7}$$

We have therefore established the following result.

Theorem 5.1. *Given G as in* (5.4)*, we can find incoming and outgoing Fatou coordinates for the respective incoming and outgoing basins at infinity.* □

We also obtain information on the behavior of G on the regions W^{a} and W^{b}, since we can apply Theorem 4.2 to the maps G_{a} and G_{b} respectively.

In Theorem 4.2 we proved that on the region W^{a} the map $\Psi^{\mathrm{a}} = \lim_{n\to\infty} T_{(n,-n)} \circ G_{\mathrm{a}}^n \circ T_{(-2n,0)}$ satisfies $\Psi^{\mathrm{a}} \circ (-1, g_\infty) = T_{(-1,1)} \circ \Psi^{\mathrm{a}}$. Since $G_{\mathrm{a}} = (\theta_{\mathrm{a}})^{-1} \circ G_2 \circ \theta_{\mathrm{a}}$ holds, we get $G_{\mathrm{a}}^n = (\theta_{\mathrm{a}})^{-1} \circ (\Psi_2)^{-1} \circ G^n \circ \Psi_2 \circ \theta_{\mathrm{a}}$, and so $\Psi^{\mathrm{a}} = \lim_{n\to\infty} T_{(n,-n)} \circ (\theta_{\mathrm{a}})^{-1} \circ (\Psi_2)^{-1} \circ G^n \circ \Psi_2 \circ \theta_{\mathrm{a}} \circ T_{(-2n,0)}$ converges and fits into a corresponding commutative diagram.

Similarly, on the region W^{b}, the map $\Psi^{\mathrm{b}} = \lim_{n\to\infty} T_{(-2n,0)} \circ G_{\mathrm{b}}^n \circ T_{(n,-n)}$ satisfies $\Psi^{\mathrm{b}} \circ (-1, g_\infty) = T_{(-1,1)} \circ \Psi^{\mathrm{b}}$. From $G_{\mathrm{b}} = (\theta_{\mathrm{b}})^{-1} \circ G_1 \circ \theta_{\mathrm{b}}$, we obtain $G_{\mathrm{b}}^n = (\theta_{\mathrm{b}})^{-1} \circ (\Psi_1)^{-1} \circ G^n \circ \Psi_1 \circ \theta_{\mathrm{b}}$, and thus $\Psi^{\mathrm{b}} = \lim_{n\to\infty} T_{(-2n,0)} \circ (\theta_{\mathrm{b}})^{-1} \circ (\Psi_1)^{-1} \circ G^n \circ \Psi_1 \circ \theta_{\mathrm{b}} \circ T_{(n,-n)}$ converges and fits again into the corresponding commutative diagram.

We have thus settle the following.

Theorem 5.2. *For G as in* (5.4)*, on the regions*

$$\begin{aligned} \widetilde{W^a} &= \{(u,v) \in -S_R \times S_R, |u|^{M-1} > |v|\}, \\ \widetilde{W^b} &= \{(u,v) \in S_R \times (-S_R), |u|^{M-1} > |v|\}, \end{aligned} \tag{5.8}$$

the limits: $\Psi^a = \lim_{n\to\infty} T_{(n,-n)} \circ (\theta_a)^{-1} \circ (\Psi_2)^{-1} \circ G^n \circ \Psi_2 \circ \theta_a \circ T_{(-2n,0)}$ *and* $\Psi^b = \lim_{n\to\infty} T_{(-2n,0)} \circ (\theta_b)^{-1} \circ (\Psi_1)^{-1} \circ G^n \circ \Psi_1 \circ \theta_b \circ T_{(n,-n)}$ *exist. Furthermore, the second coordinate conjugates g_∞ to the translation T_1, that is, the diagram*

$$\begin{array}{ccc} \widetilde{W^{a,b}} & \xrightarrow{(-1,g_\infty)} & \widetilde{W^{a,b}} \\ {\scriptstyle \Psi^{a,b}}\big\downarrow & & {\scriptstyle \Psi^{a,b}}\big\downarrow \\ \mathbb{C}^2 & \xrightarrow{T_{(-1,1)}} & \mathbb{C}^2 \end{array}$$

commutes for each case $\widetilde{W^a}$ and $\widetilde{W^b}$. □

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Resumen

La dinámica local en torno a vecindades de un punto fijo ha sido ampliamente estudiada tanto para gérmenes de una como de varias variables complejas. En dimensión uno disponemos de un cuadro casi completo de la trayectoria de las órbitas en torno a una vecindad del punto fijo. No obstante, en dimensiones más altas, apenas se cuenta con resultados parciales. En este trabajo analizamos un caso intermedio entre las dinámicas de una y varias variables. Consideramos aplicaciones de productos trenzados de la forma $F(z, w) = (\lambda(z), f(z, w))$ y tratamos el caso parabólico, es decir, cuando $DF(0, 0) = \mathrm{Id}$. Describimos el comportamiento de órbitas en torno a vecindades del origen. Además, establecemos fórmulas para las aplicaciones de conjugación en diferentes regiones.

Palabras clave: Aplicaciones de productos trenzados, coordenadas de Fatou.

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