On the intersection of two longest paths in k-connected graphs

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Abstract

We show that every pair of longest paths in a k-connected graph on n vertices intersect each other in at least $\min\{n, (8k - n + 2)/5\}$ vertices. We also show that, in a 4-connected graph, every pair of longest paths intersect each other in at least four vertices. This confirms a conjecture of Hippchen for k-connected graphs when $k \leq 4$ or $k \geq (n - 2)/3$.

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1. Introduction

It is known that every pair of longest paths in a connected graph intersect each other in at least a vertex. Hippchen [5, Conjecture 2.2.4] conjectured that, for k-connected graphs, every pair of longest paths intersect each other in at least k vertices. A similar conjecture, for cycles instead of paths, was proposed by Grötschel and attributed to Scott Smith [4, Conjecture 5.2].

Smith's conjecture has been verified up to $k = 6$ [4], and for a general k, it was proved that every pair of longest cycles intersect in at least $ck^{3/5}$ vertices, for a constant $c \approx 0.2615$ [2]. However, for Hippchen's conjecture, the only nontrivial result is for $k = 3$ and was proved by Hippchen himself [5, Lemma 2.2.3]. In this paper, we verify Hippchen's conjecture for $k = 4$.

For $k \geq 5$, Hippchen's conjecture seems hard to prove. Hence, it is natural to ask for lower bounds on the intersection of two longest paths in k-connected graphs. Note that, if the graph has a Hamiltonian path, then it is clear that we have a lower bound of k. As Hamiltonian paths appear in highly connected graphs, this motivates us to study cases in which k is a fraction of n. In this paper, we show that any two longest paths intersect in at least $\min\{n, (8k - n + 2)/5\}$ vertices.

Finally, we exhibit, for any k, an infinite family of graphs that make Hippchen's conjecture tight.

2. Preliminaries

In this paper all graphs are simple (without loops or parallel edges) and the notation and terminology are standard. When we refer to paths, we mean simple paths (without repetitions of edges or vertices). The **length** of a path P is the number of edges it has, and it is denoted by $|P|$. A **longest path** in a graph is a path with maximum length among all paths. Given a path P and two vertices x and y in P, we denote by $P[x, y]$ the subpath of P with extremes x and y. Also, we denote the length of $P[x, y]$ by $\mathrm{dist}_P(x, y)$.

Given two set of vertices S and T in a graph G, an $\boldsymbol{S}$-$\boldsymbol{T}$ **path** is a path with one extreme in S, the other extreme in T, and whose internal vertices are neither in S nor T. If $S = \{v\}$, we also say that an S-T path is a v-T path. When we refer to the intersection of two paths in a graph, we mean vertex-intersection, that is, the set of vertices the paths share. Two paths are **internally disjoint** if they have no internal vertices in common.

A graph G is $\boldsymbol{k}$**-connected** if, for any two distinct vertices u and v in G, there exists a set of k u-v internally disjoint paths. It is easy to see that for a k-connected graph on n vertices, we have $k \leq n - 1$.

Our proofs rely in two well-known facts, that we state in the following propositions. The first proposition is also known as Fan lemma.

Proposition 2.1 ([1, Proposition 9.5])**.** *Let G be a k-connected graph. Let $v \in V(G)$ and $S \subseteq V(G) \setminus \{v\}$. If $|S| \geq k$, then there exists a set of k v-S internally disjoint paths. Moreover, every two paths in this set have $\{v\}$ as their intersection.*

The second proposition is an easy corollary of the following result of Dirac.

Theorem 2.2 ([3, Theorems 3 and 4])**.** *If G is a 2-connected graph on n vertices with minimum degree k, then G has a longest cycle of length at least* $\min\{2k, n\}$.

Proposition 2.3. *The length of a longest path in a k-connected graph on n vertices is at least* $\min\{2k, n - 1\}$.

Proof. If $k = 1$ then the proof is trivial, so let us assume $k \geq 2$. As G is k-connected, every vertex in G has degree at least k. Hence, by Theorem 2.2, there exists a cycle C in G with length at least $\min\{2k, n\}$. If $|C| = n$, then, by removing an edge from C, we obtain a path of length $n - 1$. If $|C| < n$, then, as G is connected, there exists an edge uv such that $u \in V(C)$ and $v \in V(G - C)$. Let w be a vertex adjacent to u

in C. Then $C - uw + uv$ is a path in G of length $|C| \geq \min\{2k, n\} \geq \min\{2k, n-1\}$. $\square$

3. High connectivity

In this section we show an interesting result for k-connected graphs. We begin with a simple observation.

Proposition 3.1. *Let G be a k-connected graph on n vertices. Let L be the length of a longest path in G. If P and Q are two longest paths in G, then $|V(P) \cap V(Q)| \geq 2L + 2 - n$.*

Proof. It suffices to note

$$|V(P) \cap V(Q)| = |V(P)| + |V(Q)| - |V(P) \cup V(Q)| \geq 2L + 2 - n,$$

as we want. $\square$

Proposition 3.1 together with Proposition 2.3 are enough to give a nontrivial result on Hippchen's conjecture.

Corollary 3.2. *Let G be a k-connected graph on n vertices. If $k \geq (n-2)/3$, then every two longest paths intersect in at least k vertices.*

Moreover, a stronger result can be derived from these two propositions: every pair of longest paths intersect in at least $\min\{n, 4k + 2 - n\}$ vertices. The rest of this section is devoted to improve this result when $k < \frac{n-2}{3}$. The improvement relies in the following lemma. Its proof is given after Theorem 3.4.

Lemma 3.3. *Let G be a k-connected graph with $k < (n-1)/2$. Let L be the length of a longest path in G. If P and Q are two longest paths in G, then $|V(P) \cap V(Q)| \geq 2k - L/2$.*

With that lemma at hand, it is easy to settle the main result of this section.

Theorem 3.4. *Let G be a k-connected graph on n vertices. If P and Q are two longest paths in G, then $|V(P) \cap V(Q)| \geq \min\{n, (8k-n+2)/5\}$.*

Proof. Let L be the length of a longest path in G. If $k \geq (n-1)/2$ then, by Proposition 2.3, we have $L \geq n-1$. So, $|V(P) \cap V(Q)| = n \geq \min\{n, (8k-n+2)/5\}$. Hence, we may assume $k < (n-1)/2$.

By Proposition 3.1 and Lemma 3.3, we have

$$\begin{aligned} |V(P) \cap V(Q)| &\geq \max\{2L+2-n, 2k-L/2\} \\ &\geq \frac{1}{5} \cdot (2L+2-n) + \frac{4}{5} \cdot (2k-L/2) \\ &= (8k-n+2)/5, \end{aligned}$$

as we want. □

We now proceed with the proof of Lemma 3.3.

Proof of Lemma 3.3. Let $X = V(P) \cap V(Q)$. Let q be an extreme of Q. Suppose for a moment that we have $q \in X$. As G is k-connected, q has at least k neighbors in Q. Let X' be the set of vertices of Q that are next to a vertex in X considering the order of the path starting at q. That is, $X' = \{x' \in V(Q) : \text{ there exists a vertex } x \in X \text{ with } Q[q,x'] = Q[q,x] + xx'\}$. If every neighbor of q is in X', then, as $L \geq \min\{n-1, 2k\} = 2k$ by Proposition 2.3, we have $|X| \geq |X'| \geq k \geq 2k - L/2$ and we are done.

Hence, there exists a neighbor r of q in $V(Q) \setminus X'$. Let q' be the vertex adjacent to r in Q that is closer to q in Q. In that situation, the path $Q' = Q + rq - rq'$ is a longest path, with q' as one of its extremes (this interchange is known as Pósa's rotation [6]). Since $r \notin X'$, we have $q' \notin X$. Hence, as $V(Q') = V(Q)$, from now on, we may assume $q \notin X$.

By Proposition 2.1, as $|V(P)| \geq k$, there exists a set, say $\mathcal{R}$, of k q-$V(P)$ internally disjoint paths that end at different vertices of P. Let $\mathcal{R}_A$ be the set of paths in $\mathcal{R}$ that have an extreme in X. That is, $\mathcal{R}_A = \{R \in \mathcal{R} : V(R) \cap X \neq \emptyset\}$. Let $\mathcal{R}_B = \mathcal{R} \setminus \mathcal{R}_A$. Let A and B be the set of corresponding extremes of $\mathcal{R}_A$ and $\mathcal{R}_B$, respectively; that is, $A = \{a \in V(P) \cap V(R) : R \in \mathcal{R}_A\}$ and $B = \{b \in V(P) \cap V(R) : R \in \mathcal{R}_B\}$.

Claim 1. *If $R \in \mathcal{R}_B$, then $|R| \geq 2$.*

Proof. Suppose by contradiction that $|R| = 1$. Let b be the extreme of R different from q. As $b \notin V(Q)$, the path $Q + qb$ is longer than Q, a contradiction. $\square$

Let p_1 and p_2 be the two extremes of P. Put $\mathcal{R} = \{R_1, R_2, \ldots, R_k\}$, and, for $1 \leq i \leq k$, let v_i be the corresponding extreme of R_i that is in P. Moreover, we may assume that $\mathrm{dist}_P(p_1, v_i) < \mathrm{dist}_P(p_1, v_{i+1})$ holds for $1 \leq i \leq k-1$.

Claim 2. *We have $dist_P(p_1, v_1) \geq |R_1|$ and $dist_P(v_k, p_2) \geq |R_k|$.*

Proof. It suffices to note that $P - P[p_1, v_1] + R_1$ and $P - P[v_k, p_2] + R_k$ are paths. $\square$

Claim 3. *For $1 \leq i \leq k-1$, we have $dist_P(v_i, v_{i+1}) \geq |R_i| + |R_{i+1}|$.*

Proof. It suffices to note that $P - P[v_i, v_{i+1}] + R_i + R_{i+1}$ is a path. $\square$

By Claims 1, 2 and 3, we have

$$\begin{aligned}
L &= |E(P)| \\
&= \mathrm{dist}_P(p_1, v_1) + \sum_{i=1}^{k-1} \mathrm{dist}_P(v_i, v_{i+1}) + \mathrm{dist}_P(v_k, p_2) \\
&\geq |R_1| + \sum_{i=1}^{k-1} (|R_i| + |R_{i+1}|) + |R_k| \\
&= 2\sum_{i=1}^{k} |R_i| \\
&= 2\sum_{R \in \mathcal{R}_{\mathcal{A}}} |R_A| + 2\sum_{R \in \mathcal{R}_{\mathcal{B}}} |R_B| \\
&\geq 2|A| + 4|B| \\
&= 4k - 2|A| \\
&= 4k - 2|X|.
\end{aligned}$$

Hence, we conclude

$$|V(P) \cap V(Q)| = |X| \geq 2k - L/2,$$

as we want. $\square$

4. Low connectivity

In this section, we show Hippchen's conjecture for $k = 4$. We begin with a useful lemma.

Lemma 4.1. *Let P and Q be two longest paths in a graph G. Let $u \in V(P) \cap V(Q)$, $v \in V(P) \setminus V(Q)$, and $w \in V(Q) \setminus V(P)$. If $P[u, v]$ is internally disjoint from Q and $Q[u, w]$ is internally disjoint from P, then there is no vw-path internally disjoint from both P and Q.*

Proof. Suppose by contradiction that there is a vw-path R internally disjoint from P and Q. If, for instance, P has an extreme x in R, then $P + R[x, w]$ is a path longer than P, a contradiction. Hence, we may assume that R has no vertices in neither P nor Q apart from v and w. Note that $P - P[u, v] + R + Q[u, w]$ and $Q - Q[u, w] + R + P[u, v]$ are both paths, whose lengths sum $|P| + |Q| + 2|R|$, a contradiction. $\square$

We now proceed to prove the main result of this section. An **independent set** in a graph is a set of pairwise non-adjacent vertices.

Theorem 4.2. *Every pair of longest paths in a 4-connected graph intersect in at least four vertices.*

Proof. Let G be a 4-connected graph and let P and Q be two longest paths in G. Suppose by contradiction that P and Q do not intersect in at least four vertices. As G is 3-connected, P and Q intersect in at least three vertices [5, Lemma 2.2.3]. Hence, P and Q intersect in exactly three vertices, say a, b and c. Let p_1 and p_2 be the extremes of P. Suppose, without loss of generality, that abc is a subsequence in P considering the ordering from p_1. That is, $\mathrm{dist}_P(p_1, a) < \mathrm{dist}_P(p_1, b) < \mathrm{dist}_P(p_1, c)$.

For simplicity of notation, we set $P_a = P[p_1, a]$, $P_{ab} = P[a, b]$, $P_{bc} = P[b, c]$ and $P_c = P[c, p_2]$. We also let $S = \{a, b, c\}$, $G' = G - S$, $P' = P - S$, $P'_a = P_a - S$, $P'_{ab} = P_{ab} - S$, $P'_{bc} = P_{bc} - S$ and $P'_c = P_c - S$. Without loss of generality, we have two cases, depending on the order in which a, b and c appear in Q. In each of these cases, we assume similar notation to the subpaths of Q and $Q - S$ as we did for P.

Case 1: abc **is a subsequence in** Q**.**

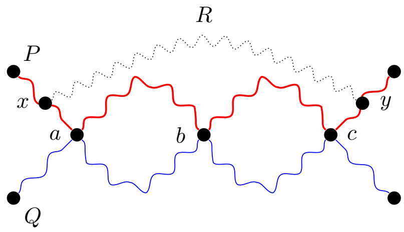
It is easy to see that $|P_a| = |Q_a|$, $|P_{ab}| = |Q_{ab}|$, $|P_{bc}| = |Q_{bc}|$, and $|P_c| = |Q_c|$. Hence, $P - P_a + Q_a$, $Q - Q_a + P_a$, $P - P_{ab} + Q_{ab}$, $Q - Q_{ab} + P_{ab}$, $P - P_{bc} + Q_{bc}$, $Q - Q_{bc} + P_{bc}$, $P - P_c + Q_c$ and $Q - Q_c + P_c$ are longest paths.

Let H be an auxiliary graph given by $V(H) = \{P'_a, P'_{ab}, P'_{bc}, P'_c, Q'_a, Q'_{ab}, Q'_{bc}, Q'_c\}$ and $E(H) = \{XY$: there is a X-Y path in G' with no internal vertex in $V(P) \cup V(Q)\}$. By Lemma 4.1, the sets $\{P'_a, Q'_a, P'_{ab}, Q'_{ab}\}$, $\{P'_{ab}, Q'_{ab}, P'_{bc}, Q'_{bc}\}$, and $\{P'_{bc}, Q'_{bc}, P'_c, Q'_c\}$ are independent in H. As G is 4-connected, the graph G' is connected, which implies that H is connected.

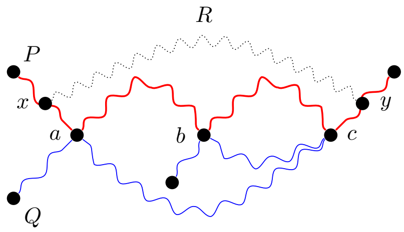
Suppose for a moment that every element of $\{P'_a, Q'_a, P'_{bc}, Q'_{bc}\}$ is empty. In that situation, a is an extreme of P. This implies that Q'_{ab} is empty; indeed, otherwise we can extend P by adding an edge aa' with $a' \in Q'_{ab}$. Analogously, P'_{ab} is also empty. Hence, as P'_c is not adjacent to Q'_c, the graph H will be either empty or disconnected, a contradiction.

Thus, the set $\{P'_a, Q'_a, P'_{bc}, Q'_{bc}\}$ has at least one nonempty element. And, analogously, the same is true for the set $\{P'_{ab}, Q'_{ab}, P'_c, Q'_c\}$. Then, in H, there is a $\{P'_a, Q'_a, P'_{bc}, Q'_{bc}\}$-$\{P'_{ab}, Q'_{ab}, P'_c, Q'_c\}$ path. Hence, one of $\{P'_aP'_c, P'_aQ'_c, Q'_aP'_c, Q'_aQ'_c\}$ is an edge of H. Without loss of generality, we may assume $P'_aP'_c \in E(H)$.

This implies that there exists a P'_a-P'_c path with no internal vertices in $V(P) \cup V(Q)$, say R, in G'. Let $\{x\} = V(R) \cap V(P'_a)$ and $\{y\} = V(R) \cap V(P'_c)$. Let P_x and P_{xa} be the corresponding subpaths of P_a. Let P_y and P_{yc} be the corresponding subpaths of P_c. Then $P - P_{xa} - P_y + R + Q_a$ and $Q - Q_a + P_{xa} + R + P_y$ are both paths, whose lengths sum $|P| + |Q| + 2|R|$, a contradiction (see Figure 1(a)).



(a)



(b)

Figure 1: Cases in the proof of Theorem 4.2.

Case 2: acb is a subsequence in Q.

It is easy to see that $|P_a| = |Q_a|$ and $|P_{bc}| = |Q_{bc}|$. Hence, $P - P_a + Q_a$, $Q - Q_a + P_a$, $P - P_{bc} + Q_{bc}$, and $Q - Q_{bc} + P_{bc}$ are longest paths. Let H be an auxiliary graph given by $V(H) = \{P'_a, P'_{ab}, P'_{bc}, P'_c, Q'_a, Q'_{ac}, Q'_{cb}, Q'_b\}$ and $E(H) = \{XY :$ there is a X-Y path in G' with no internal vertex in $V(P) \cup V(Q)\}$. By Lemma 4.1, the sets $\{P'_a, Q'_a, P'_{ab}, Q'_{ac}\}$, $\{P'_{ab}, Q'_b, P'_{bc}, Q'_{bc}\}$, and $\{P'_{bc}, Q'_{bc}, P'_c, Q'_{ac}\}$ are independent in H.

Suppose for a moment that every element of $\{P'_{ab}, Q'_{ac}, P'_c, Q'_b\}$ is empty. In this situation, c is an extreme of P. This implies that Q'_{bc} is empty; indeed, otherwise we can extend P by adding an edge cc' with $c' \in Q'_{bc}$. Analogously, P'_{bc} is also empty. Hence, as P'_a is not adjacent to Q'_a, the graph H will be either empty or disconnected, a contradiction.

Suppose now that every element of $\{P'_a, Q'_a, P'_{bc}, Q'_{bc}\}$ is empty. By a similar reasoning to the previous paragraph, P'_{ab} and Q'_{ac} are also empty.

If Q'_b is empty, then P'_c is also empty, which implies that H is empty, a contradiction. Otherwise, $P - P_{bc} + Q_{ac} + Q_b$ is a path longer than P, again a contradiction.

Therefore, in H, there is a $\{P'_{ab}, Q'_{ac}, P'_c, Q'_b\}$-$\{P'_a, Q'_a, P'_{bc}, Q'_{bc}\}$ path. Without loss of generality, we may assume $P'_a P'_c \in E(H)$. This implies that there exists a P'_a-P'_c path with no internal vertices in $V(P) \cup V(Q)$, say R, in G'. Let $\{x\} = V(R) \cap V(P'_a)$ and $\{y\} = V(R) \cap V(P'_c)$. Let P_x and P_{xa} be the corresponding subpaths of P_a. Let P_y and P_{yc} be the corresponding subpaths of P_c. Then $P - P_{xa} - P_y + R + Q_a$ and $Q - Q_a + P_{xa} + R + P_y$ are both paths, whose lengths sum $|P| + |Q| + 2|R|$, a contradiction (see Figure 1(b)).

With that, we conclude the proof of Theorem 4.2. $\square$

5. Tight families

As mentioned by Hippchen [5, Figure 2.5], the graph $K_{k,2k+2}$ (the complete bipartite graph with partitions of sizes k and $2k + 2$) makes the conjecture tight. In this section, we show that in fact, for every k, there is an infinite family of graphs that make Hippchen's conjecture tight.

Theorem 5.1. *For every k, there is an infinite family of k-connected graphs with a pair of longest paths intersecting each other in exactly k vertices.*

Proof. For any natural number n, we denote the set $\{1, \dots, n\}$ by $[n]$. Fix an arbitrary positive integer ℓ. Let $S = \{s_1, s_2, \dots, s_k\}$, and, for every $i \in [k+1]$, let $X_i = \{a_{i1}, a_{i2}, \dots, a_{i\ell}\}$ and $Y_i = \{b_{i1}, b_{i2}, \dots, b_{i\ell}\}$. We define a graph G by $V(G) = S \cup \{X_i : i \in [k+1]\} \cup \{Y_i : i \in [k+1]\}$, and $E(G) = \{sv : s \in S, v \in V(G) \setminus S\} \cup \{a_{ij}a_{i(j+1)} : i \in [k+1], j \in [\ell - 1]\} \cup \{b_{ij}b_{i(j+1)} : i \in [k+1], j \in [\ell - 1]\}$ (see Figure 2).

Note that every component of $G - S$ has size ℓ. Hence, any path in G has at most $k + \ell(k+1)$ vertices. Then

$$a_{11} \cdots a_{1\ell} s_1 a_{21} \cdots a_{2\ell} s_2 \cdots a_{k1} \cdots a_{k\ell} s_k a_{(k+1)1} \cdots a_{(k+1)\ell}$$

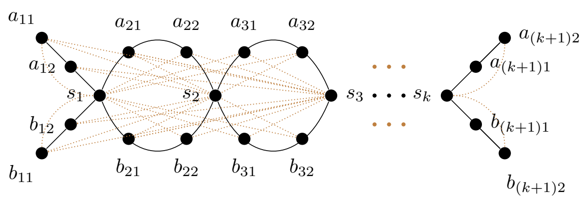


Figure 2: The graph used in the construction of Theorem 5.1 when $\ell = 2$.

and

$$b_{11} \cdots b_{1\ell} s_1 b_{21} \cdots b_{2\ell} s_2 \cdots b_{k1} \cdots b_{k\ell} s_k b_{(k+1)1} \cdots b_{(k+1)\ell}$$

are both longest paths, intersecting each other in exactly k vertices.

To finish the proof, we show that G is k-connected. Suppose by contradiction that G has a set of vertices S' of cardinality at most $k-1$ such that $G - S'$ is disconnected. Set $A = S \setminus S'$ and $B = (V(G) \setminus S) \setminus S'$. As $|S|$ and $|V(G) \setminus S|$ are at least k, both A and B are nonempty. Note that the complete bipartite graph with partitions A and B is an spanning subgraph of $G - S'$. Thus, $G - S'$ is connected, a contradiction. □

6. Conclusions and future work

In this paper, we show that every pair of longest paths in a k-connected graph intersect each other in at least $\min\{n, (8k - n + 2)/5\}$ vertices. A direct corollary of this result is that, if $k \geq n/3$, then every pair of longest paths intersect in at least $k+1$ vertices and, if $k \geq n/4$, then every pair of longest paths intersect in at least $\frac{4k+2}{5}$ vertices; in general, if $k \geq n/r$, then every pair of longest paths intersect in at least $((8-r)k+2)/5$ vertices. Thus, for every r, we propose the following conjecture.

Conjecture CHC(r). *Let G be a k-connected graph on n vertices. If $k \geq n/r$, then every pair of longest paths intersect in at least k vertices.*

In this paper, we also showed that, in a 4-connected graph, every pair of longest paths intersect in at least 4 vertices. We believe that CHC(r), for some $r > 3$, and the inconditional Hippchen conjecture for $k = 5$ can be approached with similar techniques as the ones presented here.

We also think the techniques presented here can be adapted to show similar results for cycles instead of paths; and, conversely, that the techniques used by Chen *et al.* [2] can be adapted to show similar or stronger results for paths.

Acknowlegments

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Resumen: Mostramos que cada par de caminos máximos en un grafo k-conexo con n vértices se intersecan uno al otro en por lo menos $\min\{n, (8k - n + 2)/5\}$ vértices. También mostramos que en un grafo 4-conexo cada par de caminos máximos se interseca uno al otro en por lo menos cuatro vértices. Ello confirma una conjetura de Hippchen en grafos k-conexos cuando $k \leq 4$ o $k \geq (n - 2)/3$.

Palabras clave: camino máximo, grafo k-conexo

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