A remark on the Tjurina and Milnor numbers of a foliation of second type

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Abstract

We present a relation between the Tjurina and Milnor numbers of a holomorphic foliation of secont type and the Tjurina and Milnor numbers of its union of separatrices when this last one is holomorphic.

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1. Introduction

A *germ of a singular holomorphic foliation* $\mathcal{F}_\omega$ of codimension one over $\mathbb{C}^2$ is locally given by a 1-form $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$, where $A(x,y)$, $B(x,y) \in \mathbb{C}\{x,y\}$ are coprime convergent power series called the *coefficients* of ω. The *multiplicity* of the foliation $\mathcal{F}_\omega$ is defined as $\mathrm{mult}(\omega) := \mathrm{mín}\{\nu_0(A), \nu_0(B)\}$, where $\nu_0(A)$, $\nu_0(B)$, is the algebraic multiplicity of A and B at $0 \in \mathbb{C}^2$. In this note we will consider singular holomorphic foliations.

Let $f(x,y) \in \mathbb{C}\{x,y\}$. We say that the $\mathcal{S}_f : f(x,y) = 0$ is *invariant* by $\mathcal{F}_\omega$ if $\omega \wedge \mathrm{d}f = f.\eta$, where η is a two-form (that is $\eta = g\mathrm{d}x \wedge \mathrm{d}y$, for some $g \in \mathbb{C}\{x,y\}$). If $\mathcal{S}_f$ is irreducible then we will say that $\mathcal{S}_f$ is a *holomorphic separatrix* of $\mathcal{F}_\omega : \omega = 0$. When $f(x,y) \in \mathbb{C}[[x,y]] \backslash \mathbb{C}\{x,y\}$, $\mathcal{S}_f$ is called a *formal separatrix*.

We will consider *non-dicritical* foliations, that is, foliations having a finite set of separatrices (see [3, page 158 and page 165]). Let $(\mathcal{S}_{f_j})_{j=1}^r$ be the set of all separatrices of the non-dicritical foliation $\mathcal{F}_\omega : \omega = 0$. Each separatrix $\mathcal{S}_{f_j}$ corresponds to an irreducible power series $f_j(x,y)$. Denote by $\mathcal{S}(\mathcal{F}_\omega)$ the union $\bigcup \mathcal{S}_{f_j}$ of all separatrices of the foliation $\mathcal{F}_\omega$, which we will call *union of separatrices* of $\mathcal{F}_\omega$. In the following we will denote by $\mathcal{F}_\omega$ a non-dicrital holomorphic foliation and by $\mathcal{S}(\mathcal{F}_\omega)$ its union of separatrices (convergent or formal).

The *dual vector field* associated to $\mathcal{F}_\omega$ is $X = B(x,y)\frac{\partial}{\partial x} - A(x,y)\frac{\partial}{\partial y}$. We say that the origin $(x,y) = (0,0)$ is a *simple or reduced singularity* of $\mathcal{F}_\omega$ if the matrix associated with the linear part of the vector field

$$\begin{pmatrix} \frac{\partial B(0,0)}{\partial x} & \frac{\partial B(0,0)}{\partial y} \\ -\frac{\partial A(0,0)}{\partial x} & -\frac{\partial A(0,0)}{\partial y} \end{pmatrix} \tag{1.1}$$

has two eigenvalues λ, μ, with $\frac{\lambda}{\mu} \notin \mathbb{Q}^+$.

It could happen that

a) $\lambda\mu \neq 0$ and $\frac{\lambda}{\mu} \notin \mathbb{Q}^+$ in which case we will say that the *singularity is not degenerate* or

b) $\lambda\mu = 0$ and $(\lambda, \mu) \neq (0, 0)$ in which case we will say that the singularity is a *saddle-node.*

In the *b*) case, the *strong separatrix* of a foliation with singularity P is an analytic invariant curve whose tangent at the singular point P is the eigenspace associated with the non-zero eigenvalue of the matrix given in (1.1). The zero eigenvalue is associated with a formal separatrix called *weak separatrix.*

From now on $\pi : M \to (\mathbb{C}^2, 0)$ represents *the process of singularity reduction* of $\mathcal{F}_\omega$ [12], obtained by a finite sequence of point blow-ups, where $\mathcal{D} := \pi^{-1}(0) = \bigcup_{j=1}^{n} D_j$ is the *exceptional divisor*, which is a finite union of projective lines with normal crossing (that is, they are locally described by one or two regular and transversal curves). In this process, any separatrix of $\mathcal{F}_\omega$ is smooth, disjoint and transverse to $D_j \subset \mathcal{D}$, and it does not pass through a corner (intersection of two components of the divisor $\mathcal{D}$). Let $\mathcal{F}_\omega$ be a non-dicritical formal foliation and consider the minimal reduction of singularities $\pi : M \to (\mathbb{C}^2, 0)$ of $\mathcal{F}_\omega$ (this is, a reduction with the minimal number of blow-ups that reduces the foliation). The *strict transform* of the foliation $\mathcal{F}_\omega$ is given by $\mathcal{F}'_\omega = \pi^*\mathcal{F}_\omega$ and the *exceptional divisor* is $\mathcal{D} = \pi^{-1}(0)$.

The foliation $\mathcal{F}_\omega$ is a *generalized curve* if in its reduction of singularities there are no saddle-node points.

If in the process of singularity reduction of $\mathcal{F}_\omega$, the exceptional divisor $\mathcal{D}$ at point P contains the weak invariant curve of the saddle-node, then the singularity is called *saddle-node tangent.* Otherwise we will say that $\mathcal{F}_\omega$ is a *saddle-node transverse* to $\mathcal{D}$ at point P.

The foliation $\mathcal{F}_\omega$ is of *second type* with respect to the divisor $\mathcal{D}$ if no singular points of $\mathcal{F}'_\omega$ are tangent saddle-nodes type.

Non-dicritical foliations of second type were studied by Mattei and Salem [13], also by Cano, Corral and Mol [4] and in the dicritical case by Genzmer and Mol [7] and Fernández Pérez-Mol [6].

Mattei and Salem gave the next characterization of foliations of second type in terms of the multiplicity of their union of formal separatrices:

The *intersection multiplicity* at the origin of the curves $C : h(x, y) = 0$ and $D : g(x, y) = 0$ is by definition

$$(C, D)_0 := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(g, h),$$

where (g, h) is the ideal generated by $g, h \in \mathbb{C}\{x, y\}$. We could write $(g, h)_0$ instead of $(C, D)_0$.

Let $\mathcal{S}(\mathcal{F}_\omega) : f(x, y) = 0$ be the reduced equation of the union of separatrices of the foliation $\mathcal{F}_\omega$. Now we will remember some invariants:

1. the Milnor number of $\mathcal{F}_\omega$ is $\mu(\mathcal{F}_\omega) = (A, B)_0$.
2. The Tjurina number of $\mathcal{F}_\omega$ is $\tau(\mathcal{F}_\omega) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(A, B, f)$, where (A, B, f) is the ideal generated by $A, B, f \in \mathbb{C}\{x, y\}$.
3. The Milnor number of $\mathcal{S}(\mathcal{F}_\omega)$ is $\mu(\mathcal{S}(\mathcal{F}_\omega)) = (f_x, f_y)_0$.
4. The Tjurina number of $\mathcal{S}(\mathcal{F}_\omega)$ is $\tau(\mathcal{S}(\mathcal{F}_\omega)) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f_x, f_y, f)$, where (f_x, f_y, f) is the ideal generated by $f_x, f_y, f \in \mathbb{C}\{x, y\}$.

By definition we get:

$$\mu(\mathcal{F}_\omega) \geq \tau(\mathcal{F}_\omega), \tag{1.2}$$

and

$$\mu(\mathcal{S}(\mathcal{F}_\omega)) \geq \tau(\mathcal{S}(\mathcal{F}_\omega)). \tag{1.3}$$

By [10, page 198] in the irreducible case and [16, (1.1) Lemma] in the reduced case, there are $g, h \in \mathbb{C}\{x, y\}$, with h and f coprime, and an analytic one-form η such that $g\omega = h\mathrm{d}f + f\eta$.

The *Gómez-Mont-Seade-Verjovsky index* of $\mathcal{F}_\omega$ with respect to $\mathcal{S}(\mathcal{F}_\omega)$ is

$$GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \frac{1}{2\pi i}\int_{\partial\mathcal{S}(\mathcal{F}_\omega)} \frac{g}{h} d\left(\frac{h}{g}\right).$$

When $\mathcal{S}(\mathcal{F}_\omega) : f = 0$ is irreducible we get

$$GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \mathrm{ord}_t\left(\frac{h}{g}(\gamma(t))\right),$$

where $\gamma(t)$ is a parametrization of $\mathcal{S}(\mathcal{F}_\omega)$.

2. Some remarks after Gómez-Mont theorem

The next theorem proved by Gómez-Mont gives the relation between the Gómez-Mont-Seade-Verjovsky index and the Tjurina number of a non-dicritical holomorphic foliation which union of separatrices is convergent and the Tjurina number of this last one:

Theorem 2.1 ([8, Theorem 1] and [9, Section 2.2, page 528])**.** *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic foliation with an isolated singularity at $0 \in \mathbb{C}^2$ and its union of separatrices $\mathcal{S}(\mathcal{F}_\omega)$ is convergent. We get:*

$$GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \tau(\mathcal{F}_\omega) - \tau(\mathcal{S}(\mathcal{F}_\omega)).$$

□

Now, we will get some consequences of Theorem 2.1.

Corollary 2.2. *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic foliation with an isolated singularity at $0 \in \mathbb{C}^2$ and its union of separatrices $\mathcal{S}(\mathcal{F}_\omega)$ is convergent. Then*

$$\tau(\mathcal{F}_\omega) \geq \tau(\mathcal{S}(\mathcal{F}_\omega)) \tag{2.1}$$

with equality if and only if $\mathcal{F}_\omega$ is a generalized curve foliation.

Proof. The inequality (2.1) is a consequence of $GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) \geq 0$ (see [2, Proposition 6]). By [2, Proposition 7] and [5, Théorème 3.3] we have that $\mathcal{F}_\omega$ is a generalized curve foliation if and only if $GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = 0$. We conclude the corollary by Theorem 2.1. □

Corollary 2.3. *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic foliation with an isolated singularity at $0 \in \mathbb{C}^2$ and its union of separatrices is $\mathcal{S}(\mathcal{F}_\omega)$ is convergent. If $\tau(\mathcal{F}_\omega) = \tau(\mathcal{S}(\mathcal{F}_\omega))$ then, after a change of coordinates if necessary, the Newton polygons of $\mathcal{F}_\omega$ and $\mathcal{S}(\mathcal{F}_\omega)$ are equal.*

Proof. Since $\tau(\mathcal{F}_\omega) = \tau(\mathcal{S}(\mathcal{F}_\omega))$ then $GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = 0$ so $\mathcal{F}_\omega$ is a generalized curve foliation and by Rouillé [14, Proposition 3.8] we finish the proof. □

Proposition 2.4. *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic foliation with an isolated singularity at $0 \in \mathbb{C}^2$ and its union of separatrices $\mathcal{S}(\mathcal{F}_\omega)$ is convergent. Then the values $\mu(\mathcal{S}(\mathcal{F}_\omega))$ and $\tau(\mathcal{F}_\omega)$ are in the interval $[\tau(\mathcal{S}(\mathcal{F}_\omega)), \mu(\mathcal{F}_\omega)]$. Moreover, if $\mathcal{F}_\omega$ is a generalized curve foliation then $\tau(\mathcal{F}_\omega) \leq \mu(\mathcal{S}(\mathcal{F}_\omega))$.*

Proof. By Corollary 2.2 and inequality (1.2) we get

$$\tau(\mathcal{S}(\mathcal{F}_\omega)) \leq \tau(\mathcal{F}_\omega) \leq \mu(\mathcal{F}_\omega).$$

By the inequality (1.3) and [3, Theorem 4] we get

$$\tau(\mathcal{S}(\mathcal{F}_\omega)) \leq \mu(\mathcal{S}(\mathcal{F}_\omega)) \leq \mu(\mathcal{F}_\omega).$$

So, the values $\mu(\mathcal{S}(\mathcal{F}_\omega))$ and $\tau(\mathcal{F}_\omega)$ are in the interval $[\tau(\mathcal{S}(\mathcal{F}_\omega)), \mu(\mathcal{F}_\omega)]$.

On the other hand $\mathcal{F}_\omega$ is a generalized curve foliation if and only if $\mu(\mathcal{F}_\omega) = \mu(\mathcal{S}(\mathcal{F}_\omega))$ and, by Corollary 2.2, if and only if $\tau(\mathcal{F}_\omega) = \tau(\mathcal{S}(\mathcal{F}_\omega))$. Hence $\tau(\mathcal{F}_\omega) \leq \mu(\mathcal{S}(\mathcal{F}_\omega))$ □

3. Remarks on the Tjurina number of foliations of second type

A *generic polar curve* of the foliation $\mathcal{F}_\omega$ with respect to $(a : b) \in \mathbb{P}^1_{\mathbb{C}}$ is $P_{(a:b)}(\mathcal{F}_\omega) : \omega \wedge (bdx - ady) = 0$, that is the curve of equation $aA(x, y) + bB(x, y) = 0$, where $by - ax = 0$ is not a tangent of the union of separatrices of $\mathcal{F}_\omega$.

The *polar excess number* of $\mathcal{F}_\omega$ with respect to its union of separatrices $\mathcal{S}(\mathcal{F}_\omega) : f(x, y) = 0$ is $\Delta(\mathcal{F}_\omega) := (\mathcal{P}_{(a:b)}(\mathcal{F}_\omega), \mathcal{S}(\mathcal{F}_\omega))_0 - (\mathcal{P}_{(a:b)}(\mathrm{d}f), \mathcal{S}(\mathcal{F}_\omega))_0$.

The main result in this note is:

Proposition 3.1. *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic foliation of second type with an isolated singularity at $0 \in \mathbb{C}^2$ and its union of separatrices $\mathcal{S}(\mathcal{F}_\omega)$ is convergent. Then*

$$\mu(\mathcal{F}_\omega) - \tau(\mathcal{F}_\omega) = \mu(\mathcal{S}(\mathcal{F}_\omega)) - \tau(\mathcal{S}(\mathcal{F}_\omega)).$$

Moreover if $\mu(\mathcal{F}_\omega) = \tau(\mathcal{F}_\omega) = r'$ then

$$r' \geq \mu(\mathcal{S}(\mathcal{F}_\omega)) = \tau(\mathcal{S}(\mathcal{F}_\omega)),$$

with equality if and only if $\mathcal{F}_\omega$ is a generalized curve foliation.

Proof. By [8, Theorem 1] we get

$$GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \tau(\mathcal{F}_\omega) - \tau(\mathcal{S}(\mathcal{F}_\omega))$$

and by [6, Theorem 3] we have $GSV_0(\mathcal{F}_\omega, \mathcal{S}(\mathcal{F}_\omega)) = \Delta(\mathcal{F}_\omega)$. Hence

$$\Delta(\mathcal{F}_\omega) = \tau(\mathcal{F}_\omega) - \tau(\mathcal{S}(\mathcal{F}_\omega)). \tag{3.1}$$

On the other hand by [4, Proposition 2] we have

$$(\mathcal{P}_{(a:b)}(\mathcal{F}_\omega), \mathcal{S}(\mathcal{F}_\omega))_0 = \mu(\mathcal{F}_\omega) + \mathrm{mult}(\mathcal{F}_\omega).$$

Moreover $(\mathcal{P}_{(a,b)}(\mathrm{d}f), S(\mathcal{F}_\omega))_0 = (bf_x + af_y, f)_0 = \mu(f) + \nu_0(f) - 1$, where $\mathcal{S}(\mathcal{F}_\omega) : f(x,y) = 0$ and the last equality is true by Teissier's lemma.

Hence

$$\begin{array}{rcl}\Delta(\mathcal{F}_\omega) & = & (\mathcal{P}_{(a:b)}(\mathcal{F}_\omega), \mathcal{S}(\mathcal{F}_\omega))_0 - (\mathcal{P}_{(a:b)}(\mathrm{d}f), \mathcal{S}(\mathcal{F}_\omega))_0 \\ & = & \mu(\mathcal{F}_\omega) + \mathrm{mult}(\mathcal{F}_\omega) - \mu(\mathcal{S}(\mathcal{F}_\omega)) - \nu_0(f) + 1 \\ & = & \mu(\mathcal{F}_\omega) - \mu(\mathcal{S}(\mathcal{F}_\omega)).\end{array}$$

After (3.1) we finish the proof of the first part. The second part of the statement is a consequence of [3, Theorem 4]. □

Next example illustrates Proposition 3.1:

Example 3.2. The foliation $\mathcal{F}_\omega : (xy - y^3)dx + (xy - 2x^2 + xy^2)dy$ has as union of separatrices $\mathcal{S}(\mathcal{F}_\omega) : x^2y - xy^2 = 0$. Since $\mathrm{mult}(\omega) = \mathrm{mult}(\mathcal{S}(\mathcal{F}_\omega)) - 1 = 2$, after [13, Théorème 3.1.9] we have that the foliation $\mathcal{F}_\omega$ is of second type. Moreover $\tau(\mathcal{F}_\omega) = \mu(\mathcal{F}_\omega) = 5 > 4 = \tau(\mathcal{S}(\mathcal{F}_\omega)) = \mu(\mathcal{S}(\mathcal{F}_\omega))$.

A direct consequence of Proposición 3.1 is:

Corollary 3.3. *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic foliation of second type with an isolated singularity at $0 \in \mathbb{C}^2$ and its union of separatrices $\mathcal{S}(\mathcal{F}_\omega)$ is convergent. We have*

$$\mu(\mathcal{F}_\omega) = \tau(\mathcal{F}_\omega) \textit{ if and only if } \mu(\mathcal{S}(\mathcal{F}_\omega)) = \tau(\mathcal{S}(\mathcal{F}_\omega)).$$

□

We say that the *genus* of an irreducible curve is g if the minimal system of generators of its semigroup has $g-1$ elements. After the analytical classification of branches and Corollary 3.3 we have:

Corollary 3.4. *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic generalized curve foliation with isolated singularities and its (only) separatrix $\mathcal{S}(\mathcal{F}_\omega)$ is convergent of genus g. We have*

1. $\mu(\mathcal{F}_\omega) = \tau(\mathcal{F}_\omega)$ *if and only if* $\mathcal{S}(\mathcal{F}_\omega)$ *is analytically equivalent to* $y^n + x^m = 0$ *for some natural coprime numbers* $n, m > 1$. *The normal form of* $\mathcal{F}_\omega$ *is* $\mathrm{d}(y^n + x^m) + \Delta(x, y)(nx\mathrm{d}y + my\mathrm{d}x)$, *for some* $\Delta(x, y) \in \mathbb{C}\{x, y\}$.

2. $\mu(\mathcal{F}_\omega) - \tau(\mathcal{F}_\omega) = 1$ *if and only if* $\mathcal{S}(\mathcal{F}_\omega)$ *is analytically equivalent to* $y^n - x^m + x^{m-2}y^{n-2} = 0$ *for some natural coprime numbers* $n, m > 1$.

3. $\mu(\mathcal{F}_\omega) - \tau(\mathcal{F}_\omega) = 2$ *and* $g = 2$ *if and only if* $\mathcal{S}(\mathcal{F}_\omega)$ *is analytically equivalent to* $(x^2 + y^3)^2 + xy^{(\beta+3)/2} = 0$ *for some odd natural number* $\beta > 6$.

4. $\mu(\mathcal{F}_\omega) - \tau(\mathcal{F}_\omega) = 2$ *and* $g = 1$ *then there exist coprime natural numbers* m *and* n *greater than 2 such that* $\mathcal{S}(\mathcal{F}_\omega)$ *is analytically equivalent to* $x^n - y^m + x^{n-2}y^{m-3}$ *or* $\mathcal{C}_f$ *is analytically equivalent to some member of the family* $x^n - y^m + x^{n-3}y^{m-2} + \sum_{k=2}^{2+[m/n]} a_k x^{n-2}y^{m-k}$, *for* $n \geq 4$, $m \geq 2n/(n-3)$ *and* $a_k \in \mathbb{C}$.

Proof. The first statement is a consequence of Proposition 3.3, [11] and [17, Theorem 4]. The other three statements follow from Proposition 3.3 and [1, Corollaries 8, 13, 18]. □

Proposition 3.5. *Suppose that $\mathcal{F}_\omega$ is a non-dicritical holomorphic generalized curve foliation with isolated singularities and its union of separatrices $\mathcal{S}(\mathcal{F}_\omega)$ is convergent. We have $\tau(\mathcal{F}_\omega) = \mu(\mathcal{F}_\omega)$ if and only if after a suitable holomorphic coordinate transformation $\mathcal{S}(\mathcal{F}_\omega)$ is quasi-homogeneous.*

Proof. By [15, Satz p.123, (a) and (d)] we have that after a suitable holomorphic coordinate transformation $\mathcal{S}(\mathcal{F}_\omega)$ is quasi-homogeneous is equivalent to $f \in (f_x, f_y)$, that is we get the equality of ideals $(f_x, f_y) =$

(f_x, f_y, f), or equivalently $\mu(\mathcal{S}(\mathcal{F}_\omega)) = \tau(\mathcal{S}(\mathcal{F}_\omega))$. We finish the proof applying Proposition 3.3. □

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Resumen: Presentamos una relación entre los números de Milnor y Tjurina de una foliación holomorfa de segundo tipo y los números de Milnor y Tjurina de su unión de separatrices cuando esta es holomorfa.

Palabras claves: Foliación holomorfa de segundo tipo, número de Tjurina y número de Milnor.

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