On a class of predator-prey models of Gause type with Allee effect and a square-root functional response

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Abstract

A predator-prey model of Gause type is an extension of the classical Lotka-Volterra predator-prey model. In this work, we study a predator-prey model of Gause type, where the prey growth rate is subject to an Allee effect and the action of the predator over the prey is given by a square-root functional response, which is non-differentiable at the y-axis. This kind of functional response appropriately models systems in which the prey have a strong herd structure, as the predators mostly interact with the prey on the boundary of the herd. Because of the square root term in the functional response, studying the behavior of the solutions near the origin is more subtle and interesting than other standard models.

Our study is divided into two parts: the local classification of the equilibrium points, and the behavior of the solutions in certain invariant set when the model has a strong Allee effect. In one our main results we prove, for a wide choice of parameters, that the solutions in certain invariant set approach to the y-axis. Moreover, for a certain choice of parameters, we show the existence of a separatrix curve dividing the invariant set in two regions, where in one region any solution approaches the y-axis and in the other there is a globally asymptotically stable equilibrium point. We also give conditions on the parameters to ensure the existence of a center-type equilibrium, and show the existence of a Hopf bifurcation.

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1. Introduction

A population model is a dynamical system, formed by one or several differential equations, that tries to predict the temporal evolution of the number of individuals for certain species. To do that, is necessary to deal with models that represent the interaction between the species and their relationship with the ecosystem where they live, in terms of the resources available for their survival.

A predator-prey model of Gause type [7] (see also [2, § 4]) is an extension of the well-known Lotka-Volterra predator-prey model. A generalized version of this model [6, § 4.1] is given by

$$\begin{aligned}\frac{dx}{dt} &= \alpha(x)x - h(x)y,\\ \frac{dy}{dt} &= (\rho h(x) - c)y.\end{aligned}$$

In this work, we deal with the following predator-prey model of Gause type,

$$X_\mu : \begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{x}{K}\right)(x - m)\,x - \dfrac{q\sqrt{x}}{\sqrt{x} + a}y, \\ \dfrac{dy}{dt} = \left(\dfrac{p\sqrt{x}}{\sqrt{x} + a} - c\right)y, \end{cases}$$

where $x = x(t)$ and $y = y(t)$ respectively represent the population sizes (measured as number of individuals, biomass or density per unit area or volume) of preys and predators for $t \geq 0$. In this model, the growth of the population of preys in the absence of predators is affected by the Allee effect. The (demographic) Allee effect is commonly defined as a positive relationship between the overall individual fitness and the population size or density [11, 5]. This in particular means that the per capita growth rate is an increasing function of the population density, for small values of population density. There are two classes of Allee effects: weak and strong. A *weak Allee effect* means that at low population density, the per capita population growth rate is lower, although positive, than at higher densities. If the per capita population growth rate becomes negative below certain threshold $m > 0$ (called *Allee threshold*) then we are under a *strong Allee effect*.

The involved constants have the following ecologic meanings:

- r: intrinsic prey growth rate,
- K: prey environmental carrying capacity,
- q: consuming rate per capita of the predators,
- c: intrinsic predator growth rate.

In addition, the functions that appear on the system have the following meanings:

1. The equation $\dfrac{dx}{dt} = r\left(1 - \frac{x}{K}\right)(x - m)x$ represents the growth of the population of preys affected by the Allee effect, where the term $(x - m)$ affects the logistic equation $\dfrac{dx}{dt} = rx(1 - x/K)$. The Allee effect is strong when $0 < m < K$, and weak when $-K < m \leq 0$.
2. The function $g(y) = -cy$ represents the natural mortality of the predators in absence of preys.
3. The function $h(x) = \frac{q\sqrt{x}}{\sqrt{x}+a}$ is a non-differentiable *functional response* [3, 10]. The function h, which is non-differentiable at $x = 0$, is called *square root functional response* [4], and represents a phenomenon called "herd behavior" [12, 13].

Functional responses with square root terms were considered by [1] and [4]. Similarly to these references, we will show that, under a strong Allee effect, when the population of preys is below the Allee threshold, both the preys and the predators become extinct. In particular, the predator-axis behaves like an attractor.

This paper is organized as follows: In Section 2 we introduce an equivalent model which will be our object of study. In Section 3, we determine the equilibrium points of the model introduced in Section 2, and classify the equilibrium points located in the coordinate axes. In section 4 we study the global behavior of the solutions, first under no conditions over the parameters, and then when the model has a strong Allee effect. Under the strong Allee effect, we study the existence of a center-type equilibrium, and provide an explicit example for which exists a supercritical Hopf bifurcation.

2. The model

Our object of study is the following predator-prey model of Gause type with Allee effect and non-differentiable functional response, described by the following system of nonlinear autonomous ODEs, defined on $\Omega = \{(x, y) \in \mathbb{R}^2 \,:\, x \geq 0,\, y \geq 0\}$:

$$X_\mu : \begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{x}{K}\right)(x - m)\,x - \dfrac{q\,\sqrt{x}}{\sqrt{x} + a}y \\ \dfrac{dy}{dt} = \left(\dfrac{p\,\sqrt{x}}{\sqrt{x} + a} - c\right)y, \end{cases} \tag{2.1}$$

where $x = x(t)$ e $y = y(t)$ respectively represent the population of preys and predators, and $t \geq 0$. Because of ecologic reasons, we must assume

$$\mu = (r, K, q, p, c, a, m) \in \mathbb{R}^5_{++} \times]0, K[\times] - K, K[.$$

In particular, the parameter a is positive.

As the functional response $h(x) = \dfrac{\sqrt{x}}{\sqrt{x} + a}$ and system (2.1) are non-differentiable when $x = 0$, it is required a non-usual analysis to establish all properties of the proposed model.

To simplify our calculations, we will deal with an equivalent system by making a change of variables and time rescaling. Consider the change of coordinates and time rescaling given by $\Phi : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2 \times \mathbb{R}_+$, defined as

$$(x, y, t) = \Phi(u, v, \tau) = \left(Ku, K^2 v, (\sqrt{Ku} + a)\tau/r\right). \tag{2.2}$$

Using this change of variables, we can show that system (2.1) is topologically equivalent on Ω to the system

$$Y_\eta : \begin{cases} \dfrac{du}{d\tau} = B[u(1-u)(u-M)(\sqrt{u}+A) - Q\sqrt{u}v], \\ \dfrac{dv}{d\tau} = (P\sqrt{u} - C)v, \end{cases} \tag{2.3}$$

where $B = K^{3/2}$, $Q = q/r$, $A = a/\sqrt{K}$, $M = m/K$, $P = \sqrt{K}(p-c)/r$, $C = ac/r$. See Proposition A.1 in Appendix A for more details.

Remark 2.1. In general, we have $B > 0$, $C > 0$, $Q > 0$, $A \in]0,1[$, $M \in]-1,1[$, $P \in \mathbb{R}$.

3. Classification of the equilibrium points

From now on, we will consider the variables x, y and t instead of u, v and τ in (2.3), that is

$$Y_\eta : \begin{cases} x' = B\big[x(1-x)(x-M)(\sqrt{x}+A) - Q\sqrt{x}y\big], \\ y' = (P\sqrt{x} - C)y, \end{cases} \tag{3.1}$$

The vector field Y_η is thus defined on $\Omega = \{(x,y) \,:\, x \geq 0,\, y \geq 0\}$. The equilibrium points of system (3.1) satisfy

$$B[x(1-x)(x-M)(\sqrt{x}+A) - Q\sqrt{x}y] = 0, \tag{3.2}$$

$$(P\sqrt{x} - C)y = 0. \tag{3.3}$$

From (3.3), either $y = 0$ or $P\sqrt{x} = C$. If $y = 0$, the corresponding value of x must be $x = 0$, $x = 1$ or $x = M$, giving the equilibrium points $(0,0)$, $(1,0)$ and $(M,0)$, the last one only if $M > 0$. On the other hand, $P\sqrt{x} = C$ has a solution if, and only if, $P > 0$ (and this holds, whenever $p > c$ in the original parameters). When this is the case, the equilibrium point $\mathcal{P}_e = (x_e, y_e)$ satisfies $x_e = \dfrac{C^2}{P^2}$ and y_e satisfies

$$y_e = \frac{x_e(1-x_e)(x_e-M)(C+PA)}{QC} = Hx_e(1-x_e)(x_e-M),$$

where $H = \dfrac{C + PA}{QC} = \dfrac{p}{AQc}$. When $M \leq 0$, y_e will be positive if, and only if, $C < P$; when $M > 0$, y_e will be positive if, and only if, $P\sqrt{M} < C < P$.

The following table summarizes our previous findings.

		Equilibrium points
$M \leq 0$	$P \leq 0$ or $P \leq C$	$(0,0), (1,0)$
	$0 < C < P$	$(0,0), (1,0), (x_e, y_e)$
$M > 0$	$P \leq 0$ or $C \leq P\sqrt{M}$ or $0 < P \leq C$	$(0,0), (1,0), (M,0)$
	$P\sqrt{M} < C < P$	$(0,0), (1,0), (M,0), (x_e, y_e)$

The Jacobian matrix of system (3.1) is

$$DY_\eta(x,y) = \begin{bmatrix} B\,G(x,y) & -BQ\sqrt{x} \\ \dfrac{Py}{2\sqrt{x}} & P\sqrt{x} - C \end{bmatrix} \tag{3.4}$$

whenever $x > 0$, where

$$G(x,y) = \frac{1}{2}(1-x)(x-M)\sqrt{x} - (M - 2(1+M)x + 3x^2)(A + \sqrt{x}) - \frac{y}{2\sqrt{x}}.$$

Like the original vector field X_μ, the vector field Y_η is non-differentiable at the y-axis.

We will show later that the equilibrium point $\mathcal{P}_e = (x_e, y_e)$ can be neither a saddle nor degenerate. Moreover, depending on the parameters, $\mathcal{P}_e$ can be of center type or be hyperbolic and have any other nature, see Figure 1. We will further study under which conditions $\mathcal{P}_e$ can be of center-type in Section 4.1, so we now will study the equilibrium points in the x-axis.

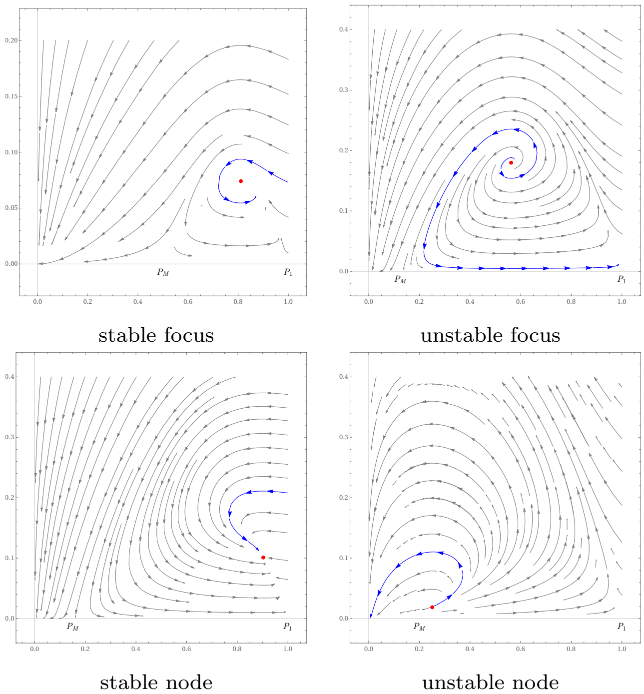

Figura 1: Possible behavior of the solutions around $\mathcal{P}_e = (x_e, y_e)$

3.1. Equilibrium points in the x-axis

Proposition 3.1. *The equilibrium point $\mathcal{P}_1 = (1, 0)$ is*

1. *A hyperbolic saddle, when $C < P$.*
2. *A hyperbolic stable node, when $P < C$.*
3. *A saddle-node, when $P = C$.*

Proof. Observe that

$$J_1 = DY_\eta(\mathcal{P}_1) = \begin{bmatrix} -B(1-M)(A+1) & -BQ \\ 0 & P-C \end{bmatrix} \tag{3.5}$$

whose eigenvalues are $-B(1-M)(A+1)<0$ and $P-C$. Therefore, when $P>C$ such eigenvalues have opposite signs and $\mathcal{P}_1$ is a hyperbolic saddle, and when $P<C$, both eigenvalues are negative, so $\mathcal{P}_1$ is a hyperbolic stable node.

When $P=C$, the Jacobian of the system at $\mathcal{P}_1$ given in (3.5) reduces to

$$J_1=\begin{bmatrix}\lambda & -BQ\\ 0 & 0\end{bmatrix}=\begin{bmatrix}Q & 1\\ \lambda/B & 0\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \lambda\end{bmatrix}\begin{bmatrix}Q & 1\\ \lambda/B & 0\end{bmatrix}^{-1}$$

where $\lambda=-B(1-M)(A+1)$ is the non-zero eigenvalue of J_1. Now consider the change of variables

$$(\hat{x},\hat{y},\tau)=\varphi(x,y,t)=\left(\frac{B}{\lambda}y,x-1-\frac{BQ}{\lambda}y,\lambda t\right)$$

which transforms system (3.1) to

$$\begin{cases}\dfrac{d\hat{x}}{d\tau}=p_2(\hat{x},\hat{y})\\ \dfrac{d\hat{y}}{d\tau}=\hat{y}+q_2(\hat{x},\hat{y})\end{cases}$$

where $p_2(\hat{x},\hat{y})=a_{20}\hat{x}^2+a_{11}\hat{x}\hat{y}+h.o.t.$, $a_{20}=\dfrac{CQ}{2\lambda}\neq 0$, $a_{11}=\dfrac{C}{2\lambda}$, and $q_2(\hat{x},\hat{y})=b_{20}\hat{x}^2+b_{11}\hat{x}\hat{y}+b_{02}\hat{y}^2+h.o.t$, with

$$\begin{aligned}b_{20}&=\frac{Q^2}{2\lambda}((2+A)BM-(4+3A)B-C),\\ b_{11}&=\frac{Q}{2\lambda}((5+3A)BM-(9+7A)B-C),\\ b_{02}&=\frac{B}{2\lambda}((3+2A)M-(5+4A)).\end{aligned}$$

By the Implicit Function Theorem, there exists a function $\hat{y}=\phi(\hat{x})=c_1\hat{x}+c_2\hat{x}^2+h.o.t.$, defined in a neighborhood of 0 such that

$$\phi(\hat{x})+q_2(\hat{x},\phi(\hat{x}))=c_1\hat{x}+(b_{20}+b_{11}c_1+b_{02}c_1^2+c_2)\hat{x}^2+h.o.t.=0,$$

which in turn implies $c_1 = 0$ and $c_2 = -b_{20}$, thus $\phi(\hat{x}) = -b_{20}\hat{x}^2 + h.o.t.$ Replace $\hat{y} = \phi(\hat{x})$ on $p_2(\hat{x}, \hat{y})$ to obtain

$$p_2(\hat{x}, \phi(\hat{x})) = a_{20}\hat{x}^2 + h.o.t.$$

We now refer to Theorem 1, Section 2.11, in [9], to conclude that $\mathcal{P}_1$ is a saddle node. □

Remark 3.2. When $P = C$, $\mathcal{P}_1$ is a saddle node equilibrium of (3.1). We will prove later (Proposition 4.1) that the x-axis is an invariant set of (3.1) and that any solution on the x-axis near $\mathcal{P}_1$ converges to $\mathcal{P}_1$. This implies that the x-axis contains two stable separatrices, which separate the parabolic and hyperbolic sectors associated to $\mathcal{P}_1$. A straightforward analysis of the sign of the components of Y_η near $\mathcal{P}_1$ allows us to conclude that the unstable separatrix associated to $\mathcal{P}_1$ cannot be in Ω (hence, it must be in the fourth quadrant). Therefore, any solution near $\mathcal{P}_1$ in Ω must converge to $\mathcal{P}_1$.

The proofs of Propositions 3.3 and 3.5 are similar to the previous one, and can be found in Appendix A.

Proposition 3.3. *The equilibrium point $\mathcal{P}_M = (M, 0)$, whenever $M > 0$, is:*

1. *A hyperbolic saddle, when $P\sqrt{M} < C$.*
2. *A hyperbolic unstable node, when $C < P\sqrt{M}$.*
3. *A saddle node, when $C = P\sqrt{M}$.* □

Remark 3.4. Similarly to Remark 3.2, when $C = P\sqrt{M}$, the x-axis near $\mathcal{P}_M$ contains two unstable separatrices, and the associated parabolic region is unstable and contained in Ω, that is, any solution near $\mathcal{P}_M$ must get away from $\mathcal{P}_M$.

Recall that the vector field Y_η associated to (3.1) is non-differentiable at every point in the y-axis, so we cannot use the Grobman-Hartman Theorem to study the behavior of the equilibrium point $\mathcal{P}_0 = (0, 0)$. To deal with this issue, consider the homeomorphism $h : \Omega_0 \to \Omega_0$ defined as $h(x, y) = (\sqrt{x}, y) = (u, v)$, where $\Omega_0 = \{(x, y) \in \mathbb{R}^2 \ : \ x > 0, \, y \geq 0\}$. Using this homeomorphism as a topological conjugation, system (3.1) is equivalent to

$$Z : \begin{cases} \dfrac{du}{dt} = \dfrac{B}{2}[u(1-u^2)(u^2-M)(u+A) - Qv], \\ \dfrac{dv}{dt} = (Pu - C)v. \end{cases} \tag{3.6}$$

Note that vector field Z has polynomial components, so it is C^∞ on Ω.

Proposition 3.5. *The equilibrium point* $\widehat{\mathcal{P}}_0 = (0, 0)$ *of* (3.6)*, is:*

1. *A hyperbolic saddle, when* $M < 0$*.*
2. *A hyperbolic stable node, when* $M > 0$*.*
3. *A topological saddle, when* $M = 0$*.* □

Corollary 3.6. *For system* (3.6)*, when* $M \leq 0$*, in a neighborhood of* $\hat{\mathcal{P}}_0$*, there exist two separatrices* $\widehat{\Sigma}_s$ *and* $\widehat{\Sigma}_u$*, stable and unstable, respectively, such that* $\widehat{\Sigma}_u$ *is contained in the u-axis and* $\widehat{\Sigma}_s$ *passes through the first and third quadrants in the uv-plane.*

Proof. From Proposition 3.5, $\widehat{\mathcal{P}}_0$ is a saddle when $M \leq 0$, so there exist stable and unstable separatrices $\widehat{\Sigma}_s$ and $\widehat{\Sigma}_u$.

Note that the u-axis is an invariant line of (3.6) and, when u is small enough, $\dfrac{du}{d\tau}$ has the same sign as u. Therefore, there is a small neighborhood $\widehat{V}_0$ of $\widehat{\mathcal{P}}_0$, where $\widehat{\Sigma}_u$ is contained in the u-axis.

On the other hand, from the proof of Proposition 3.5, $\hat{\lambda} = -C$ is the negative eigenvalue of the Jacobian matrix $\widehat{J}_0$ of (3.6) at $\widehat{\mathcal{P}}_0$, with associated eigenspace generated by $\hat{v} = (QB, 2C - BMA)$. Since $\hat{v}$ has positive components, due to $M \leq 0$, the tangent line of $\widehat{\Sigma}_s$ at $\widehat{\mathcal{P}}_0$ must be the eigenspace of $\hat{\lambda}$. Therefore, reducing $\widehat{V}_0$ if necessary, $\widehat{\Sigma}_s$ must be contained in the first and third quadrants in the uv-plane. The corollary now follows. □

Theorem 3.7. *For system* (3.1)*, we have the following possibilities for the equilibrium point* $\mathcal{P}_0 = (0, 0)$*:*

1. *When* $M \leq 0$*, there exists a stable separatrix* Σ *passing through* $\mathcal{P}_0$ *and a neighborhood* V_0 *of* $\mathcal{P}_0$ *such that* Σ *divides* $W_0 = V_0 \cap \Omega_0$ *into two regions* R_1 *and* R_2*, where* R_2 *is an hyperbolic sector and every solution in* R_1 *approaches the* y*-axis.*

2. *When* $M > 0$*, there exists a neighborhood* V_0 *of* $\mathcal{P}_0$ *such that every solution in* $W_0 = V_0 \cap \Omega_0$ *approaches the* y*-axis.*

Proof. First assume $M \leq 0$. By Proposition 3.5 and Corollary 3.6, the equilibrium point $\widehat{\mathcal{P}}_0$ of (3.6) is a saddle and there exist a neighborhood $\widehat{V}_0$ of $\widehat{\mathcal{P}}_0$, and stable and unstable separatrices $\widehat{\Sigma}_s$ and $\widehat{\Sigma}_u$, with $\widehat{\Sigma}_u$ contained in the u-axis and $\widehat{\Sigma}_s$ passing through the first and third quadrants of the uv-plane. This implies that $\widehat{\Sigma}_s$ divides $\widehat{V}_0 \cap \Omega_0$ into two regions: S_1, where every solution in S_1 exits Ω_0 through the positive v-axis, and S_2, where every solution remains in Ω_0.

We now recall that $h : \Omega_0 \to \Omega_0$ is a topological conjugation between (3.1) and (3.6). Define V_0 as any neighborhood of $\mathcal{P}_0$ such that $W_0 := V_0 \cap \Omega_0 = h^{-1}(\widehat{V}_0)$ and let $\Sigma = h^{-1}(\widehat{\Sigma}_s)$. Thus, Σ divides W_0 in two regions: $R_1 = h^{-1}(S_1)$ and $R_2 = h^{-1}(S_2)$. Therefore, any solution in R_1 must approach the y-axis. Moreover, since $\hat{\mathcal{P}}_0$ is a saddle of (3.6), R_2 is an hyperbolic sector.

Now assume that $M > 0$, so $\widehat{\mathcal{P}}_0$ is a stable node of (3.6). In this case, there exists $\widehat{V}_0$ such that every solution of (3.6) starting in $\widehat{V}_0$ approaches $\widehat{\mathcal{P}}_0$ in infinite time. We now can divide the solutions in two: solutions that remain in Ω_0, and solutions that escape Ω_0. Since $\dfrac{du}{d\tau} < 0$ when $u = 0$ and $v > 0$, then any solution that starts in $\hat{V}_0$ that escape Ω_0 cannot enter Ω_0 again.

Finally, let V_0 be any neighborhood of $\mathcal{P}_0$ such that $W_0 := V_0 \cap \Omega_0 = h^{-1}(\hat{V}_0)$. Then, any solution of (3.1) must approach the y-axis. □

Remark 3.8. Theorem 3.7 implies that, when the population of predators and preys are small enough, under a strong Allee effect, both species become extinct. On the other hand, under a weak Allee effect, there are three possibilities:

- Both the predators and preys become extinct, but preys first and then predators.
- Both prey and predators extinct roughly simultaneously. This happens when the populations start at the separatrix Σ.
- Both species cannot become extinct at the same time. In fact, the preys cannot become extinct in any case.

4. Global behavior of the solutions

Throughout this section, we will use the notation $Y_\eta = (x', y')$ to specify the components of Y_η.

We begin this section by studying the solutions contained in the coordinate axes.

Proposition 4.1. *The coordinate axes are invariant sets of* (3.1). *Moreover,*

1. *any solution starting at the y-axis converges to the origin;*
2. *when $M \leq 0$, any solution on the x-axis with initial point $(R, 0)$ with $R > 0$ converges to $\mathcal{P}_1$; and*
3. *when $M > 0$, any solution on the x-axis with initial point $(R, 0)$, $R \neq M$, converges either to the origin, when $0 < R < M$, or to $\mathcal{P}_1$, when $M < R$.*

Proof. Indeed, note that

$$Y_\eta(x, 0) = (f(x), 0), \quad Y_\eta(0, y) = (0, -Cy),$$

where $f(x) = Bx(1 - x)(x - M)(\sqrt{x} + A)$. Hence, any solution starting in an axis stays in such axis, both in forward and backward time. Items

1, 2 and 3 are consequence of the signs that y' and x' take, respectively, on the y and x axis. □

Remark 4.2. The previous proposition can be expressed in ecological terms. Item 1 means that the predators in absence of preys become extinct. Item 2 means that under a weak Allee effect, the population of preys in absence of predators approaches to its carrying capacity. Finally, item 3 means that under a strong Allee effect, preys become extinct when their population is below the Allee threshold, otherwise their population also approaches to its carrying capacity.

Theorem 4.3 (Boundedness of the solutions)**.** *The solutions of the system* (3.1) *are bounded.*

Proof. From Proposition 4.1, the coordinates axes are invariant, and any solution contained in them is bounded.

Recall that the homeomorphism $h : \Omega_0 \to \Omega_0$ defined as $h(x, y) = (\sqrt{x}, y) = (u, v)$, where $\Omega_0 = \{(x, y) \in \mathbb{R}^2 \;:\; x > 0,\, y \geq 0\}$, serves as a topological conjugation between (3.1) and (3.6), which is given by

$$Z : \begin{cases} \dfrac{du}{d\tau} = \dfrac{B}{2}[u(1-u^2)(u^2-M)(u+A) - Qv], \\ \dfrac{dv}{d\tau} = (Pu - C)v. \end{cases} \tag{4.1}$$

To show that the solutions of (4.1) in Ω_0 are bounded we apply the Poincaré compactification technique to Z, by using the change of coordinates,

$$(u, v) = \left(\frac{r}{s}, \frac{1}{s}\right), \quad r > 0,\ s > 0.$$

to obtain

$$Z_\infty : \begin{cases} \dfrac{dr}{d\tau} = -Br^6 - ABr^5s + Br^4s^2 + BMr^4s^2 + AB(1+M)r^3s^3 \\ \qquad\quad - BMr^2s^4 - 2Pr^2 - BQs^5 - ABMrs^5 \\ \dfrac{ds}{d\tau} = -2Prs^5 + 2Cs^6, \end{cases} \tag{4.2}$$

which can extended to the coordinate axes rs. Thus, to show that the solutions of (4.1) are bounded, we need to show that there are no stable equilibrium points of (4.2) on the infinity line $s = 0$. First note that the only equilibrium point of (4.2) in $s = 0$, is the origin $\mathcal{P}_\infty = (0, 0)$, whose associated Jacobian matrix is the null matrix. To deal with this, consider the horizontal blow-up on $\mathcal{P}_\infty$, given by the map,

$$(r, s) = (x_1, x_1 x_2),$$

so we obtain the system

$$X_1 : \begin{cases} \dfrac{dx_1}{d\tau} = -By_1[Qx_2^5 + x_1 + Ax_1y_1 - x_1y_1^2 - Mx_1y_1^2 \\ \qquad\qquad - AMx_1y_1^3 + Mx_1y_1^4 + AMx_1y_1^5], \\ \dfrac{dy_1}{d\tau} = -x_1[BQy_1^5 + Bx_1 + ABx_1y_1 - B(1+M)x_1y_1^2 \\ \qquad\qquad - AB(1+M)x_1y_1^3 - (BM+2p)x_1y_1^4 + (AB-2C)x_1y_1^5]. \end{cases} \tag{4.3}$$

System (4.3) has only a equilibrium point $Q_1 = (0, 0)$ on the line $x_1 = 0$, however, the Jacobian matrix of X_1 at Q_1 is again the null matrix. On the other hand, the vertical blow-up (4.2) does not have equilibrium points.

We now do a blow-up at Q_1, to obtain that the horizontal blow-up has a hyperbolic saddle and the vertical blow-up has a degenerate equilibrium point. If we repeat this process three more times, we obtain in each case two equilibrium points: a hyperbolic saddle in the horizontal blow-up and a degenerate equilibrium in the vertical one. Finally, at the fourth blow-up, we obtain two hyperbolic saddles and a third equilibrium point outside the first quadrant. Therefore, no equilibrium points of (4.2) at the infinity line $s = 0$ are stable and, in particular, the solutions of (4.1) in Ω_0 are bounded. The theorem follows due to the topological conjugation between (3.1) and (4.1). □

From now on, given $R > 0$, Ω_R will denote the set

$$\Omega_R = \{(x, y) \,:\, 0 \le x \le R,\, y \ge 0\}.$$

Lemma 4.4. *Let $R > 0$ such that, x' in* (3.1) *is negative when $x = R$ and $y > 0$, then the set Ω_R is a positively invariant region of system* (3.1).

Proof. Due to the continuity of Y_η, $x' \leq 0$ at $(R, 0)$. Since $x' < 0$ when $x = R$ and $y > 0$, any solution with initial condition in Ω_R, cannot exit this set through this vertical line. This, along with the fact that the coordinate axes are invariant, implies the lemma. □

Proposition 4.5. *For any choice of parameters η, the set*

$$\Omega_1 = \{(x, y) \,:\, 0 \leq x \leq 1,\, y \geq 0\}.$$

is a positively invariant region of (3.1).

Proof. Note that $Y_\eta(1, y) = (-Qy, (P - C)y)$, hence $x' < 0$ on the line $x = 1$, $y > 0$. The proposition now follows from Lemma 4.4. □

4.1. Behavior of the solutions under a strong Allee effect

In this section, we consider (2.1) under a strong Allee effect. This implies that the parameter M is positive on system (3.1).

The following Lemma provides sufficient conditions to guarantee that, when the population of preys is below the Allee threshold, both the preys and the predators become extinct.

Lemma 4.6. *Assume $M > 0$ and let $0 < R \leq M$ such that y' in* (3.1) *is non-positive on the line $x = R$, $y > 0$. Then any solution of* (3.1) *starting in $\Omega_R \setminus \mathcal{P}_M$ approaches the y-axis.*

Proof. Since $M > 0$ and the definition of Y_η, $x' < 0$ at any point in Ω_R minus the y-axis and possibly the equilibrium point $\mathcal{P}_M$, and $y' < 0$ at any point in Ω_R minus the x-axis. Therefore, by Lemma 4.4, Ω_R is a positively invariant region.

Let $\phi(t) = (x(t), y(t))$ be a solution of (3.1), with initial point $\phi(0) = (x_0, y_0) \in \Omega_R \setminus \mathcal{P}_M$ and maximal domain $]\beta_-, \beta_+[$. If $\phi(0)$ belongs to the coordinate axes then it approaches the origin, by Proposition 4.1. If $x_0 > 0$ and $y_0 > 0$ then $x(t)$ and $y(t)$ are strictly decreasing, due to x' and y' being negative on Ω_R, hence, without loss of generality, we may assume that $x_0 < M$. Moreover, $\phi(t)$ is bounded. Therefore, there exists $\bar{x}$ and $\bar{y}$ such that

$$\lim_{t\to\beta_+} x(t) = \bar{x} \geq 0, \qquad \lim_{t\to\beta_+} y(t) = \bar{y} \geq 0.$$

We now claim that $\bar{x} = 0$. Otherwise, $\phi([0, \beta_+[)$ would be contained in the compact set $\tilde{C} = [\bar{x}, x_0] \times [0, y_0]$, with Y_η being C^1 on some open set containing $\tilde{C}$. By the Poincaré-Bendixson Theorem, the solution ϕ must exit $\tilde{C}$, a contradiction. The lemma now follows. □

Lemma 4.7. *Assume $M > 0$, $P > 0$ and $C < P\sqrt{M}$. Then, any solution of* (3.1) *starting at (x_0, y_0), with $\frac{C^2}{P^2} < x_0 \leq 1$ and $y_0 > 0$, enters Ω_{C^2/P^2}.*

Proof. From Proposition 4.5, Ω_1 is a positively invariant region (3.1). Let $\phi(t)$ be a solution such that $\phi(0) = (x_0, y_0)$, and assume that $x(t) \geq C^2/P^2$ for all $t > 0$. Since $\phi(t)$ is bounded, there exists $\hat{y} > 0$ such that $\phi([0, +\infty[)$ is contained on the compact set $\hat{C} = [C^2/P^2, 1] \times [0, \hat{y}]$. Note that the only equilibrium points in $\hat{C}$ are $\mathcal{P}_M$ and $\mathcal{P}_1$ which are an unstable node and a saddle point, respectively, and the stable manifold of $\mathcal{P}_1$ is contained in the x-axis. Thus, since Y_η is C^1 on some open set containing $\hat{C}$, by the Poincaré-Bendixson Theorem, $\phi(t)$ must exit $\hat{C}$. This proves the lemma. □

Theorem 4.8. *If $M > 0$ then any solution of* (3.1) *starting in $\Omega_M \setminus \mathcal{P}_M$ approaches the y-axis.*

Proof. If $P \leq 0$ or $0 < P\sqrt{M} \leq C$, then $y' \leq 0$ on the line $x = M$ and $y > 0$ and the theorem follows from Lemma 4.6.

Now assume that $P > 0$ and $C < P\sqrt{M}$. As $y' = 0$ on the line $x = C^2/P^2$, $y > 0$, by Lemma 4.6, any solution starting in Ω_{C^2/P^2} approaches the y-axis. We now use Lemma 4.7, and the theorem follows. □

We now state the main theorem of this section.

Theorem 4.9. *Assume $M > 0$.*

1. *If $P \leq C$ then M is a saddle and there exists a separatrix Σ, determined by the stable manifold of $\mathcal{P}_M$, that divides Ω_1 in two regions D_1 and D_2 such that, any solution starting at D_1 approaches the y-axis and any solution starting at D_2 converges to $\mathcal{P}_1$.*
2. *If $P > 0$ and $C \leq P\sqrt{M}$ then any solution starting at Ω_1 approaches the y-axis.*
3. *If $P > 0$ and $P\sqrt{M} < C < P$ then $\mathcal{P}_e$ is a center-type equilibrium if, and only if, $4C^3 + 3AC^2P - 2CP^2 - AP^3 \neq 0$ and*

$$0 < \frac{C^2(6C^3 + 5AC^2P - 4CP^2 - 3AP^3)}{P^2(4C^3 + 3AC^2P - 2CP^2 - AP^3)} < 1$$

The proof of Theorem 4.9 will be divided in three separate propositions.

Proposition 4.10. *When $M > 0$ and $P \leq C$, $\mathcal{P}_M$ is a saddle and there exists a separatrix Σ that divides Ω_1 in two regions D_1 and D_2, such that, any solution starting at D_1 approaches the y-axis and any solution starting at D_2 converges to $\mathcal{P}_1$.*

Proof. Since $M < 1$, by Proposition 3.3, $\mathcal{P}_M$ is a hyperbolic saddle. Let Σ be the separatrix curve determined by the stable manifold $W^s(\mathcal{P}_M)$. Note that Theorem 4.8 implies that $W^s(\mathcal{P}_M)$ cannot enter the region Ω_M.

We claim that Σ must intersect the vertical line $x = 1$, $y > 0$, by Poincaré-Bendixson theorem, since Σ is bounded, $\mathcal{P}_1$ is a stable equilibrium point (even when $P = C$, see Remark 3.2), there are not other equilibrium points in the interior of $\Omega_1 \setminus \Omega_M$ and system (3.1) is C^1 in $\Omega_1 \setminus \Omega_M$. Thus, Σ divides Ω_1 in two invariant regions: D_1 and D_2.

Using a similar argument, since $\mathcal{P}_M$ is not stable, any solution starting at the interior of D_1 must reach Ω_M and, by Theorem 4.8, we are done. On the other hand, as (3.1) is C^1 in an open set containing $\overline{D_2}$, any solution starting at D_2 must converge to $\mathcal{P}_1$, again by the Poincaré-Bendixson Theorem. The proposition now follows. □

Remark 4.11. Condition $P \leq C$ implies that the population of predators must decrease, whenever the population of preys is saturated at its carrying capacity. When this is the case, under a strong Allee effect, depending on the initial population of predators, one of the following conclusions hold:

- Both the predators and the preys become extinct.
- The predators become extinct and the preys survive, but their population approaches to the Allee threshold. This occurs when the populations start at the separatrix Σ.
- The predators become extinct and the preys saturate again at its carrying capacity.

See Figure 2 for a graphical representation of the conclusions of Proposition 4.10. The separatrix Σ is represented by the blue curve.

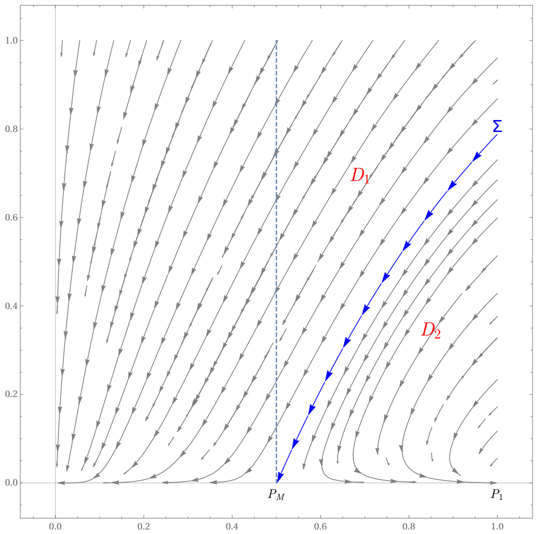

Figura 2: $A = 1/4$, $B = Q = P = 1$, $M = 1/2$, $C = 2$

Proposition 4.12. *If $P > 0$ and $C \leq P\sqrt{M}$, then any solution starting at the interior of Ω_1 approaches the y-axis.*

Proof. Due to Lemma 4.7, it is enough to proof that any solution $\phi(t)$, with starting point $\phi(0) = (x_0, y_0)$ satisfying $M < x_0 < 1$, enters Ω_M. Assume otherwise, that is, $x(t) > M$, for all $t > 0$. Now, note that $\phi(t)$ is bounded and Ω_1 is an invariant region, so there exists $\hat{y} > 0$ such that $\phi([0, +\infty[)$ is contained in the compact set $\hat{C} = [M, 1] \times [0, \hat{y}]$. By the Poincaré-Bendixson Theorem, $\phi(t)$ must converge to $\mathcal{P}_1$, a contradiction since $\mathcal{P}_1$ is unstable. □

Remark 4.13. Conditions $P > 0$ and $C \leq P\sqrt{M}$, imply that $p > c$ and $c \leq p\dfrac{\sqrt{m}}{a + \sqrt{m}}$. This means in particular that the population of predators is non-decreasing when the population of preys is at the Allee threshold. When this happens, Proposition 4.12 implies that both species become extinct, whenever the population of preys is smaller than its carrying capacity.

See Figure 3 for a graphical representation of the conclusions of Proposition 4.12.

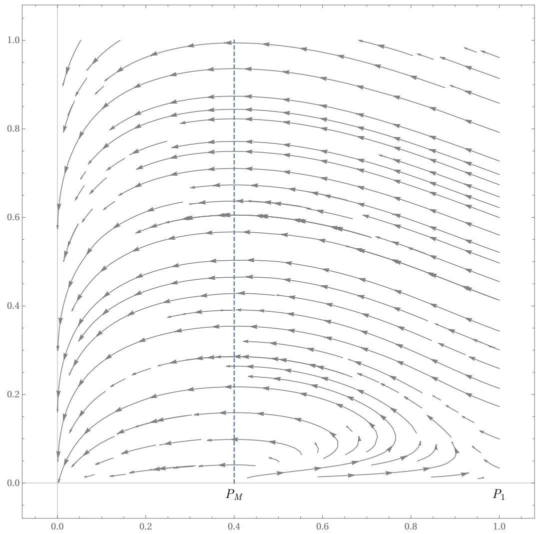

Figura 3: $A = 1/4$, $B = Q = P = 1$, $M = 4/9$, $C = 2/3$

Proposition 4.14. *If $P > 0$ and $P\sqrt{M} < C < P$ then $\mathcal{P}_e$ is a center-type equilibrium if, and only if, $4C^3 + 3AC^2P - 2CP^2 - AP^3 \neq 0$ and*

$$0 < \frac{C^2(6C^3 + 5AC^2P - 4CP^2 - 3AP^3)}{P^2(4C^3 + 3AC^2P - 2CP^2 - AP^3)} < 1$$

Proof. Since $x_e = \dfrac{C^2}{P^2}$, in (3.4) we obtain

$$DY_\eta(\mathcal{P}_e) = \begin{bmatrix} B\,G(x_e, y_e) & -BQ\sqrt{x_e} \\ P\dfrac{y_e}{2\sqrt{x_e}} & 0 \end{bmatrix}.$$

Hence, the determinant of DY_η at $\mathcal{P}_e$ is $\det(DY_\eta(\mathcal{P}_e)) = \dfrac{1}{2}BQPy_e > 0$. Remains to prove that the trace of the Jacobian matrix at $\mathcal{P}_e$ is zero. Indeed, again from (3.4), we obtain

$$\begin{aligned}\operatorname{tr}(DY_\eta(\mathcal{P}_e)) = -\frac{B}{2P^5}\Big[&6C^5 + 5AC^4P - 4C^3(1+M)P^2 \\ &- 3AC^2(1+M)P^3 + 2CMP^4 + AMP^5\Big],\end{aligned}$$

which is zero if we choose

$$M = \frac{C^2(6C^3 + 5AC^2P - 4CP^2 - 3AP^3)}{P^2(4C^3 + 3AC^2P - 2CP^2 - AP^3)} \tag{4.4}$$

The proposition now follows. □

In view of Proposition 4.14, we obtain conditions to guarantee when the equilibrium $\mathcal{P}_e$ is of center-type. This motivates us to apply a linear perturbation on the model under these conditions in order to obtain a Hopf perturbation. In the next section, we follow the steps of [8, Section 3.5] in an specific example, to guarantee the existence of a limit cycle.

Example and Hopf Bifurcation Consider the following choice of parameters:

$$A = \frac{1}{16}, \quad P = 6, \quad C = 5, \quad B = Q = 1.$$

Using (4.4), we choose $M = \dfrac{925}{4764}$, so

$$\mathcal{P}_e = \left(\frac{25}{36}, \frac{2556125}{18522432}\right) \approx (0{,}694, 0{,}138)$$

is a center-type equilibrium. Using the parameters above, we now perturb (3.1) considering $P = 6 + \alpha$, where α is a perturbation parameter, so we obtain

$$\begin{aligned} x' &= \left(\frac{1}{4} + \sqrt{x}\right)(1-x)x\left(-\frac{925}{4764} + x\right) - \sqrt{x}\,y, \\ y' &= \big((6+\alpha)\sqrt{x} - 5\big)\,y. \end{aligned}$$

Note that the above system has equilibrium points

$$x_e(\alpha) = \frac{25}{(6+\alpha)^2},$$

$$y_e(\alpha) = -\frac{125(1+\alpha)(11+\alpha)(26+\alpha)(37\alpha^2+444\alpha-3432)}{19056(6+\alpha)^6}.$$

We consider the change of coordinates $(x, y) = (u + x_e(\alpha), v + y_e(\alpha))$ to obtain the following system

$$\begin{aligned} u' &= a(\alpha)u + b(\alpha)v + f(u,v,\alpha), \\ v' &= c(\alpha)u + d(\alpha)v + g(u,v,\alpha), \end{aligned} \tag{4.5}$$

where

$$a(\alpha) = -\frac{25\alpha(-5190696 - 362726\alpha + 31773\alpha^2 + 2590\alpha^3 + 37\alpha^4}{38112(6+\alpha)^5}$$

$$c(\alpha) = -\frac{25(1+\alpha)(11+\alpha)(26+\alpha)(-3432+444\alpha+37\alpha^2)}{38112(6+\alpha)^4}$$

$$b(\alpha) = -\frac{5}{6+\alpha}, \qquad d(\alpha) = 0,$$

and f, g are smooth functions which have Taylor expansions in (u, v) starting with at least quadratic terms. Let

$$A(\alpha) = \begin{bmatrix} a(\alpha) & b(\alpha) \\ c(\alpha) & d(\alpha) \end{bmatrix}, \qquad F(u,v,\alpha) = (f,g),$$

so system (4.5) can be rewritten as

$$\vec{x}\,' = A(\alpha)\vec{x} + F(\vec{x},\alpha), \quad \vec{x} = (u,v).$$

Note that the eigenvalues of $A(\alpha)$ take the form

$$\lambda_1(\alpha) = \lambda(\alpha), \qquad \lambda_2(\alpha) = \overline{\lambda(\alpha)}$$

where $\lambda(\alpha) = \mu(\alpha) + \omega(\alpha)i$,

$$\mu(\alpha) = -\frac{25\alpha(37\alpha^4 + 2590\alpha^3 + 31773\alpha^2 - 362726\alpha - 5190696)}{76224(6+\alpha)^5}$$
$$\omega(\alpha) = \frac{1}{2}\sqrt{4\det(A(\alpha)) - \operatorname{tr}(A(\alpha))^2} = \frac{5\sqrt{5}}{76224(6+\alpha)^5)}\Big[1163566731165696 + 2133205673803776\alpha + 1130185096986816\alpha^2 + 304862575757472\alpha^3 + 24309457093660\alpha^4 - 6637949269140\alpha^5 - 1991153106709\alpha^6 - 234121358224\alpha^7 - 14407575038\alpha^8 - 452204380\alpha^9 - 5647421\alpha^{10}\Big]^{1/2}.$$

Note that $\mu'(0) = \dfrac{600775}{2744064} \neq 0$. We now follow the steps given in [8, Section 3.5] and apply Theorem 3.3. For this, remains to verify that the first Lyapunov coefficient, called $l_1(0)$ in [8], is non-null.

Let $q(\alpha) \in \mathbb{C}^2$ be the eigenvector of $A(\alpha)$ associated to the eigenvalue $\lambda(\alpha)$ and let $p(\alpha) \in \mathbb{C}^2$ be the eigenvector of $A(\alpha)^\top$ associated to the eigenvalue $\overline{\lambda(\alpha)}$, and assume, by rescaling $p(\alpha)$, that $\langle p(\alpha), q(\alpha)\rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the usual inner product of $\mathbb{C}^2$. We may now define

$$g(z, \bar{z}, \alpha) = \langle p(\alpha), F(zq(\alpha) + \bar{z}\overline{q(\alpha)})\rangle = \sum_{k+l\geq 2} \frac{g_{kl}(\alpha)}{k!l!} z^k \bar{z}^l,$$

that is $g_{kl}(\alpha) = \dfrac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} g(z, \bar{z}, \alpha)\Big|_{z=0}$. Denote $\omega_0 = \omega(0)$ and $\hat{g}_{kl} = g_{kl}(0)$, then, from [8, Equation 3.20], the first Lyapunov coefficient is given by:

$$l_1(0) = \frac{1}{2\omega_0^2} Re(i\hat{g}_{20}\hat{g}_{11} + \omega_0 \hat{g}_{21}).$$

In our example, we obtained the following values:

$$\omega_0 = \frac{715}{72}\sqrt{\frac{5}{1191}}, \quad \hat{g}_{11} = \frac{17667936}{\sqrt{5}} - 371134764\sqrt{\frac{3}{397}}\, i,$$
$$\hat{g}_{20} = \frac{17667936}{\sqrt{5}} - \frac{5207546916}{5}\sqrt{\frac{3}{397}}\, i,$$
$$\hat{g}_{21} = -\frac{897806862453676032}{125} + \frac{628677742657536}{25}\sqrt{\frac{1191}{5}}\, i,$$

hence,

$$l_1(0) = -\frac{35713462758285312}{125}\sqrt{\frac{1191}{5}} \neq 0.$$

Thus, in virtue of Theorems 3.3 and 3.4 and Section 3.4 in [8], we obtain a supercritical Hopf bifurcation. In particular, for $\alpha > 0$ small enough, we can guarantee the existence of limit cycles, see Figure 4. This implies that, under this choice of parameters, there is coexistence between the predators and the preys.

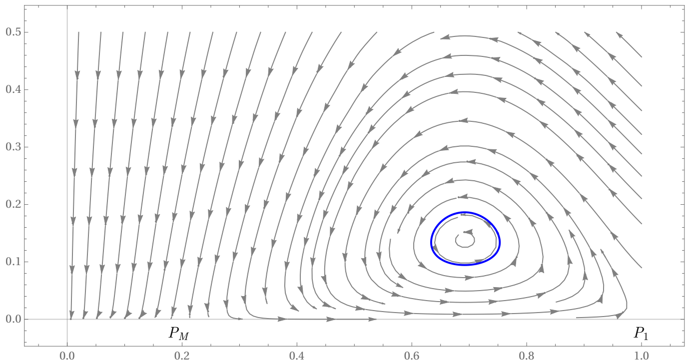

Figura 4: Limit cycle when $A = \frac{1}{16}$, $P = 6$, $C = 5$, $B = Q = 1$ and $\alpha = 0{,}1$

5. Further comments and conclusions

In this work we studied a predator-prey model of Gause type, where the preys have a non-differentiable square root functional response and are affected by an Allee effect. We discovered that the dynamic around the origin is similar to that the works of Braza [4] and Ajraldi [1]: when the population of predators and preys are small enough, either both species become extinct, or the preys always survives.

Moreover, under a strong Allee effect, for an wide choice of parameters, the predators are doomed to extinction. This extinction in fact happens for all parameters, whenever the population of preys is below the Allee threshold. We provide necessary conditions for the survival of both species, and under these conditions, we characterized the existence of a type-center equilibrium. Finally, we provide an explicit example when both species coexist, proving the existence of a limit cycle by means of a Hopf bifurcation.

6. Acknowledgements

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A. Complementary results and proofs

Proposition A.1. *The vector field associated to system* (2.1) *is topologically equivalent to the system*

$$Y_\eta : \begin{cases} \dfrac{du}{d\tau} = B[u(1-u)(u-M)(\sqrt{u}+A) - Q\sqrt{u}v] \\ \dfrac{dv}{d\tau} = (P\sqrt{u}-C)v, \end{cases}$$

where $B = K^{3/2}$, $Q = q/r$, $A = a/\sqrt{K}$, $M = m/K$, $P = \sqrt{K}(p-c)/r$, $C = ac/r$.

Proof. Consider the change of variables and time rescaling defined as in (2.2):

$$(x,y,t)=\Phi(u,v,\tau)=\left(Ku,K^2v,(\sqrt{Ku}+a)\tau/r\right).$$

Note that Φ is a diffeomorphism and the Jacobian matrix of Φ is

$$D\Phi(u,v,\tau)=\begin{bmatrix} K & 0 & 0\\ 0 & K^2 & 0\\ \dfrac{\sqrt{K}\tau}{2r\sqrt{u}} & 0 & \dfrac{\sqrt{Ku}+a}{r}\end{bmatrix},$$

whose determinant is $\det(D\Phi(u,v,\tau))=\dfrac{K^3}{r}(\sqrt{Ku}+a)\neq 0$. Moreover, $\dfrac{du}{dx}=\dfrac{1}{K}$, $\dfrac{dv}{dy}=\dfrac{1}{K^2}$, $\dfrac{dt}{d\tau}=(\sqrt{Ku}+a)/r$, thus

$$\begin{aligned}\frac{du}{d\tau}&=\frac{du}{dx}\frac{dx}{dt}\frac{dt}{d\tau}\\ &=\frac{1}{K}\left[r(1-u)(Ku-m)Ku-\frac{q\sqrt{Ku}}{\sqrt{Ku}+a}K^2v\right]\frac{\sqrt{Ku}+a}{r}\\ &=(1-u)(Ku-m)u(\sqrt{Ku}+a)-\frac{q\sqrt{Ku}}{r}Kv\\ &=K^{3/2}[u(1-u)(u-m/K)(\sqrt{u}+a/\sqrt{K})-\frac{q}{r}\sqrt{u}v]\\ &=B[u(1-u)(u-M)(\sqrt{u}+A)-Q\sqrt{u}v].\end{aligned}$$

In the same way

$$\begin{aligned}\frac{dv}{d\tau} &= \frac{dv}{dy}\frac{dy}{dt}\frac{dt}{d\tau} = \frac{1}{K^2}\left[\frac{p\sqrt{x}}{\sqrt{x}+a} - c\right] y \frac{\sqrt{Ku}+a}{r} \\ &= \frac{1}{K^2}\left[\frac{p\sqrt{Ku}}{\sqrt{Ku}+a} - c\right] K^2 v \frac{\sqrt{Ku}+a}{r} \\ &= \left[\frac{p}{r}\sqrt{Ku} - \frac{c}{r}(\sqrt{Ku}+a)\right] v \\ &= \left[\frac{p-c}{r}\sqrt{Ku} - \frac{ca}{r}\right] v \\ &= [P\sqrt{u} - C]v.\end{aligned}$$

□

Proof of Proposition 3.3. We will follow the same steps done in the proof of Proposition 3.1. The Jacobian matrix at $\mathcal{P}_M$ is

$$J_M = DY_\eta(\mathcal{P}_M) = \begin{bmatrix} B(1-M)M(A+\sqrt{M}) & -BQ\sqrt{M} \\ 0 & P\sqrt{M}-C \end{bmatrix}, \tag{A.1}$$

with eigenvalues $B(1-M)M(A+\sqrt{M}) > 0$ and $P\sqrt{M} - C$. Items 1 and 2 in the proposition follow from here. On the other hand, when $\sqrt{M}P = C$, (3.5) reduces to

$$J_M = \begin{bmatrix} \lambda & -BQ\sqrt{M} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} BQ\sqrt{M} & 1 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} BQ\sqrt{M} & 1 \\ \lambda & 0 \end{bmatrix}^{-1}$$

where $\lambda = B(1-M)M(A+\sqrt{M})$ is the non-zero eigenvalue of J_M. Now consider the change of variables

$$(\hat{x}, \hat{y}, \tau) = \varphi(x, y, t) = \left(\frac{y}{\lambda}, x - M - \frac{B\sqrt{M}Q}{\lambda} y, \lambda t\right)$$

which transforms system (3.1) to

$$\begin{cases} \dfrac{d\hat{x}}{dt} = p_2(\hat{x}, \hat{y}) \\ \dfrac{d\hat{y}}{dt} = \hat{y} + q_2(\hat{x}, \hat{y}) \end{cases}$$

where $p_2(\hat{x},\hat{y}) = a_{20}\hat{x}^2 + a_{11}\hat{x}\hat{y} + h.o.t.$, $a_{20} \neq 0$, and $q_2(\hat{x},\hat{y}) = b_{20}\hat{x}^2 + b_{11}\hat{x}\hat{y} + b_{02}\hat{y}^2 + h.o.t.$ We again use the Implicit Function Theorem, to conclude that there exists a function $\hat{y} = \phi(\hat{x}) = c_2\hat{x}^2 + h.o.t$, defined in a neighborhood of 0 such that

$$\phi(\hat{x}) + q_2(\hat{x},\phi(\hat{x})) = 0,$$

and

$$p_2(\hat{x},\phi(\hat{x})) = a_{20}\hat{x}^2 + h.o.t.$$

As in the proof of Proposition 3.1, we now refer again to Theorem 1, Section 2.11, in [9], to conclude that $\mathcal{P}_M$ is a saddle node. $\square$

Proof of Proposition 3.5. The Jacobian matrix of system (3.6) at the origin is

$$\widehat{J}_0 = DZ(\widehat{\mathcal{P}}_0) = \begin{bmatrix} -\frac{1}{2}BMA & -\frac{1}{2}QB \\ 0 & -C \end{bmatrix}.$$

The eigenvalues of $\widehat{J}_0$ are $-C < 0$ and $-\frac{1}{2}BMA$. Thus, if $M < 0$ then $\widehat{\mathcal{P}}_0$ is a hyperbolic saddle, and if $M > 0$ then it is a hyperbolic stable node.

When $M = 0$, $\widehat{J}_0$ reduces to

$$\widehat{J}_0 = \begin{bmatrix} 0 & -\frac{1}{2}BQ \\ 0 & -C \end{bmatrix} = \begin{bmatrix} 1 & QB \\ 0 & 2C \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -C \end{bmatrix} \begin{bmatrix} 1 & BQ \\ 0 & 2C \end{bmatrix}^{-1}.$$

Now consider the change of variables

$$(\hat{x},\hat{y},\tau) = \varphi(u,v,t) = \left(u - \frac{BQ}{2C}v, \frac{v}{2C}, -Ct\right)$$

which transforms system (3.1) into

$$\begin{cases} \dfrac{d\hat{x}}{d\tau} = p_2(\hat{x},\hat{y}), \\ \dfrac{d\hat{y}}{d\tau} = \hat{y} + q_2(\hat{x},\hat{y}), \end{cases}$$

where $p_2(\hat{x}, \hat{y}) = a_{11}\hat{x}\hat{y} + a_{02}\hat{y}^2 + h.o.t.$ and

$$q_2(\hat{x}, \hat{y}) = -\frac{P}{C}\hat{x}\hat{y} - \frac{BPQ}{C}\hat{y}^2.$$

Since $\hat{y}$ divides $\hat{y} + q_2(\hat{x}, \hat{y})$, the function $\phi(\hat{x}) = 0$, defined in a neighborhood of 0, satisfies

$$\phi(\hat{x}) + q_2(\hat{x}, \phi(\hat{x})) = 0.$$

Replace $\hat{y} = \phi(\hat{x}) = 0$ on $p_2(\hat{x}, \hat{y})$ to obtain

$$p_2(\hat{x}, \phi(\hat{x})) = -\frac{AB}{2C}\hat{x}^3 + h.o.t.$$

We now refer to Theorem 1, Section 2.11 in [9], to conclude that $\widehat{\mathcal{P}}_0$ is a topological saddle. □

Referencias

[1] V. Ajraldi, M. Pittavino, and E. Venturino. Modeling herd behavior in population systems. *Nonlinear Analysis: Real World Applications*, 12(4):2319–2338, Aug. 2011.

[2] A. D. Bazykin, A. I. Khibnik, and B. Krauskopf. *Nonlinear Dynamics of Interacting Populations*, volume 11 of *World Scientific Series on Nonlinear Science Series A*. World Scientific, May 1998.

[3] J. L. Bravo, M. Fernández, M. Gámez, B. Granados, and A. Tineo. Existence of a polycycle in non-lipschitz gause-type predator?prey models. *Journal of Mathematical Analysis and Applications*, 373(2):512–520, 2011.

[4] P. A. Braza. Predator-prey dynamics with square root functional responses. *Nonlinear Analysis: Real World Applications*, 13(4):1837–1843, 2012.

[5] F. Courchamp, L. Berec, and J. Gascoigne. *Allee Effects in Ecology and Conservation.* Oxford University Press, Feb. 2008.

[6] H. I. Freedman. *Deterministic mathematical models in population ecology.* Pure and applied mathematics ; v. 57. M. Dekker, New York, 1980.

[7] G. F. Gause, N. P. Smaragdova, and A. A. Witt. Further studies of interaction between predators and prey. *Journal of Animal Ecology*, 5(1):1–18, 1936.

[8] Y. A. Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences.* Springer New York, New York, NY, 2004.

[9] L. Perko. *Differential equations and dynamical systems.* Number 7 in Texts in applied mathematics. Springer, New York, 2nd ed edition, 1996.

[10] E. Sáez and I. Szántó. A polycycle and limit cycles in a non-differentiable predator-prey model. *Proceedings Mathematical Sciences*, 117(2):219–231, May 2007.

[11] P. A. Stephens, W. J. Sutherland, and R. P. Freckleton. What is the Allee effect? *Oikos*, 87(1):185–190, 1999.

[12] K. Vilches-Ponce. Dinámicas de un modelo de depredación del tipo Gause con respuesta funcional no diferenciable. Master's thesis, Pontificia Universidad Católica de Valparaíso, Chile, 2009.

[13] K. Vilches-Ponce, E. González-Olivares, and A. Rojas-Palma. Prey herd behavior modeled by a generic non-differentiable functional response. *Math. Model. Nat. Phenom.*, 13(3):26, 2018.

Resumen: Un modelo depredador-presa de tipo Gause es una extensión del clásico modelo depredador-presa de Lotka-Volterra. En este trabajo estudiamos un modelo depredador-presa de tipo Gause, donde el crecimiento de las presas es sujeto a un efecto Allee y la acción del depredador sobre la presa es dada por una funcional de respuesta de raíz cuadrada, la cual no es diferenciable en el eje y. Este tipo de respuesta funcional modela apropiadamente sistemas en los cuales la presa posee un fuerte comportamiento de rebaño, pues los depredadores interactúan con las presas mayormente en la frontera del rebaño. Debido al término de raíz cuadrada en la respuesta funcional, el estudio del comportamiento de las soluciones cerca al origen es más sutil e interesante que en otros modelos.

Nuestro estudio es dividido en dos partes: la clasificación local de los puntos de equilibrio, y el comportamiento de las soluciones en cierto conjunto invariante cuando el modelo tiene un efecto Allee fuerte. En uno de nuestros resultados principales probamos, para una amplia selección de parámetros, que las soluciones en cierto conjunto invariante se aproximan al eje y. Además, para cierta elección de parámetros, probamos la existencia de una curva separatriz que divide el conjunto invariante en dos regiones: una donde toda solución se aproxima al eje y, y otra donde hay un punto de equilibrio global y asintóticamente estable. También damos condiciones para asegurar la existencia de un equilibrio de tipo centro, y mostramos la existencia de una bifurcación de Hopf.

Palabras claves: Modelos depredador-presa, Modelos de Gause, Efecto Allee, Funcional de respuesta de raíz cuadrada.

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