Singular traces of an integral operator related to the Riemann Hypothesis

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Abstract

It is shown that if the integral operator (Hilbert-Schmidt, non nuclear and non-normal) on $L^2(0,1)$, $(A_\rho f)(\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)dx$, where ρ is the fractional part function, belongs to a geometrically stable ideal J, then $\tau(A_\rho) = 0$ for every non-trivial singular trace τ on J.

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1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operator on a separable complex Hilbert Space H. The adjoint of an operator $T \in B(H)$ is denoted by T^*. Denote by $\{s_n(T)\}_{n\geq 1}$ the sequence of singular values of a compact operator $T \in B(H)$. By an ideal we mean a two-sided ideal in $B(H)$. A linear functional τ from the ideal J into $\mathbb{C}$ is said to be a trace, if:

i) $\tau(U^*TU) = \tau(T)$ for every $T \in J$ and $U \in B(H)$ unitary,

ii) $\tau(T) \geq 0$ for every $T \in J$ with T a non-negative operator. We denote by $T \geq 0$ when T is a non-negative operator.

Therefore a trace is a positive unitary invariant linear functional. Obviously, the usual trace is an example of a trace on the ideal $S^1(H)$ of nuclear operators.

A trace τ on an ideal J will be call singular if it vanishes on the set $\mathcal{F}(H)$ of finite rank operators. This definition makes sense, since by the Calkin Theorem [7], each proper ideal in $B(H)$ contains the finite rank operators and is contained in the ideal $\mathcal{K}(H)$ of the compact linear operators on H.

In 1966, J. Dixmier proved the existence of singular traces [10]. These traces are called Dixmier traces, and its importance is due to their applications in noncommutative geometry [9]. Other examples of singular traces were given by N. Kalton [13]. J. Varga [20] and S. Albeverio et al. [1].

The condition (i) of trace coincides with the notion that a trace vanishes on the commutator subspace of J,

$$\mathrm{Com}(J) = \mathrm{span}\{[A, B] : A \in J, B \in B(H)\},$$

where $[A, B] = AB - BA$. The commutator subspace has been characterizes in terms of arithmetic means of monotone sequences by N. Kalton [14]; note that N. Kalton works with the special case of geometrically stable ideals.

In order to study the asymptotic distribution of prime numbers among the natural ones, Riemann introduced the function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad s = \sigma + it, \ \sigma > 1, \ t \in \mathbb{R}$$

afterwards known as the Riemann zeta function. He proved that the analytic continuation of this function, defined on $\mathbb{C}\backslash\{1\}$, has an infinite number of zeros in the strip $\{s : 0 \leq \sigma \leq 1\}$, all of them being non-real and symmetrically distributed with respect to the lines $\{s : t = 0\}$ and $\{s : \sigma = \frac{1}{2}\}$. Concerning the distribution of these zeros he formulated the famous Riemann Hypothesis: All the zeros of ζ in the strip $\{s : 0 \leq \sigma \leq 1\}$ lie on the line $\{s : \sigma = \frac{1}{2}\}$.

In [2], J. Alcántara-Bode has reformulated the Riemann Hypothesis as a problem of functional analysis by means of the following theorem.

Theorem 1.1. *Let* $(A_\rho f)(\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)dx$*, where* $\rho(x) = x - [x]$*,* $x \in \mathbb{R}$*,* $[x] \in \mathbb{Z}$*,* $[x] \leq x < [x] + 1$*, be considered as an operator on* $L^2(0,1)$*. Then the Riemann Hypothesis holds if and only if* $\ker(A_\rho) = \{0\}$*, or if and only if* $h \notin Ran(A_\rho)$ *where* $h(x) = x$*.* □

The aim of this paper is to show that if $A_\rho \in J$, where J is a geometrically stable ideal (see Definition 2.10) of $L^2(0,1)$, then $\tau(A_\rho) = 0$ for every τ non-trivial singular trace on J.

2. Singular traces and the commutator subspace

Let ℓ^∞ the space of all bounded sequences of complex numbers and w a dilation invariant extended limit on ℓ^∞, that is, w is an extended limit on ℓ^∞ [18] and

$$w(\{x_1, x_2, \cdots\}) = w(\{x_1, x_1, x_2, x_2, \cdots\}) \text{ for all } x = \{x_1, x_2, \cdots\} \in \ell^\infty.$$

The Dixmier trace of $T \in M^{1,\infty}(H)$ with $T \geq 0$ is the number

$$\mathrm{Tr}_w(T) := w\left(\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n} s_k(T)\right\}\right),$$

where

$$M^{1,\infty}(H) = \left\{T \in \mathcal{K}(H) : \|T\|_{1,\infty} := \sup_{n\geq 1}\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n} s_k(T)\right\} < \infty\right\}.$$

It was shown in [16, Theorem 10.22] that the weight Tr_w defines a positive, unitarily invariant, additive and positive homogeneous function on the positive cone of $M^{1,\infty}(H)$, that can uniquely be extended to a singular trace on all of $M^{1,\infty}(H)$, i.e., for an arbitrary $T \in M^{1,\infty}(H)$, its Dixmier trace is defined by

$$\mathrm{Tr}_w(T) := w\left(\left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n} s_k(T_1) - s_k(T_2) + is_k(T_3) - is_k(T_4)\right\}\right),$$

where $T = T_1 - T_2 + iT_3 - iT_4$, $\ 0 \leq T_j \in M^{1,\infty}(H)$, $\ j = 1, 2, 3, 4$. In addition to this, the Dixmier trace vanishes on the ideal $S^1(H)$ and it is continuous in the norm $\|\cdot\|_{1,\infty}$, more precisely,

$$|\mathrm{Tr}_w(T)| \leq \|T\|_{1,\infty} \text{ for all } T \in M^{1,\infty}(H).$$

A smaller subclass of Dixmier traces was suggested by A. Connes in [9]. He observed that for any extended limit γ on ℓ^∞ the functional $w := \gamma \circ M$ is a dilation invariant extended limit. Here, the operator $M : \ell^\infty \to \ell^\infty$ is defined by

$$M(\{x_1, x_2, \cdots\}) = \left\{\frac{1}{\log(n+1)}\sum_{k=1}^{n}\frac{x_k}{k}\right\}.$$

The Dixmier trace associated to this dilation invariant extended limit w is called a Connes-Dixmier trace.

The notion of measurable operators was introduced by A. Connes [9] and is the following.

Definition 2.1. An operator $T \in M^{1,\infty}(H)$ is called Dixmier measurable if $\mathrm{Tr}_w(T)$ is independent of the choice of the dilation invariant extended limit w on ℓ^∞.

Definition 2.2. An operator $T \in M^{1,\infty}(H)$ is called Connes-Dixmier measurable if $\mathrm{Tr}_w(T)$ is independent of the choice of the dilation invariant extended limit $w = \gamma \circ M$, where γ is an extended limit on ℓ^∞.

Evidently, every Dixmier measurable operator is Connes- Dixmier measurable. For positive operators, by [8, Theorems 6.6 and 6.7] a characterization of Dixmier measurable operators and Connes-Dixmier measurable operators is given by the following.

Theorem 2.3. *For a positive operator T in $M^{1,\infty}(H)$, the following statements are equivalent:*

a) *T is Dixmier measurable;*

b) *T is Connes-Dixmier measurable;*

c) *the limit* $\displaystyle \lim_{n\to+\infty} \frac{1}{\log(n+1)} \sum_{k=1}^{n} s_k(T)$ *exists.* □

In general, for an arbitrary $T \in M^{1,\infty}(H)$, the next theorem [18, Theorem 4] gives a characterization of Dixmier measurability.

Theorem 2.4. *An operator $T \in M^{1,\infty}(H)$ is Dixmier measurable if and only if the limit*

$$\lim_{n\to+\infty} \frac{1}{\log(n+1)} \sum_{k=1}^{n} \frac{1}{k\log(k+1)} \sum_{i=1}^{k} \tilde{s}_{[\alpha i]}(T)$$

exists uniformly in $\alpha \geq 1$, here

$$\tilde{s}_k(T) := s_k(T_1) - s_k(T_2) + i s_k(T_3) - i s_k(T_4),$$

with $T = T_1 - T_2 + iT_3 - iT_4$, $0 \leq T_j \in M^{1,\infty}(H)$, $j = 1, 2, 3, 4$. □

The Theorem 2.3 tell us that on the positive cone of $M^{1,\infty}(H)$ the notions of Dixmier and Connes-Dixmier measurable coincide. In Addition to this, according to [19, Theorem 7.7], these notions are the same on the ideal $\mathscr{L}^{1,\infty}(H) \subset M^{1,\infty}(H)$, where

$$\mathscr{L}^{1,\infty}(H) = \left\{ T \in \mathcal{K}(H) : \sup_{n\geq 1}\{ns_n(T)\} < \infty \right\};$$

and by [15, Theorem 9.7.5], a characterization of Dixmier measurable elements is given in the following theorem.

Theorem 2.5. *An operator $T \in \mathscr{L}^{1,\infty}(H)$ is Dixmier measurable if and only if the limit $\lim_{n\to+\infty} \frac{1}{\log(n+1)} \sum_{k=1}^{n} \lambda_k(T)$ exists. Here, $\{\lambda_n(T)\}_{n\geq 1}$ is the sequence of nonzero eigenvalues of T, ordered in such a way that $|\lambda_n(T)| \geq |\lambda_{n+1}(T)|\ \forall n \in \mathbb{N}$.* □

Observe that the previous limit is equal to the Dixmier trace of T [15, Theorem 7.3.1].

The existence of a singular trace which is non-trivial on a compact operator T, i.e., on the two-sided ideal generated by T,

$$(T) = \cup_{r=1}^{\infty}\{\sum_{i=1}^{r} X_i T Y_i;\ X_i, Y_i \in B(H)\},$$

was studied by J. Varga [20], and it has been completely characterized in [1]. For this reason, the notion of irregular, eccentric and generalized eccentric operator arises.

Definition 2.6. We say that a compact operator $T \in B(H)$ is

a) regular if $\sum_{k=1}^{n} s_k(T) = O(ns_n(T))\ (n \to \infty)$

b) irregular if it is not regular

c) eccentric if it is irregular but not nuclear

d) generalized eccentric if 1 is a limit point of the sequence $\left\{ \dfrac{S_{2n}(T)}{S_n(T)} \right\}$,

where

$$S_n(T) = \begin{cases} \sum_{k=1}^{n} s_k(T) & ,T \notin S^1(H) \\ \sum_{k=1}^{n} s_k(T) - \mathrm{Tr}(|T|), & T \in S^1(H) \end{cases} .$$

Remark 2.7. By [20, Lemma 1], the class of generalized eccentric operators which are not nuclear coincides with the class of eccentric operators.

By [1, Lemma 2.6], that an operator is generalized eccentric can be reformulated as follows.

Lemma 2.8. *Let $T \in B(H)$ be a compact operator. Then T is generalized eccentric if and only if there exists an increasing sequence of natural numbers $\{p_n\}$ such that* $\lim\limits_{k \to +\infty} \dfrac{S_{kp_k}(T)}{S_{p_k}(T)} = 1.$ □

In this context, the main result in [1] is the following.

Theorem 2.9. *Let $T \in B(H)$ be a compact operator. Then the following are equivalent:*

a) *There exists a singular trace τ such that $0 < \tau(|T|) < +\infty$.*

b) *T is generalized eccentric.* □

The process to construct the singular trace given by (a) is as follows:

We introduce a triple $\Omega = (T, w, \{n_k\})$, where T is a generalized eccentric operator, w is an extended limit and $n_k = np_k,\ k \in \mathbb{N}$, where $\{p_n\}$ is the sequence given in Lemma 2.8. Associated with the triple Ω, on the positive part of the ideal (T), we defined the functional

$$\tau_\Omega(A) := w\left(\left\{ \frac{S_{n_k}(A)}{S_{n_k}(T)} \right\} \right),\ A \in (T)_+,$$

and by [1, Theorem 2.11] this functional extends linearly to a singular trace on the ideal (T).

Now we concentrate on the commutator subspace of symmetrically normed ideals and geometrically stable ideals, terminologies used in [15] and [14] respectively.

Definition 2.10. An ideal J of $B(H)$ is called a symmetrically normed ideal if it is a Banach space for a norm $\|\cdot\|_J$ with the following property:

$$\|BAC\|_J \leq \|B\|\|A\|_J\|C\| \text{ for all } A \in J \text{ and } B, C \in B(H).$$

Definition 2.11. An ideal J of $B(H)$ is called geometrically stable if a diagonal operator $\text{diag}\{s_1, s_2, \cdots\} \in J$, where $s_1 \geq s_2 \geq \cdots \geq 0$, then we have $\text{diag}\{u_1, u_2, \cdots\} \in J$, where $u_n = (s_1 \cdots s_2 \cdots s_n)^{1/n}$.

Remark 2.12. By [15, Lemma 5.5.9], every symmetrically normed ideal is a geometrically stable ideal.

The following theorem [14, Theorem 3.3] characterizes the commutator subspace for the special case of geometrically stable ideals.

Theorem 2.13. *Suppose that J is a geometrically stable ideal of $B(H)$. Then the following conditions are equivalent:*

a) $S \in Com(J)$;

b) $diag\left\{\frac{1}{n}(\lambda_1(S) + \cdots + \lambda_n(S))\right\} \in J$;

c) *there exists $T \in J$ so that $\frac{1}{n}|\lambda_1(S) + \cdots + \lambda_n(S)| \leq s_n(T)$ for each $n \in \mathbb{N}$.* □

Example 2.14. Let $V : L^2(0,1) \to L^2(0,1)$ be the integral operator $(Vf)(t) = 2i\int_0^t f(s)ds$. Is known that the operator V is Volterra, it means that V is compact and V has no eigenvalues. An easy calculation shows that $s_n(V) = \frac{4}{(2n-1)\pi}$, $n \in \mathbb{N}$. Then, by Remark 2.7, V is a generalized eccentric operator. Finally, $V \in M^{1,\infty}(L^2(0,1))$ and by the Theorem 2.13, $V \in \text{Com}(M^{1,\infty}(L^2(0,1)))$. Hence V is Dixmier measurable and $\text{Tr}_w(V) = 0$ for all w dilation invariant extended limit on ℓ^∞.

3. The integral operador A_ρ

To study the Riemann Hypothesis, J. Alcántara-Bode introduced the integral operator $A_\rho : L^2(0,1) \to L^2(0,1)$, $(A_\rho f)(\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)dx$. By [2], the Riemann Hypothesis holds if and only if $\ker(A_\rho) = \{0\}$, or if and only if $h \notin \mathrm{Ran}(A_\rho)$ where $h(x) = x$ for all $x \in [0,1]$.

The problem of verifying that $h \in \mathrm{Ran}(A_\rho)$ leads to an ill posed problem [12], for this reason, J. Alcántara-Bode regularizes this problem replacing A_ρ by

$$(A_\rho(\alpha)f)(\theta) = \int_0^1 \rho\left(\frac{\alpha\theta}{x}\right) f(x)dx,\ 0 < \alpha \leq 1,\ f \in L^2(0,1).$$

Note that $A_\rho = A_\rho(1)$. The symbol I stands for identity maps. The operator $V_\alpha \in B(L^2(0,1))$ defined by $(V_\alpha f)(x) = f\left(\frac{x}{\alpha}\right)\xi_{[0,\alpha]}(x)$, where ξ_C denotes the characteristic function of C, was introduced in [3], and we have the relations

$$\begin{aligned} (V_\alpha^* f)(x) &= \alpha f(\alpha x),\ V_\alpha^* V_\alpha = \alpha I, \\ V_\alpha V_\alpha^* &= \alpha \xi_{[0,\alpha]},\ V_\alpha^* A_\rho V_\alpha = \alpha^2 A_\rho \\ A_\rho(\alpha) &= \frac{1}{\alpha} V_\alpha^* A_\rho,\ A_\rho(\alpha) V_\alpha = \alpha A_\rho. \end{aligned}$$

We briefly summarize properties of $A_\rho(\alpha)$ established in [2, 3, 4]:

i) $A_\rho(\alpha)$, $0 < \alpha \leq 1$, is Hilbert-Schmidt, but neither nuclear, nor normal, nor monotone (or accretivo).

ii) $\lambda \in \sigma(A_\rho(\alpha))\backslash\{0\}$ ($\sigma(A_\rho(\alpha))$ is the spectrum of $A_\rho(\alpha)$), $0 < \alpha \leq 1$, if and only if $T_\alpha(\lambda^{-1}) = 0$ where

$$T_\alpha(u) = 1 - \alpha u + \sum_{r=1}^{+\infty} (-1)^{r+1} \frac{\alpha^{(r+1)(r+2)/2}}{(r+1)!(r+1)} \prod_{\ell=1}^{r} \zeta(\ell+1) u^{r+1}$$

is an entire function with an infinite number of zeros; moreover the multiplicity of a zero of T_α coincides with the algebraic multiplicity of the corresponding eigenvalue.

iii) $T_1(u)$ is an entire function of order one and type one. For $0 < \alpha < 1$, $T_\alpha(u)$ is an entire function of order zero.

iv) For $0 < \alpha < 1$, $\sum_{n=1}^{+\infty} |\lambda_n(A_\rho(\alpha))|^r < \infty$, $\forall r > 0$. Moreover, $\sum_{n=1}^{+\infty} |\lambda_n(A_\rho)| = +\infty$ and

$$|\lambda_n(A_\rho)| \leq \frac{e}{n}, \ \forall n \in \mathbb{N}. \tag{3.1}$$

4. Results

The following result is thanks to Julio Alcántara-Bode (private communication).

Theorem 4.1. *If* $0 < \alpha \leq \beta \leq 1$ *then*

$$A_\rho(\alpha) = A_\rho(\beta) V^*_{\frac{\alpha}{\beta}} + \alpha \left\langle \cdot, \xi_{[\frac{\alpha}{\beta},1]} \frac{1}{h} \right\rangle h. \tag{4.1}$$

Proof. By the Müntz-Szasz Theorem [6], it is sufficient to verify (4.1) for h^r with $r \in \mathbb{N}$, where $h(x) = x$. To this end, we use the identity [5],

$$\int_0^1 \rho\left(\frac{\theta}{x}\right) x^r dx = \frac{\theta}{r} - \frac{\zeta(r+1)}{r+1}\theta^{r+1}, \ \mathrm{Re}(r) > -1. \tag{4.2}$$

Evaluating the right-hand side of (4.1) we have

$$\begin{aligned} A_\rho(\beta) V^*_{\frac{\alpha}{\beta}}(h^r)(\theta) + \alpha \left(\int_0^1 h^r(x) \xi_{[\frac{\alpha}{\beta},1]} \frac{1}{h(x)} dx \right) h(\theta) \\ = \frac{\alpha^{r+1}}{\beta^{r+1}} \int_0^1 \rho\left(\frac{\beta\theta}{x}\right) x^r dx + \frac{\alpha}{r}\left(1 - \frac{\alpha^r}{\beta^r}\right)\theta. \end{aligned}$$

It follows from (4.2) that

$$\frac{\alpha^{r+1}}{\beta^{r+1}}\int_0^1 \rho\left(\frac{\beta\theta}{x}\right)x^r dx+\frac{\alpha}{r}\left(1-\frac{\alpha^r}{\beta^r}\right)\theta=\frac{\alpha^{r+1}}{\beta^{r+1}}\left(\frac{\beta\theta}{r}-\frac{\zeta(r+1)}{r+1}(\beta\theta)^{r+1}\right)$$
$$+\frac{\alpha}{r}\left(1-\frac{\alpha^r}{\beta^r}\right)\theta=(A_\rho(\alpha))(h^r)(\theta).$$

Hence, (4.1) is true in h^r with $r\in\mathbb{N}$. □

Remark 4.2. If $0<\alpha,\beta\leq 1$ and $\lambda>0$ with $\lambda\alpha,\lambda\beta\in\langle 0,1]$, we get from (4.1) that

$$A_\rho(\lambda\alpha)=A_\rho(\lambda\beta)V^*_{\frac{\alpha}{\beta}}+\lambda\alpha\left\langle\cdot,\xi_{[\frac{\alpha}{\beta},1]}\frac{1}{h}\right\rangle h.$$

For a given operator $T\notin S^1(H)$, where H is a separable complex Hilbert space, the following theorem shows the existence of a non-trivial singular trace defined on a geometrically stable ideal taking the value of zero on T.

Theorem 4.3. *For every operator $T\notin S^1(H)$ there exists a geometrically stable ideal J and a non-trivial singular trace τ defined on J, such that $T\in J$ and $\tau(T)=0$.*

Proof. By [11, Theorem 3.1], there exists a generalized eccentric operator B such that $T\in(B)_0\subset(B)$. Here $(B)_0$ is called the kernel of (B) (see [11]). As we explained in the construction of the singular trace in the Theorem 2.9, we can take the triple $\Omega=(B,w,\{n_k\})$. Associated with Ω, on the positive part of (B), we have the functional

$$t_\Omega(A):=w\left(\left\{\frac{S_{n_k}(A)}{\sum_{i=1}^{n_k}s_i(B)}\right\}\right)=w\left(\left\{\frac{\sum_{i=1}^{n_k}s_i(A)}{\sum_{i=1}^{n_k}s_i(B)}\right\}\right);A\in(B)_+$$

that extends linearly to a singular trace on the ideal (B). We also denote this extension by t_Ω. Clearly, $t_\Omega(|B|)=1$. It is easy to see that t_Ω is bounded with the norm $\|\cdot\|_B$, where

$$\|A\|_B = \sup_{n\in\mathbb{N}} \left\{ \frac{\sum_{k=1}^{n} s_k(A)}{\sum_{k=1}^{n} s_k(B)} \right\},$$

so it extends by continuity to a singular trace $\tilde{t}_\Omega : \overline{(B)}^{\|\cdot\|_B} \to \mathbb{C}$. By Remark 2.12, the ideal $\overline{(B)}^{\|\cdot\|_B}$ is geometrically stable. Finally, by [11, Proposition 2.7], we get that $\tilde{t}_\Omega(T) = 0$. □

Finally, we have the main result.

Theorem 4.4. *If $A_\rho \in J$, where J is a geometrically stable ideal of $L^2(0,1)$, then $\tau(A_\rho) = 0$ for every τ non-trivial singular trace on J.*

Proof. Firstly, we prove that for $0 < \alpha, \beta \leq 1$ the compact operators $\frac{1}{\alpha}A_\rho(\alpha) - \frac{1}{\beta}A_\rho(\beta)$ have no eigenvalues, it means that they are Volterra (J. Alcántara-Bode, private communication). Indeed, similar to the proof [3, Theorem 3], it can be shown that the eigenfunctions $\varphi^{\alpha,\beta}$ associated to the non-zero eigenvalues $\lambda^{\alpha,\beta}$ of $\dfrac{1}{\alpha}A_\rho(\alpha) - \dfrac{1}{\beta}A_\rho(\beta)$ are analytic in $[0,1]$, and they can be extended to entire functions with $\varphi^{\alpha,\beta}(x) = \sum_{r=1}^{+\infty} c_r x^r$, where $c_r = c_r(\lambda^{\alpha,\beta})$. Substituting this series into

$$\left(\frac{1}{\alpha}A_\rho(\alpha) - \frac{1}{\beta}A_\rho(\beta)\right)\varphi^{\alpha,\beta} = \lambda^{\alpha,\beta}\varphi^{\alpha,\beta}, \tag{4.3}$$

we get from (4.2) that

$$\sum_{r=1}^{+\infty} \frac{\zeta(r+1)}{r+1} c_r(\beta^r - \alpha^r)x^{r+1} = \sum_{r=1}^{+\infty} \lambda^{\alpha,\beta} c_r x^r.$$

It gives that $c_r = 0$ for all $r \geq 1$ and then $\varphi^{\alpha,\beta} = 0$. Hence, the operators

$\dfrac{1}{\alpha}A_\rho(\alpha) - \dfrac{1}{\beta}A_\rho(\beta)$ are Volterra. Taking $\beta = 1$ in (4.1) give us

$$A_\rho(\alpha) = \frac{1}{\alpha} V_\alpha^* A_\rho = A_\rho V_\alpha^* + \alpha \left\langle \cdot, \xi_{[\alpha,1]} \frac{1}{h} \right\rangle. \tag{4.4}$$

Let τ a non-trivial singular trace on J (the existence of J and a non-trivial singular trace on J is guaranteed by the Theorem 4.2). Applying τ in (4.4), we obtain

$$\frac{1}{\alpha}\tau(V_\alpha^* A_\rho) = \tau(A_\rho V_\alpha^*) = \tau(V_\alpha^* A_\rho).$$

It implies that for $0 < \alpha < 1$ we have that $\tau(V_\alpha^* A_\rho) = 0$ and therefore $\tau(A_\rho(\alpha)) = 0$. Since $A_\rho - \frac{1}{\alpha}A_\rho(\alpha)$, $0 < \alpha < 1$ is Volterra, it follows from the Theorem 2.13 that $A_\rho - \frac{1}{\alpha}A_\rho(\alpha) \in Com(J)$, and hence $\tau(A_\rho - \frac{1}{\alpha}A_\rho(\alpha)) = \tau(A_\rho) - \frac{1}{\alpha}\tau(A_\rho(\alpha)) = 0$. From this equality, we conclude that $\tau(A_\rho) = 0$. □

Taking $J = \mathscr{L}^{1,\infty}(L^2(0,1))$ and $J = M^{1,\infty}(L^2(0,1))$ in the previous theorem, we have the following corollaries.

Corollary 4.5. *If $A_\rho \in \mathscr{L}^{1,\infty}(L^2(0,1))$ then A_ρ is Dixmier measurable and* $\displaystyle\lim_{n\to+\infty} \frac{1}{\log(n+1)} \sum_{k=1}^{n} \lambda_k(A_\rho) = 0$.

Proof. Since $\mathscr{L}^{1,\infty}(L^2(0,1))$ is a quasi-Banach ideal, by [14, Proposition 3.2], $\mathscr{L}^{1,\infty}(L^2(0,1))$ is geometrically stable. Therefore, the corollary follows from the Theorems 2.5 and 4.4. □

Corollary 4.6. *If $A_\rho \in M^{1,\infty}(L^2[0,1])$ then A_ρ is Dixmier measurable and the limit*

$$\lim_{n\to+\infty} \frac{1}{\log(n+1)} \sum_{k=1}^{n} \frac{1}{k\log(k+1)} \sum_{i=1}^{k} \tilde{s}_{[\alpha i]}(A_\rho)$$

exists uniformly in $\alpha \geq 1$.

Proof. It follows from the Theorems 2.4, 4.4 and the Remark 2.12. □

Remark 4.7.

i) We known that for $0 < \alpha, \beta \leq 1$ the operators $\frac{1}{\alpha}A_\rho(\alpha) - \frac{1}{\beta}A_\rho(\beta)$ are Volterra. Similarly, it can be show that for $0 < \alpha + \beta \leq 1$ the operators $A_\rho(\alpha + \beta) - A_\rho(\alpha) - A_\rho(\beta)$ are Volterra.

ii) It is not difficult to show that for each $n \in \mathbb{N}$ we have

$$\left(\frac{\alpha}{\beta}\right)^{1/2} \leq \frac{s_n(A_\rho(\alpha))}{s_n(A_\rho(\beta))} \leq \left(\frac{\alpha}{\beta}\right)^{-1/2}, \ 0 < \alpha \leq \beta \leq 1.$$

Therefore, from this inequality and the Remark 2.7, we obtain that A_ρ is a generalized eccentric operator if and only if $A_\rho(\alpha)$ is a generalized eccentric operator for some $\alpha \in \langle 0, 1]$.

iii) By [17], if H is a separable complex Hilbert space and $A \in \mathcal{K}(H)$, then there exists a compact normal operator N and a Volterra operator Q such that $A = N + Q$ and $\{\lambda_n(A)\} = \{\lambda_n(N)\}$. We call this decomposition, the Ringrose decomposition of A. If $A_\rho = N + Q$ is the Ringrose decomposition of A_ρ, by (3.1) and the Remark 2.7, N is a generalized eccentric operator and it belongs to $\mathscr{L}^{1,\infty}(L^2(0,1))$. Therefore, by the Theorem 2.9, there exists a singular trace τ on (N) such that $\tau(|A_\rho - Q|) = 1$.

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Resumen: Se ha probado que si el operador integral Hilbert-Schmidt, no nuclear y no normal en $L^2(0,1)$, $(A_\rho f)(\theta) = \int_0^1 \rho\left(\frac{\theta}{x}\right) f(x)dx$, donde ρ es la función parte fraccionaria, pertenece a un ideal geométricamente estable J, entonces $\tau(A_\rho) = 0$ para toda traza singular no trivial τ en I.

Palabras claves: traza singular, operador exéntrico generalizado, operador Volterra.

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