ON FRACTIONAL INTEGRAL OPERATORS OF THREE VARIABLES AND INTEGRAL TRANSFORMS

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Abstract

The present paper is a continuation to authors paper [11] where three variable analogues of certain fractional integral operators of M. Saigo were investigated. This paper deals with the effect of operating three variable analogues of Mellin and Laplace transforms on these three variable analogues of fractional integral operators of the earlier paper.

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1 Introduction

In 1978, M. Saigo [17] defined certain integral operators involving the Gauss hypergeometric function as follows:

Let $\alpha > \beta$ and η be real numbers. The fractional integral operator $I_x^{\alpha,\beta,\eta}$, which acts on certain functions $f(x)$ on the interval $(0, \infty)$ was defined as

$$I_x^{\alpha,\beta,\eta} f = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt) \quad (1.1)$$

Under the same assumptions in defining (1.1), he also defined the integral operator $J_x^{\alpha,\beta,\eta}$ as

$$J_x^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt) \quad (1.2)$$

Later on in 1988, Saigo and Raina [19] obtained the generalized fractional integrals and derivatives introduced by Saigo [17-18] of the system $S_q^n(x)$, where the general system of polynomials

$$S_q^n(x) = \sum_{r=0}^{[\frac{n}{q}]} \frac{(-n)_{qr}}{r!} A_{n,r} x^r$$

were defined by Srivastave [20], where $q > 0$ and $n \geq 0$ are integers, and $A_{n,r}$ are arbitrary sequence of real or complex numbers.

In an earlier communication the present authors [11] defined and studied certain three variables analogues of (1.1) and (1.2) which are as given below:

I. Let $c > 0$, $c' > 0$, $c'' > 0$, a, b, b', b'' be real numbers. A three variable analogue of fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ due to M. Saigo is defined as

$$ {}_1I_{0,x;0,y;0,z}^{a,b,b',b'';c,c',c''} \; f(x,y,z) \; = $$

$$ \frac{x^{-a}y^{-a}z^{-a}}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int_0^x \int_0^y \int_0^z (x-u)^{c-1} \; (y-v)^{c'-1} \; (z-w)^{c''-1} $$

$$ F_A^{(3)} \left[\begin{array}{l} a, b, b', b''; 1-\frac{u}{x}, 1-\frac{v}{y}, 1-\frac{w}{z} \\ c, c', c'' \quad ; \end{array} \right] f(u,v,w) \; dw \; dv \; du \quad (1.3) $$

where $F_A^{(3)}$ is a Lauricella function of three variables defined by

$$ F_A^{(3)} \left[\begin{array}{l} a, b, b', b''; x, y, z \\ c, c', c'' \quad ; \end{array} \right] = $$

$$ \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{n+r+s} \; (b)_n \; (b')_r \; (b'')_s}{n! \; r! \; s! \; (c)_n \; (c')_r \; (c'')_s} \; x^n \; y^r \; z^s, $$

$$ |x| + |y| + |z| < 1. $$

Special Cases :

i) For $a = b = b' = b'' = 0,\ c = \alpha,\ c' = \beta,\ c'' = \gamma$, (1.3) reduces to

$$ {}_1I^{0,0,0,0;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}_1R^{\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) $$

$$ = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\beta-1}(z-w)^{\gamma-1} f(u,v,w)\, dw\, dv\, du \tag{1.4} $$

Here (1.4) may be taken as a three variable analogue of Riemann-Liouville fractional integral operator $R^{\alpha}_{0,x}$.

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = b'' = 0,\ c' = \beta,\ c'' = \gamma$, (1.3) becomes

$$ {}_1I^{\alpha,-\eta,0,0;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{x}_{1}E^{\alpha,\beta,\gamma,\eta}_{0,x;0,y;0,z}\ f(x,y,z) = $$

$$ \frac{x^{-\alpha-\eta}\ y^{-\alpha}\ z^{-\alpha}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}\ (y-v)^{\beta-1}\ (z-w)^{\gamma-1}\ u^{\eta}\ f(u,v,w)\ dw\ dv\ du \tag{1.5} $$

(iii) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta,\ c' = \beta,\ c'' = \gamma$, (1.3) gives

$$ {}_1I^{\alpha,0,-\eta,0;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{y}_{1}E^{\alpha,\beta,\gamma,\eta}_{0,x;0,y;0,z}\ f(x,y,z) = $$

$$ \frac{x^{-\alpha}\ y^{-\alpha-\eta}\ z^{-\alpha}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}\ (y-v)^{\beta-1}\ (z-w)^{\gamma-1}\ v^{\eta}\ f(u,v,w)\ dw\ dv\ du \tag{1.6} $$

(iv) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta,\ c' = \beta,\ c'' = \gamma$, (1.3) yields

$$ {}_1I^{\alpha,0,0,-\eta;\alpha,\beta,\gamma}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{z}_{1}E^{\alpha,\beta,\gamma,\eta}_{0,x;0,y;0,z}\ f(x,y,z) = $$

$$\frac{x^{-\alpha}\ y^{-\alpha}\ z^{-\alpha-\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}\ (y-v)^{\beta-1}\ (z-w)^{\gamma-1}\ w^{\eta}$$
$$f(u,v,w)\ dw\ dv\ du \tag{1.7}$$

Here (1.5), (1.6) and (1.7) may be regarded as three variable analogue of Erdelyi Kober fractional integral operator.

Under the same conditions of (1.3), a three variable analogues of $J_{x,\infty}^{\alpha,\beta,\gamma}$ is as defined below:

$$_1J_{x,\infty;y,\infty;z,\infty}^{a,b,b',b'';c,c',c''}\ f(x,y,z)\ =$$
$$\frac{1}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{c-1}\ (v-y)^{c'-1}\ (w-z)^{c''-1}$$
$$F_A^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{x}{u},1-\frac{y}{v},1-\frac{z}{w} \\ c,c',c'' \quad ; \end{array}\right] u^{-a}v^{-a}w^{-a}f(u,v,w)dw\ dv\ du \tag{1.8}$$

Special Cases:

(i) For $a=b=b'=b''=0,\ c=\alpha,\ c'=\beta,\ c''=\gamma$, (1.8) reduces to

$$_1J_{x,\infty;y,\infty;z,\infty}^{0,0,0,0;\alpha,\beta,\gamma}\ f(x,y,z)\ =\ _1L_{x,\infty;y,\infty;z,\infty}^{\alpha,\beta,\gamma}\ f(x,y,z) =$$
$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}\ (v-y)^{\beta-1}\ (w-z)^{\gamma-1}$$
$$f(u,v,w)\ dw\ dv\ du \tag{1.9}$$

We may consider (1.9) as a three variable analogue of Weyl fractional integral operator $L_{x,\infty}^{\alpha}$.

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = b'' = 0,\ c' = \beta,\ c'' = \gamma$, (1.8) reduces to

$$
{}_1J^{\alpha,-\eta,0,0;\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ =\ {}^{x}_{1}K^{\alpha,\beta,\gamma,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) =
$$

$$
\frac{x^{\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{\alpha-1}(v-y)^{\beta-1}(w-z)^{\gamma-1}u^{-\alpha-\eta}v^{-\alpha}\ w^{-\alpha}
$$

$$
f(u,v,w)\ dw\ dv\ du \tag{1.10}
$$

(iii) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta,\ c' = \beta,\ c'' = \gamma$, (1.8) becomes

$$
{}_1J^{\alpha,0,-\eta,0;\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ =\ {}^{y}_{1}K^{\alpha,\beta,\gamma,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) =
$$

$$
\frac{y^{\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{\alpha-1}(v-y)^{\beta-1}(w-z)^{\gamma-1}u^{-\alpha}\ v^{-\alpha-\eta}\ w^{-\alpha}
$$

$$
f(u,v,w)\ dw\ dv\ du \tag{1.11}
$$

(iv) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta,\ c' = \beta,\ c'' = \gamma$, (1.8) gives

$$
{}_1J^{\alpha,0,0,-\eta;\alpha,\beta,\gamma}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ =\ {}^{z}_{1}K^{\alpha,\beta,\gamma,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) =
$$

$$
\frac{z^{\eta}}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{\alpha-1}(v-y)^{\beta-1}(w-z)^{\gamma-1}\ u^{-\alpha}\ v^{-\alpha}w^{-\alpha-\eta}
$$

$$
f(u,v,w)\ dw\ dv\ du \tag{1.12}
$$

We may consider (1.10), (1.11) and (1.12) as three variable analogues of Erdelyi-Kober fractional integral operator $K^{\eta,\alpha}_{x,\infty}$.

II. Let $c > 0,\ a,\ a',\ a'',\ b,\ b',\ b''$ be real numbers. Then a second three variable analogue of $I^{\alpha,\beta,\gamma}_{0,x}$ is as follows:

$$
{}_2I_{0,x;0,y;0,z}^{a,a',a'',b,b',b'';c}\ f(x,y,z)
$$

$$
= \frac{x^{-a}y^{-a'}z^{-a''}}{\{\Gamma(c)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{c-1}\ (y-v)^{c-1}\ (z-w)^{c-1}
$$

$$
F_A^{(3)}\left[\begin{array}{l} a,a',a'',b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad\qquad\quad ; \end{array}\right] f(u,v,w)\ dw\ dv\ du \tag{1.13}
$$

where

$$
F_A^{(3)}\left[\begin{array}{l} a,a',a'',b,b',b'';x,y,z \\ c \qquad\qquad\qquad\quad ; \end{array}\right]
$$

$$
= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_n\ (a')_r\ (a'')_s\ (b)_n\ (b')_r\ (b'')_s}{n!\ r!\ s!\ (c)_{n+r+s}}\ x^n y^r z^s
$$

Special Cases:

(i) For $a=a'=a''=0,\ c=\alpha$, (1.13) reduces to

$$
{}_2I_{0,x;0,y;0,z}^{0,0,0;b,b',b'';\alpha}\ f(x,y,z)\ =\ {}_2R_{0,x;0,y;0,z}^{\alpha}\ f(x,y,z)
$$

$$
= \frac{1}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}\ (z-w)^{\alpha-1}\ f(u,v,w)\ dw\ dv\ du \tag{1.14}
$$

Here (1.14) may be regarded as a three variable analogue of Riemann-Liouville fractional integral operator $R_{0,x}^{\alpha}$.

(ii) For $a=c=\alpha,\ a'=a''=0, b=-\eta$, (1.13) becomes

$$
{}_2I_{0,x;0,y;0,z}^{\alpha,0,0;-\eta,b',b'';\alpha}\ f(x,y,z)\ =\ {}_2^{x}E_{0,x;0,y;0,z}^{\alpha,\eta}\ f(x,y,z)
$$

$$= \frac{x^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}\ (z-w)^{\alpha-1}u^{\eta}f(u,v,w)dw\ dv\ du \tag{1.15}$$

(*iii*) For $a = a'' = 0,\ a' = c = \alpha,\ b' = -\eta$, (1.13) gives

$${}_2I_{0,x;0,y;0,z}^{0,\alpha,0,b,-\eta,b'';\alpha}\ f(x,y,z)\ =\ {}_2^{y}E_{0,x;0,y;0,z}^{\alpha,\eta}\ f(x,y,z)$$

$$= \frac{y^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}v^{\eta}f(u,v,w)\ dw\ dv\ du \tag{1.16}$$

(*iv*) For $a = a' = 0,\ a'' = c = \alpha,\ b'' = -\eta$, (1.13) becomes

$${}_2I_{0,x;0,y;0,z}^{0,0,\alpha,b,b',-\eta;\alpha}\ f(x,y,z)\ =\ {}_2^{z}E_{0,x;0,y;0,z}^{\alpha,\eta}\ f(x,y,z)$$

$$= \frac{z^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}\ (z-w)^{\alpha-1}\ w^{\eta}f(u,v,w)dw\ dv\ du \tag{1.17}$$

Here (1.15), (1.16) and (1.17) may be thought of as the second three variable analogues of Erdelyi-Kober fractional integral operator $E_{0,x}^{\alpha,\eta}$.

Under the same conditions of (1.13), a second three variable analogues of $J_{x,\infty}^{\alpha,\beta,\gamma}$ is as defined below:

$${}_2J_{x,\infty;y,\infty;z,\infty}^{a,a',a'',b,b',b'';c} f(x,y,z) = \frac{1}{\{\Gamma(c)\}^3} \int_x^{\infty} \int_y^{\infty} \int_z^{\infty} (u-x)^{c-1}(v-y)^{c-1}(w-z)^{c-1}$$

$$F_A^{(3)} \left[\begin{array}{ll} a, a', a'', b, b', b''; 1-\frac{u}{x}, 1-\frac{v}{y}, 1-\frac{w}{z} \\ c \qquad\qquad\qquad\qquad ; \end{array} \right] u^{-a}\ v^{-a'}\ w^{-a''}$$

$$f(u,v,w)\ dw\ dv\ du \tag{1.18}$$

Special Cases:

(i) For $a = a' = a'' = 0,\ c = \alpha$, (1.18) reduces to

$$ {}_2J^{0,0,0;b,b',b'';\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) \ = \ {}_2L^{\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) $$

$$ = \frac{1}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}\,(v-y)^{\alpha-1}\,(w-z)^{\alpha-1}\, f(u,v,w)\, dw\, dv\, du \tag{1.19} $$

It can be considered as a three variable analogue of Weyl fractional integral operator $L^{\alpha}_{x,\infty}$.

(ii) For $a' = a'' = 0,\ a = c = \alpha,\ b = -\eta$, (1.18) becomes

$$ {}_2J^{\alpha,0,0,-\eta;b',b'';\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) \ = \ {}^{x}_{2}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = $$

$$ \frac{x^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1} u^{-\alpha-\eta} f(u,v,w) dw dv du \tag{1.20} $$

(iii) For $a = a'' = 0,\ a' = c = \alpha,\ b' = -\eta$, (1.18) gives

$$ {}_2J^{0,\alpha,0,b,-\eta,b'';\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) \ = \ {}^{y}_{2}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = $$

$$ \frac{y^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1} v^{-\alpha-\eta} f(u,v,w) dw dv du \tag{1.21} $$

(iv) For $a = a' = 0,\ a'' = c = \alpha,\ b'' = -\eta$, (1.18) yields

$$ {}_2J^{0,0,\alpha,b,b',-\eta;\alpha}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = {}^{z}_{2}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty} f(x,y,z) = $$

$$ \frac{z^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1} w^{-\alpha-\eta} f(u,v,w) dw dv du \tag{1.22} $$

Here (1.20), (1.21) and (1.22) may be taken of as the second three variable analogues of Erdelyi-Kober fractional integral operator $K^{\eta,\alpha}_{x,\infty}$.

III. Let $c > 0$, $c' > 0$, $c'' > 0$, a, b be real numbers. Then a third three variable analogue of $I^{\alpha,\beta,\eta}_{0,x}$ is as defined below:

$$
{}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z} f(x,y,z) =
$$

$$
\frac{x^{-a}y^{-a'}z^{-a''}}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int\limits_0^x \int\limits_0^y \int\limits_0^z (x-u)^{c-1}(y-v)^{c'-1}(z-w)^{c''-1}
$$

$$
F^{(3)}_C \left[\begin{array}{l} a, b \quad ; 1-\frac{u}{x}, 1-\frac{v}{y}, 1-\frac{w}{z} \\ c, c', c''; \end{array} \right] f(u,v,w)\, dw\, dv\, du \quad (1.23)
$$

where

$$
F^{(3)}_A \left[\begin{array}{l} a, b \quad ; x, y, z \\ c, c', c''; \end{array} \right] = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{n+r+s}\ (b)_{n+r+s}}{n!\ r!\ s!\ (c)_n (c')_r (c'')_s}\ x^n y^r z^s
$$

Under the same conditions of (1.23), a third three variable analogue of $J^{\alpha,\beta,\eta}_{n,\infty}$ is as given below:

$$
{}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) =
$$

$$
\frac{1}{\Gamma(c)\Gamma(c')\Gamma(c'')} \int\limits_x^\infty \int\limits_y^\infty \int\limits_z^\infty (u-x)^{c-1}\ (v-y)^{c'-1}\ (w-z)^{c''-1}
$$

$$
F^{(3)}_C \left[\begin{array}{l} a, b \quad ; 1-\frac{x}{u}, 1-\frac{y}{v}, 1-\frac{z}{w} \\ c, c', c''; \end{array} \right] f(u,v,w)\, dw\, dv\, du \quad (1.24)
$$

IV. Let $c > 0$, a, b, b', b'' be real numbers. A fourth three variable analogue of fractional integral operator $I^{\alpha,\beta,\eta}_{0,x}$ due to M. Saigo is defined as:

$$ {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\, f(x,y,z) = \frac{x^{-a}y^{-a}z^{-a}}{\{\Gamma(c)\}^3} \int_0^x\int_0^y\int_0^z (x-u)^{c-1}(y-v)^{c-1}(z-w)^{c-1} $$

$$ F_D^{(3)}\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right] f(u,v,w)\, dw\, dv\, du \quad (1.25) $$

where

$$ F_D^{(3)}\left[\begin{array}{l} a,b \qquad ;x,y,z \\ c,c',c''; \end{array}\right] = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(a)_{n+r+s}\,(b)_n(b')_r(b'')_s}{n!\;r!\;s!\;(c)_{n+r+s}}\; x^n y^r z^s $$

Special Cases:

(i) For $a=b=b'=b''=0,\ c=\alpha$, (1.25) reduces to

$$ {}_4I^{0,0,0,0;\alpha}_{0,x;0,y;0,z}\, f(x,y,z) \;=\; {}_2R^{\alpha}_{0,x;0,y;0,z}\, f(x,y,z) = $$

$$ \frac{1}{\{\Gamma(\alpha)\}^3} \int_0^x\int_0^y\int_0^z (x-u)^{\alpha-1}\,(y-v)^{\alpha-1}\,(z-w)^{\alpha-1}\, f(u,v,w)\, dw\, dv\, du \quad (1.26) $$

Which is (1.14) i.e. a three variable analogue of Riemann-Liouville fractional integral operator $R^{\alpha}_{0,x}$.

(ii) For $a=c=\alpha,\ b=-\eta,\ b'=b''=0$, (1.25) becomes

$$ {}_4I^{\alpha,-\eta,0,0;\alpha}_{0,x;0,y;0,z}\, f(x,y,z) \;=\; {}^{x}_{3}E^{\alpha,\eta}_{0,x;0,y;0,z}\, f(x,y,z) = $$

$$ \frac{x^{-\alpha-\eta}y^{-\alpha}z^{-\alpha}}{\{\Gamma(\alpha)\}^3} \int_0^x\int_0^y\int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}u^{\eta} f(u,v,w)dwdvdu \quad (1.27) $$

(*iii*) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta$, (1.25) gives

$$ {}_4I^{\alpha,0,-\eta,0;\alpha}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{y}_{3}E^{\alpha,\eta}_{0,x;0,y;0,z}\ f(x,y,z) = $$

$$ \frac{x^{-\alpha}y^{-\alpha-\eta}z^{-\alpha}}{\{\Gamma(\alpha)\}^3}\int\limits_0^x\int\limits_0^y\int\limits_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}v^{\eta}f(u,v,w)dwdvdu \tag{1.28} $$

(*iv*) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta$, (1.25) yields

$$ {}_4I^{\alpha,0,0,-\eta;\alpha}_{0,x;0,y;0,z}\ f(x,y,z) \ = \ {}^{z}_{3}E^{\alpha,\eta}_{0,x;0,y;0,z}\ f(x,y,z) = $$

$$ \frac{x^{-\alpha}y^{-\alpha}z^{-\alpha-\eta}}{\{\Gamma(\alpha)\}^3}\int\limits_0^x\int\limits_0^y\int\limits_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1}w^{\eta}f(u,v,w)dwdvdu \tag{1.29} $$

Here (1.27), (1.28) and (1.29) may be considered as third three variable analogues of Erdelyi-Kober fractional integral operator $E^{\alpha,\eta}_{0,x}$.

It may be remarked here that (1.27), (1.28) and (1.29) can also be obtained from (1.5), (1.6) and (1.7) respectively by taking $\alpha = \beta = \gamma$.

Under the same condition of (1.25), a fourth three variable analogue of another fractional integral operator $J^{\alpha,\beta,\gamma}_{x,\infty}$ due to M. Saigo is defined as follows:

$$ {}_4I^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = \frac{1}{\{\Gamma(c)\}^3}\int\limits_x^\infty\int\limits_y^\infty\int\limits_z^\infty (u-x)^{c-1}(v-y)^{c-1}(w-z)^{c-1} $$

$$ F^{(3)}_D\left[\begin{array}{l} a,b,b',b'';1-\frac{u}{x},1-\frac{v}{y},1-\frac{w}{z} \\ c \qquad\qquad ; \end{array}\right] u^{-a}v^{-a}w^{-a}f(u,v,w)\ dw\ dv\ du \tag{1.30} $$

Special Cases:

(i) For $a = b = b' = b'' = 0,\ c = \alpha$, (1.30) reduces to

$$ {}_4I^{0,0,0,0;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ =\ {}_2L^{\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) $$

$$ = \frac{1}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}\,(v-y)^{\alpha-1}\,(w-z)^{\alpha-1}\,f(u,v,w)\,dw\,dv\,du \tag{1.31} $$

Which is (1.19) i.e. a second three variable analogue of Weyl fractional integral operator $L^{\alpha}_{x,\infty}$. It can be obtained from (1.9) by taking $\alpha = \beta = \gamma$.

(ii) For $a = c = \alpha,\ b = -\eta,\ b' = b'' = 0$, (1.30) becomes

$$ {}_4I^{\alpha,-\eta,0,0;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ =\ {}^{x}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = $$

$$ \frac{x^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1}u^{-\alpha-\eta}v^{-\alpha}w^{-\alpha} f(u,v,w)\ dw\ dv\ du \tag{1.32} $$

(iii) For $a = c = \alpha,\ b = b'' = 0,\ b' = -\eta$, (1.30) gives

$$ {}_4I^{\alpha,0,-\eta,0;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ =\ {}^{y}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = $$

$$ \frac{y^\eta}{\{\Gamma(\alpha)\}^3} \int_x^\infty \int_y^\infty \int_z^\infty (u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1}u^{-\alpha}v^{-\alpha-\eta}w^{-\alpha} f(u,v,w)\ dw\ dv\ du \tag{1.33} $$

(iv) For $a = c = \alpha,\ b = b' = 0,\ b'' = -\eta$, (1.30) yields

$$ {}_4I^{\alpha,0,0,-\eta;\alpha}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ =\ {}^{z}_{3}K^{\alpha,\eta}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z) = $$

$$\frac{z^{\eta}}{\{\Gamma(\alpha)\}^3}\int_x^{\infty}\int_y^{\infty}\int_z^{\infty}(u-x)^{\alpha-1}(v-y)^{\alpha-1}(w-z)^{\alpha-1}u^{-\alpha}v^{-\alpha}w^{-\alpha-\eta}$$
$$f(u,v,w)\,dw\,dv\,du \tag{1.34}$$

Here (1.32), (1.33) and (1.34) may be considered as third three variable analogues of Erdelyi-Kober fractional integral operator $K^{\alpha,\eta}_{x,\infty}$.

Further (1.32), (1.33) and (1.34) can also be obtained from (1.10), (1.11) and (1.12) respectively by taking $\alpha=\beta=\gamma$.

The aim of the present paper is to study the effects of integral transforms say the Mellin and Laplace transforms on the three variable analogues of fractional integral operators introduced in [11] and reproduced here through (1.3), (1.8), (1.13), (1.18), (1.23), (1.24), (1.25)and (1.30).

Theorem 1.1 : For functions $f(x,y,z)$, $g(x,y,z)$, $f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ and $g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ defined for $0\le x<\infty$, $0\le y<\infty$, $0\le z<\infty$ and $c>0$, $c'>0$, $c''>0$, we have

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}x^{a-c-1}y^{a-c'-1}z^{a-c''-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}$$
$$g(x,y,z)dzdydx$$

$$=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}x^{a-c-1}y^{a-c'-1}z^{a-c''-1}g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}$$
$$f(x,y,z)dzdydx \tag{1.38}$$

provided that each triple integral exists.

Theorem 1.2 : Under the conditions stated in theorem 1.1, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}$$

$$g(x,y,z)dzdydx$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a-c'-1}z^{a-c''-1}g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}$$

$$f(x,y,z)dzdydx \tag{1.39}$$

provided that each triple integral exists.

Theorem 1.3 : For functions $f(x,y,z)$, $g(x,y,z)$, $f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ and $g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ defined for $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$ and $c > 0$, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a'-c-1}z^{a''-c-1}f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}$$

$$g(x,y,z)dzdydx$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty x^{a-c-1}y^{a'-c-1}z^{a''-c-1}g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}$$

$$f(x,y,z)dzdydx \tag{1.40}$$

provided that each triple integral exists.

Theorem 1.4 : Under the conditions stated in theorem 1.3, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} z^{a''-c-1} f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}$$

$$g(x,y,z)dzdydx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty x^{a-c-1} y^{a'-c-1} z^{a''-c-1} g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}$$

$$f(x,y,z)dzdydx \tag{1.41}$$

provided that each triple integral exists.

Theorem 1.5 : For functions $f(x,y,z)$, $g(x,y,z)$, $f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ and $g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ defined for $0 \le x < \infty$, $0 \le y < \infty$, $0 \le z < \infty$ and $c > 0$, $c' > 0$, $c'' > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} z^{a-c''-1} f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}\ g(x,y,z)dzdydx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} z^{a-c''-1} g\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z} f(x,y,z)dzdydx \tag{1.42}$$

provided that each triple integral exists.

Theorem 1.6 : Under the conditions stated in theorem 1.5, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} z^{a-c''-1} f\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) {}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}$$

$$g(x,y,z)dzdydx$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty x^{a-c-1} y^{a-c'-1} z^{a-c''-1} g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}$$
$$f(x,y,z)dzdydx \tag{1.43}$$

provided that each triple integral exists.

Theorem 1.7 : For functions $f(x,y,z)$, $g(x,y,z)$, $f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ and $g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ defined for $0 \leq x < \infty$, $0 \leq y < \infty$, $0 \leq z < \infty$ and $c > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty (xyz)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\, g(x,y,z)dzdydx$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty (xyz)^{a-c-1} g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\, f(x,y,z)dzdydx \tag{1.44}$$

provided that each triple integral exists.

Theorem 1.8 : Under the conditions stated in theorem 1.7, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty (xyz)^{a-c-1} f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\, g(x,y,z)dzdydx$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty (xyz)^{a-c-1} g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right) {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\, f(x,y,z)dzdydx \tag{1.45}$$

provided that each triple integral exists.

Theorem 1.9 : For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0$,

$c' > 0$, $c'' > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_1I_{0,x;0,y;0,z}^{a,b,b',b'';c,c',c''}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_1J_{x,\infty;y,\infty;z,\infty}^{a,b,b',b'';c,c',c''}\ f(x,y,z)\ dz\ dy\ dx \qquad (1.46)$$

provided that each triple integral exists.

Theorem 1.10 : For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_2I_{0,x;0,y;0,z}^{a,a',a'',b,b',b'';c}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_2J_{x,\infty;y,\infty;z,\infty}^{a,a',a'',b,b',b'';c}\ f(x,y,z)\ dz\ dy\ dx \qquad (1.47)$$

provided that each triple integral exists.

Theorem 1.11 : For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0$ $c' > 0$, $c'' > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_3I_{0,x;0,y;0,z}^{a,b;c,c',c''}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_3J_{x,\infty;y,\infty;z,\infty}^{a,b;c,c',c''}\ f(x,y,z)\ dz\ dy\ dx \qquad (1.48)$$

provided that each triple integral exists.

Theorem 1.12 : For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of the three dimensional space and $c > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\ {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ g(x,y,z)\ dz\ dy\ dx$$

$$= \int_0^\infty \int_0^\infty \int_0^\infty g(x,y,z)\ {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z)\ dz\ dy\ dx \tag{1.49}$$

provided that each triple integral exists.

2 Mellin Transformation

In this section we shall study the effect of operating three variable analogue of Mellin transform on the above defined operators. A three variable analogue of Mellin transform of a function $f(x,y,z)$ of three variables $x,\ y$ and z is defined as follows:

$$M\{f(u,v,w) : r,s,t\} = \int_0^\infty \int_0^\infty \int_0^\infty u^{r-1} v^{s-1} w^{s-1} f(u,v,w) dw dv du \tag{2.1}$$

The effects of operating (2.1) on the operators (1.3), (1.8), (1.13), (1.18), (1.23), (1.24), (1.25) and (1.30) are given in the form of the following theorems:

Theorem 2.1 : For $c > 0,\ c' > 0,\ c'' > 0,\ Rl(r) > 0,\ Rl(s) > 0, Rl(t) > 0$, we have

$$M\left\{ {}_1J^{a,b,b',b'';c_1,c,c',c''}_{x,\infty;y,\infty;z,\infty} f(x,y,z) : r,s,t \right\}$$

$$= \frac{\Gamma(r)\Gamma(s)\Gamma(t)}{\Gamma(r+c)\Gamma(s+c')\Gamma(t+c'')}$$

$$\times F^{(3)}\left[\begin{array}{l} a :: -; -; - : b; b'; b''; 1, 1, 1 \\ - :: -; -; - : c + r; c' + s; c'' + t; \end{array}\right]$$

$$\times M\left\{x^{c-a} y^{c'-a} z^{c''-a} f(x, y, z) : r, s, t\right\} \tag{2.2}$$

provided that term by term integration is valid and $F^{(3)}[x, y, z]$ is given by (1.35).

Theorem 2.2 : For $c > 0$, $Rl(r) > 0$, $Rl(s) > 0$, $Rl(t) > 0$, we have

$$M\left\{{}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty} f(x, y, z) : r, s, t\right\}$$

$$= \frac{\Gamma(r)\Gamma(s)\Gamma(t)}{\Gamma(r + c)\Gamma(s + c)\Gamma(t + c)}$$

$$\times F^{(3)}\left[\begin{array}{l} - :: -; -; - : a, b, c; a', b', c; a'', b'', c; 1, 1, 1 \\ c :: -; -; - : r + c; s + c; t + c \qquad ; \end{array}\right]$$

$$\times M\left\{x^{c-a} y^{c-a'} z^{c-a''} f(x, y, z) : r, s, t\right\} \tag{2.3}$$

provided that term by term integration is valid and $F^{(3)}[x, y, z]$ is given by (1.35).

Theorem 2.3 : For $c > 0,\ c' > 0, c'' > 0,\ Rl(r) > 0,\ Rl(s) > 0, Rl(t) > 0$, we have

$$M\left\{{}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty} f(x, y, z) : r, s, t\right\}$$

$$= \frac{\Gamma(r)\Gamma(s)\Gamma(t)}{\Gamma(r+c)\Gamma(s+c')\Gamma(t+c'')}$$

$$\times F^{(3)} \left[\begin{array}{l} a, b :: -; -; - : -; -; -; 1, 1, 1 \\ - :: -; -; - : r+c; s+c'; t+c''; \end{array} \right]$$

$$\times M \left\{ x^{c-a} y^{c'-a} z^{c''-a} f(x,y,z) : r, s, t \right\} \qquad (2.4)$$

provided that term by term integration is valid and $F^{(3)}[x,y,z]$ is given by (1.35).

Theorem 2.4 : For $c > 0$, $Rl(r) > 0$, $Rl(s) > 0$, $Rl(t) > 0$, we have

$$M \left\{ {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty} f(x,y,z) : r, s, t \right\}$$

$$= \frac{\Gamma(r)\Gamma(s)\Gamma(t)}{\Gamma(r+c)\Gamma(s+c)\Gamma(t+c)}$$

$$\times F^{(3)} \left[\begin{array}{l} a :: -; -; - : b, c; b', c; b'', c; 1, 1, 1 \\ c :: -; -; - : r+c; s+c; t+c \qquad ; \end{array} \right]$$

$$\times M \left\{ (xyz)^{-} a f(x,y,z) : r, s, t \right\} \qquad (2.5)$$

provided that term by term integration is valid and $F^{(3)}[x,y,z]$ is given by (1.35).

Theorem 2.5 : For a function of three variables $f(x,y,z)$ defined in the positive octant of three dimensional xyz-space and $c>0,\ c'>0, c''>0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\,{}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z} \ \left\{x^{r-1}, y^{s-1}, z^{t-1}\right\} dzdydx$$

$$= M\left\{{}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty} f(x,y,z) : r,s,t\right\} \tag{2.6}$$

provided that the triple integrals involved exist.

Theorem 2.6 : Under the conditions stated in theorem 2.5, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\,{}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty} \ \left\{x^{r-1}, y^{s-1}, z^{t-1}\right\} dzdydx$$

$$= M\left\{{}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z} f(x,y,z) : r,s,t\right\} \tag{2.7}$$

provided that the triple integrals involved exist.

Theorem 2.7 : For a function of three variables $f(x,y,z)$ defined in the positive octant of three dimensional xyz-space and $c > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\,{}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z} \ \left\{x^{r-1}, y^{s-1}, z^{t-1}\right\} dzdydx$$

$$= M\left\{{}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty} f(x,y,z) : r,s,t\right\} \tag{2.8}$$

provided that the triple integrals involved exist.

Theorem 2.8 : Under the conditions stated in theorem 2.7, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\,{}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty} \ \left\{x^{r-1}, y^{s-1}, z^{t-1}\right\} dzdydx$$

$$= M\left\{ {}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z} f(x,y,z) : r,s,t \right\} \tag{2.9}$$

provided that the triple integrals involved exist.

Theorem 2.9 : Under the conditions stated in theorem 2.5, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\, {}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z} \left\{ x^{r-1}, y^{s-1}, z^{t-1} \right\} dzdydx$$

$$= M\left\{ {}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty} f(x,y,z) : r,s,t \right\} \tag{2.10}$$

provided that the triple integrals involved exist.

Theorem 2.10 : Under the conditions stated in theorem 2.5, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\, {}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty} \left\{ x^{r-1}, y^{s-1}, z^{t-1} \right\} dzdydx$$

$$= M\left\{ {}_3J^{a,b;c,c',c''}_{0,x;0,y;0,z} f(x,y,z) : r,s,t \right\} \tag{2.11}$$

provided that the triple integrals involved exist.

Theorem 2.11 : Under the conditions stated in theorem 2.7, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\, {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z} \left\{ x^{r-1}, y^{s-1}, z^{t-1} \right\} dzdydx$$

$$= M\left\{ {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty} f(x,y,z) : r,s,t \right\} \tag{2.12}$$

provided that the triple integrals involved exist.

Theorem 2.12 : Under the conditions stated in theorem 2.7, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)_4 J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty} \left\{x^{r-1}, y^{s-1}, z^{t-1}\right\} dzdydx$$

$$= M\left\{ {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z} f(x,y,z) : r,s,t \right\} \tag{2.13}$$

provided that the triple integrals involved exist.

Theorem 2.13: For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of three dimensional xyz-space and $c > 0$, $c' > 0, c'' > 0$, we have

$$M\left[f(x,y,z)_1 I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z} \left\{x^{r-1}, y^{s-1}, z^{t-1}\ g(x,y,z)\right\} : r,s,t \right]$$

$$= M\left[g(x,y,z)_1 J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty} \left\{x^{r-1}, y^{s-1}, z^{t-1}\ f(x,y,z)\right\} : r,s,t \right] \tag{2.14}$$

Theorem 2.14: For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of three dimensional xyz-space and $c > 0$, we have

$$M\left[f(x,y,z)_2 I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z} \left\{x^{r-1}, y^{s-1}, z^{t-1}\ g(x,y,z)\right\} : r,s,t \right]$$

$$= M\left[g(x,y,z)_2 J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty} \left\{x^{r-1}, y^{s-1}, z^{t-1}\ f(x,y,z)\right\} : r,s,t \right] \tag{2.15}$$

Theorem 2.15: Under the conditions stated in theorem 2.13, we have

$$M\left[f(x,y,z)_3 I^{a,b;c,c',c''}_{0,x;0,y;0,z} \left\{x^{r-1}, y^{s-1}, z^{t-1}\ g(x,y,z)\right\} : r,s,t \right]$$

$$= M\left[g(x,y,z)_3 J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty} \left\{x^{r-1}, y^{s-1}, z^{t-1}\ f(x,y,z)\right\} : r,s,t \right] \tag{2.16}$$

Theorem 2.16: Under the conditions stated in theorem 2.14, we have

$$M\left[f(x,y,z){}_3I^{a,b,b',b'';c}_{0,x;0,y;0,z}\left\{x^{r-1},y^{s-1},z^{t-1}\ g(x,y,z)\right\}:r,s,t\right]$$
$$=M\left[g(x,y,z){}_3J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\left\{x^{r-1},y^{s-1},z^{t-1}\ f(x,y,z)\right\}:r,s,t\right] \tag{2.18}$$

It is interesting to note that in terms of triple Mellin transforms the results (1.38), (1.39), (1.40), (1.41), (1.42), (1.43), (1.44), and (1.45) can respectively be written as

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}\ g(x,y,z):a-c,a-c',a-c''\right]$$
$$=M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}\ f(x,y,z):a-c,a-c',a-c''\right] \tag{2.18}$$

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}\ g(x,y,z):a-c,a-c',a-c''\right]$$
$$=M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z):a-c,a-c',a-c''\right] \tag{2.19}$$

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}\ g(x,y,z):a-c,a'-c,a''-c\right]$$
$$=M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z}\ f(x,y,z):a-c,a'-c,a''-c\right] \tag{2.20}$$

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ g(x,y,z):a-c,a'-c,a''-c\right]$$

$$= M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_1J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z):a-c,a'-c,a''-c\right] \tag{2.21}$$

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}\ g(x,y,z):a-c,a-c',a-c''\right]$$

$$= M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_3I^{a,b;c,c',c''}_{0,x;0,y;0,z}\ f(x,y,z):a-c,a-c',a-c''\right] \tag{2.22}$$

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\ g(x,y,z):a-c,a-c',a-c''\right]$$

$$= M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_3J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z):a-c,a-c',a-c''\right] \tag{2.23}$$

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ g(x,y,z):a-c,a-c,a-c\right]$$

$$= M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_2I^{a,b,b',b'';c}_{0,x;0,y;0,z}\ f(x,y,z):a-c,a-c,a-c\right] \tag{2.24}$$

$$M\left[f\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ g(x,y,z):a-c,a-c,a-c\right]$$

$$= M\left[g\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right){}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\ f(x,y,z):a-c,a-c,a-c\right] \tag{2.25}$$

3 Laplace Transformation

The triple Laplace transform of a function of three variables $f(x, y, z)$ defined in the positive octant of the three dimensional xyz-space is defined by the equation

$$L\{f(x,y,z):r,s,t\} = \int_0^\infty \int_0^\infty \int_0^\infty e^{-rx-sy-tz}\ f(x,y,z)dzdydx \quad (3.1)$$

Making use of results of theorems (1.9), (1.10), (1.11), and (1.12), the relationships of (3.1) with the operators (1.3), (1.8), (1.13), (1.18), (1.23), (1.24), (1.25), and (1.30) are given in the form of the following theorems:

Theorem 3.1: For a function of three variables $f(x, y, z)$ defined in the positive octant of three dimensional xyz-space and $c > 0,\ c' > 0, c'' > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\, {}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z}\ \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty} f(x,y,z):r,s,t\right] \quad (3.2)$$

provided that the triple integrals involved exist.

Theorem 3.2: Under the conditions stated in theorem 3.1, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\, {}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty}\ \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z} f(x,y,z):r,s,t\right] \quad (3.3)$$

provided that the triple integrals involved exist.

Theorem 3.3: For a function of three variables $f(x, y, z)$ defined in the positive octant of three dimensional xyz-space and $c > 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) {}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z} \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty} f(x, y, z) : r, s, t\right] \tag{3.4}$$

provided that the triple integrals involved exist.

Theorem 3.4: Under the conditions stated in theorem 3.3, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) {}_2J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty} \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_2I^{a,a',a'',b,b',b'';c}_{0,x;0,y;0,z} f(x, y, z) : r, s, t\right] \tag{3.5}$$

provided that the triple integrals involved exist.

Theorem 3.5: Under the conditions stated in theorem 3.1, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) {}_3I^{a,b,;c,c',c''}_{0,x;0,y;0,z} \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_3J^{a,b,;c,c',c''}_{x,\infty;y,\infty;z,\infty} f(x, y, z) : r, s, t\right] \tag{3.6}$$

provided that the triple integrals involved exist.

Theorem 3.6: Under the conditions stated in theorem 3.1, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, z) {}_3J^{a,b,;c,c',c''}_{x,\infty;y,\infty;z,\infty} \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_3I^{a,b,;c,c',c''}_{0,x;0,y;0,z}\, f(x,y,z) : r,s,t\right] \tag{3.7}$$

provided that the triple integrals involved exist.

Theorem 3.7: Under the conditions stated in theorem 3.3, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\, {}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z} \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty} f(x,y,z) : r,s,t\right] \tag{3.8}$$

provided that the triple integrals involved exist.

Theorem 3.8: Under the conditions stated in theorem 3.3, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty f(x,y,z)\, {}_4J^{a,b,b',b'';c}_{x,\infty;y,\infty;z,\infty} \left[e^{rx-sy-tz}\right] dzdydx$$

$$= L\left[{}_4I^{a,b,b',b'';c}_{0,x;0,y;0,z} f(x,y,z) : r,s,t\right] \tag{3.9}$$

provided that the triple integrals involved exist.

We further give relationships among triple Laplace transform, triple Mellin transform and the operators (1.3), (1.8), (1.13), (1.18), (1.23), (1.24), (1.25), and (1.30) in the form of the following theorems:

Theorem 3.9: For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of three dimensional xyz-space and $c > 0$, $c' > 0, c'' > 0$, we have

$$M\left[f(x,y,z)\, {}_1I^{a,b,b',b'';c,c',c''}_{0,x;0,y;0,z} \left\{e^{rx-sy-tz}\ g(x,y,z)\right\} : r,s,t\right]$$

$$= L\left[g(x,y,z)\, {}_1J^{a,b,b',b'';c,c',c''}_{x,\infty;y,\infty;z,\infty} \left\{x^{r-1}, y^{s-1}, z^{t-1}\ f(x,y,z)\right\} : r,s,t\right] \tag{3.10}$$

Theorem 3.10: For functions of three variables $f(x,y,z)$ and $g(x,y,z)$ defined in the positive octant of three dimensional xyz-space and $c > 0$, we have

$$M\left[f(x,y,z)_2I_{0,x;0,y;0,z}^{a,a',a'',b,b',b'';c}\left\{e^{-rx-sy-tz}\ g(x,y,z)\right\}:r,s,t\right]$$

$$=L\left[g(x,y,z)_2J_{x,\infty;y,\infty;z,\infty}^{a,a',a'',b,b',b'';c}\left\{x^{r-1},y^{s-1},z^{t-1}\ f(x,y,z)\right\}:r,s,t\right] \tag{3.11}$$

Theorem 3.11: Under the conditions stated in theorem 3.9, we have

$$M\left[f(x,y,z)_3I_{0,x;0,y;0,z}^{a,b;c,c',c''}\left\{e^{-rx-sy-tz}\ g(x,y,z)\right\}:r,s,t\right]$$

$$=L\left[g(x,y,z)_3J_{x,\infty;y,\infty;z,\infty}^{a,b;c,c',c''}\left\{x^{r-1},y^{s-1},z^{t-1}\ f(x,y,z)\right\}:r,s,t\right] \tag{3.12}$$

Theorem 3.12: Under the conditions stated in theorem 3.10, we have

$$M\left[f(x,y,z)_4I_{0,x;0,y;0,z}^{a,b,b',b'';c}\left\{e^{-rx-sy-tz}\ g(x,y,z)\right\}:r,s,t\right]$$

$$=L\left[g(x,y,z)_3J_{x,\infty;y,\infty;z,\infty}^{a,b;c,c',c''}\left\{x^{r-1},y^{s-1},z^{t-1}\ f(x,y,z)\right\}:r,s,t\right] \tag{3.13}$$

Theorem 3.13: Under the conditions stated in theorem 3.9, we have

$$L\left[f(x,y,z)_1I_{0,x;0,y;0,z}^{a,b,b',b'';c,c',c''}\left\{e^{-rx-sy-tz}\ g(x,y,z)\right\}:r,s,t\right]$$

$$=L\left[g(x,y,z)_1I_{0,x;0,y;0,z}^{a,b,b',b'';c,c',c''}\left\{e^{-rx-sy-tz}\ f(x,y,z)\right\}:r,s,t\right] \tag{3.14}$$

Theorem 3.14: Under the conditions stated in theorem 3.10, we have

$$L\left[f(x,y,z)_2I_{0,x;0,y;0,z}^{a,a',a'',b,b',b'';c}\left\{e^{-rx-sy-tz}\ g(x,y,z)\right\}:r,s,t\right]$$

$$= L\left[g(x,y,z)_2 J^{a,a',a'',b,b',b'';c}_{x,\infty;y,\infty;z,\infty}\left\{e^{-rx-sy-tz}\ f(x,y,z)\right\} : r,s,t\right] \quad (3.15)$$

Theorem 3.15: Under the conditions stated in theorem 3.9, we have

$$L\left[f(x,y,z)_3 I^{a,b;c,c',c''}_{0,x;0,y;0,z}\left\{e^{-rx-sy-tz}\ g(x,y,z)\right\} : r,s,t\right]$$

$$= L\left[g(x,y,z)_3 J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\left\{e^{-rx-sy-tz}\ f(x,y,z)\right\} : r,s,t\right] \quad (3.16)$$

Theorem 3.16: Under the conditions stated in theorem 3.10, we have

$$L\left[f(x,y,z)_4 I^{a,b,b',b'';c}_{0,x;0,y;0,z}\left\{e^{-rx-sy-tz}\ g(x,y,z)\right\} : r,s,t\right]$$

$$= L\left[g(x,y,z)_4 J^{a,b;c,c',c''}_{x,\infty;y,\infty;z,\infty}\left\{e^{-rx-sy-tz}\ f(x,y,z)\right\} : r,s,t\right] \quad (3.17)$$

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Resumen

Este trabajo es una continuación del artículo de los autores [11] donde se analizaron los análogos en tres variables de ciertos operadores de integración fraccionaria de M. Saigo.

El presente trabajo trata sobre el efecto de operar con los análogos en tres variables de las transformadas de Mellin y de Laplace en los análogos en tres variables de los operadores de integración fraccionaria del trabajo citado.

Palabras Clave: Análogos en tres variables de operadores de integración fraccionaria de M. Saigo, Transformada de Laplace en tres variables, Transformada de Mellin en tres variables.

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