CHARACTERIZATIONS OF HARDLY e-OPEN FUNCTIONS

Miguel Caldas[1]

April, 2011

Abstract

A function is defined to be hardly e-open provided that the inverse image of each e-dense subset of the codomain that is contained in a proper open set is e-dense in the domain. Characterizations and properties of hardly e-open functions are presented.

MSC(2010): 54A40.

Keywords: *Topological spaces, e-open sets, e-dense sets, hardly e-open functions.*

1. *Departamento de Matemática Aplicada, UFF, Brasil.*

1. Characterizations of Hardly e-open Functions

Recent progress in the study of characterizations and generalizations of closedness has been done by means of several generalized closed sets. The first step of generalizing closed set was done by Levine in 1970 [15]. The notion of generalized closed sets has been studied extensively in recent years by many topologist because generalized closed sets are the only natural generalization of closed sets. More importantly, they also suggest several new properties of topological spaces. As a generalization of closed sets, e-closed sets were introduced and studied by E. Ekici ([10], [11], [12], [3], [5]). In this paper hardly e-open function is introduced and get results which similar the results for hardly open functions [2].

Throughout the paper (X, τ) and (Y, σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$, $int(A)$ and $X \backslash A$ denote the closure of A, the interior of A and the complement of A in X, respectively. A subset A of a space (X, τ) is called semi-open [14] (resp. α-open [17], preopen [16], β-open [1], γ-open [13], δ-preopen [18]) if $A \subset cl(Int(A))$ (resp. $A \subset int(cl(int(A)))$, $A \subset int(cl(A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(int(A)) \cup int(cl(A))$, $A \subset int(\delta cl(A))$). A point $x \in X$ is called a δ-cluster point of A [19] if $int(cl(V)) \cap A \neq \emptyset$ for every open set V of X containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta cl(A)$. If $A = \delta cl(A))$, then A is said to be δ-closed. The complement of a δ-closed set is said to be δ-open. The union of all δ-open sets contained in a subset A is called the δ-interior of A and is denoted by $\delta int(A)$.

A subset A of a topological space X is said to be e-open [7] if $A \subset int(\delta cl(A)) \cup cl(\delta int(A))$. The complement of an e-open set is said to be e-closed [7]. The intersection (union) of all e-closed (e-open) sets containing (contained in) A in X is called the e-closure [7] (e-interior

[7]) of A and is denoted by $cl_e(A)$ (resp. $int_e(A)$).

Lemma 1.1. *([7], [9], [8]). The following properties holds for the e-closure of a set in a space X:*

(1) Arbitrary union (intersection) of e-open (e-closed) sets in X is e-open (resp. e-closed).

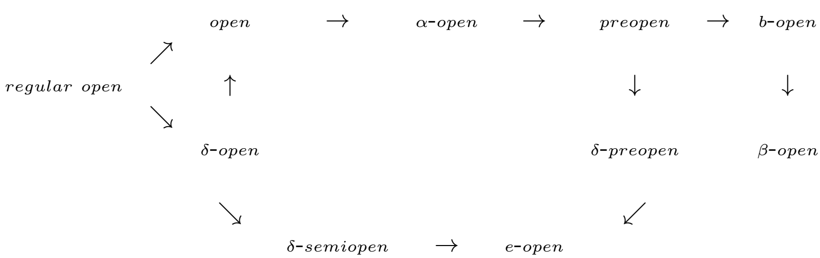
(2) A is e-closed in X if and only if $A = cl_e(A)$.

(3) $cl_e(A) \subset cl_e(B)$ whenever $A \subset B(\subset X)$.

(4) $cl_e(A)$ is e-closed in X.

(5) $cl_e(A) = \{x \in X \mid U \cap A \neq \emptyset$ for every e-open set U containing $x\}$.

We have the following diagram in which the converses of implications need not be true, as is showed in ([7], [9], [8]).



Recall that, a subset E of a topological space (X, τ) is said to be e-dense in X ([8], [4]) if $cl_e(E) = X$, equivalently if there is no proper e-closed set C in X such that $E \subset C \subset X$.

Definition 1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be hardly e-open provided that for each e-dense subset A of Y that is contained in a proper open set, $f^{-1}(A)$ is e-dense in X.*

For T_1 spaces every proper set is contained in a proper open set. Therefore we have the following result.

Theorem 1.2. *If Y is an T_1 space, then a function $f : (X, \tau) \to (Y, \sigma)$ is hardly e-open if and only if for each e-dense subset A of Y, $f^{-1}(A)$ is e-dense in X.*

Theorem 1.2 can be strengthened by replacing the T_1 requirement with the following weaker condition.

A subset A of a topological space (X, τ) is called a generalized closed set (briefly g-closed set) [15] if $Cl(A) \subset U$ whenever $A \subset U$ and U is open in (X, τ). A space X is called a $T_{1/2}$ if every g-closed set is closed.

The following result due to Dunham [6] provides a useful characterizations for $T_{1/2}$-spaces.

Theorem 1.3. *For a topological space (X, τ) the following conditions are equivalent:*

(i) X is $T_{1/2}$,

(ii) For each $x \in X$, $\{x\}$ is either closed or open.

We have the following characterizations of hardly e-openness.

Theorem 1.4. *If Y is a $T_{1/2}$-space, then a function $f : (X, \tau) \to (Y, \sigma)$ is hardly e-open if and only if for each e-dense subset A of Y, $f^{-1}(A)$ is e-dense in X.*

Proof. Assume f is hardly e-open. Let A be an e-dense subset of Y. Suppose $y \in Y \backslash A$. Since A is e-dense, hence dense, $int(Y \backslash A) = \emptyset$ and therefore $\{y\}$ is not open. Since Y is a $T_{1/2}$ space, $\{y\}$ is closed. Therefore A is contained in the proper open subset $Y \backslash \{y\}$ and since f is hardly e-open, $f^{-1}(A)$ is e-dense in X. The converse is clear.

Theorem 1.5. *A function $f : (X, \tau) \to (Y, \sigma)$ is hardly e-open if and only if $int_e(f^{-1}(A)) = \emptyset$ for each set $A \subset Y$ having the property that $int_e(A) = \emptyset$ and A containing a nonempty closed set.*

Proof. Assume f is hardly e-open. Let $A \subset Y$ such that $int_e(A) = \emptyset$ and let F be a nonempty closed set contained in A. Since $int_e(A) = \emptyset$, $Y \backslash A$ is e-dense in Y. Because $F \subset A$, $Y \backslash A \subset Y \backslash F \neq Y$. Therefore $f^{-1}(Y \backslash A)$ is e-dense in X. Thus $X = cl_e(f^{-1}(Y \backslash A)) = cl_e(X \backslash f^{-1}(A)) = X \backslash int_e(f^{-1}(A))$ which implies that $int_e(f^{-1}(A)) = \emptyset$.
For the converse implication assume that $int_e(f^{-1}(A)) = \emptyset$ for every $A \subset Y$ having the property that $int_e(A) = \emptyset$ and A contains a nonempty closed set. Let A be an e-dense subset of Y, that is contained in the proper open set U. Then $int_e(Y \backslash A) = \emptyset$ and $\emptyset \neq Y \backslash U \subset Y \backslash A$. Thus $Y \backslash A$ contains a nonempty closed set and hence $int_e(f^{-1}(Y \backslash A)) = \emptyset$. Then $\emptyset = int_e(f^{-1}(Y \backslash A)) = int_e(X \backslash f^{-1}(A)) = X \backslash cl_e(f^{-1}(A))$ and hence $f^{-1}(A)$ is e-dense in X.

Theorem 1.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $int_e(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_e(A) \neq \emptyset$ and there exists a nonempty closed set F for which $f^{-1}(F) \subset A$ then f is hardly e-open.*

Proof. Let $B \subset U \subset Y$ where B is e-dense and U is a proper open set. Let $A = f^{-1}(Y \backslash B)$ and $F = Y \backslash U$, obviously $f^{-1}(F) = f^{-1}(Y \backslash U) \subset f^{-1}(Y \backslash B) = A$. Also $int_e(f(A)) = int_e(f(f^{-1}(Y \backslash B))) \subset int_e(Y \backslash B) = \emptyset$. Therefore we must have that $\emptyset = int_e(A) = int_e(f^{-1}(Y \backslash B)) = int_e(X \backslash f^{-1}(B))$ which implies that $f^{-1}(B)$ is e-dense. It follows that f is hardly e-open.

Theorem 1.7. *If $f : (X, \tau) \to (Y, \sigma)$ is hardly e-open, then $int_e(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_e(A) \neq \emptyset$ and $f(A)$ contains a nonempty closed set.*

Proof. Let $A \subset X$ such that $int_e(A) \neq \emptyset$ and let F be a nonempty closed set for which $F \subset f(A)$. Suppose $int_e(f(A)) = \emptyset$. Then $Y \backslash f(A)$

is e-dense in Y and $Y\backslash f(A) \subset Y\backslash F$ where $Y\backslash F$ is a proper open set. Since f is hardly e-open, $f^{-1}(Y\backslash f(A))$ is e-dense in X. But $f^{-1}(Y\backslash f(A)) = X\backslash f^{-1}(f(A))$ and hence $int_e(f^{-1}(f(A))) = \emptyset$. It follows that $int_e(A) = \emptyset$ which is a contradiction.

Theorems 1.6 and 1.7 are reversible provided that f is surjective. Thus we have the following characterization for surjective hardly e-open functions.

Theorem 1.8. *If $f : (X, \tau) \to (Y, \sigma)$ is surjective, then the following conditions are equivalent:*

(i) *f is hardly e-open.*

(ii) *$int_e(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_e(A) \neq \emptyset$ and there exists a nonempty closed $F \subset Y$ such that $F \subset f(A)$.*

(iii) *$int_e(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_e(A) \neq \emptyset$ and there exists a nonempty closed set $F \subset Y$ such that $f^{-1}(F) \subset A$.*

Proof. $(i) \Rightarrow (ii)$: Theorem 1.7
$(ii) \Rightarrow (iii)$: Since f is surjective $f^{-1}(F) \subset A$ implies that $F \subset f(A)$.
$(iii) \Rightarrow (i)$: Theorem 1.6.

References

[1] M. E. Abd El-Monsef, S. N. Ei-Deeb and R. A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Assiut Univ.* 12 (1983), 77-90.

[2] O.W. Baker, Weak forms of openness base upon denseness. *Tr. J. of Mathematics*, 20(1996), 389-394.

[3] M. Caldas, On fantly e-continuous functions. To appear in Sarajevo J. Math.

[4] M. Caldas, On somewhat e-continuity. *Pro Mathematica* 47(2010), 275-284.

[5] M. Caldas and S. Jafari, On Strongly faint e-continuous functions. *Projecciones J. Math.*, 30(2011), 29-41.

[6] Dunham, A new closure operator for non-T_1 topologies, *Kyungpook Math. J.*, 22(1982), 55-60.

[7] E. Ekici, On e-open sets, DP^*-sets and DPC^*-sets and decompositions of continuity. *The Arabian J. for Sci. and Eng.*, 33 (2A)(2008), 269-282.

[8] E. Ekici, On a-open sets, A^*-sets and Decompositions of continuity and super continuity, *Annales Univ. Sci. Budapest*, 51 (2008), 39-51.

[9] E. Ekici, New forms of contra continuity, *Carpathian J. Math.*, 24 (1)(2008), 37-45.

[10] E. Ekici, On e^*-open sets and $(D, S)^*$-sets, *Math. Moravica*, Vol. 13(1)(2009), 29-36.

[11] E. Ekici, Some generalizations of almost contra-super-continuity, *Filomat*, 21(2)(2007), 31-44.

[12] E. Ekici, A note on a-open sets and e^*-sets, *Filomat*, 21(1)(2008), 89-96.

[13] A.A. El-Atik, Study of some types of mappings on topological spaces, *Bull.Masters. Thesis, Faculty of Science, Tanta University, Egypt*, (1997).

[14] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 68 (1961), 44-46.

[15] N. Levine, Generalized closed sets in topology, *Rend. Circ. Math. Palermo*, 19(1970), 89-96.

[16] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, 53 (1982), 47-53.

[17] O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.*, 15 (1965), 961-970.

[18] S. Raychaudhuri and M.N. Mukherjee, On δ-almost continuity and δ-preopen sets, *Bull. Inst. Math. Acad. Sinica*, 21 (1993), 357-366.

[19] N. V. Veličko, H-closed topological spaces, Mat. Sb., 70 (1966), 98-112; English transl., *Amer. Math. Soc. Transl.*, 78 (1968), 103-118.

Resumen

Una función se define como "hardly e-open" siempre que la imagen inversa de cada subconjunto e-denso del codominio, que esté contenido en un adecuado conjunto abierto, es e-denso en el dominio. Se estudian caracterizaciones y propiedades de las funciones "hardly e-open".

Palabras clave: Espacios topológicos, conjuntos e-abiertos, conjuntos e-densos, funciones "hardly e-open".

Miguel Caldas
Departamento de Matemática Aplicada,
Universidade Federal Fluminense,
Rua Mário Santos Braga, s/n^o
24020-140, Niterói, RJ BRAZIL.
gmamccs@vm.uff.br