MULTIPLE OBJECTIVE NETWORK FLOW PROBLEMS

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Abstract

In this work, it is presented the multiple objective network flow problems. This kind of problem is converted into single objective problem and solved by using the primal dual interior point method. The linear system associated to the interior point method is solved by using the Cholesky decomposition, implemented in MATLAB code. Networks of different dimensions are constructed and the computational results show the efficiency of the mentioned interior point method for solving multiple objective network flow problems.

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1. Introduction

The multiple objective linear programming problems (MLPP) deal with optimization problems with two or more linear objective functions. This kind of problem differs from the classical single objective optimization problem only in the expression of their respective and conflicted objective function. In a single objective problem, the goal is to find the optimal solution, that is, the feasible solution (or solutions) that gives the best value of the objective function. It is noticed that even there are alternative optimal solutions, the optimal value of the objective function is unique. Certainly, the notion of optimality must be dropped for multiple objective linear problems because a solution which minimizes one objective will not in general minimize any of the other objectives. The reason of the interest of this problem is that many practical real-world problems can be modeled as MLPP, as mentioned in [7].

The MLPP can be expressed in the following form:

$$\text{minimize} \quad \{f_1(x), f_2(x), ..., f_K(x)\}$$
$$\text{subject to :} \quad x \in S$$

where $K \geq 2$ and for linear functions, $f_i(x) = c^i x$ for $i = 1, \ldots, K$ and S is a feasible set (nonempty).

The word "*minimize*" means to minimize all the objective functions simultaneously. For simplicity of the treatment in this study, it is assumed that all the objective functions are to be minimized. If an objective function f_i is to be maximized, it is equivalent to minimize the function $-f_i$. If there no conflict between the objective functions, then a solution can be found where every objective function attains its optimum. To avoid such a trivial case, it is assumed that there does not exist such a single solution that is optimal with respect to every objective function.

A decision vector is Pareto optimal solution if it is solution that cannot be improved in one objective function without deteriorating their performance in at least one of the rest. Mathematically the Pareto solution can be described as follows. A decision vector $x^* \in S$ is Pareto optimal if there does not exist another decision vector $x \in S$ such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, \ldots, K$ and $f_j(x) < f_j(x^*)$ for at least one index j.

There are usually a lot (infinite number) of Pareto optimal solutions. In addition to Pareto optimality, several other terms are sometimes used for the optimality concept. For example, the idea of Pareto is very similar to the concept of noninferiority. By some mathematical programmers, noninferiority is called nondominance, or by others it is called efficiency.

Mathematically, every Pareto optimal point is an equally acceptable solution of the MLPP. In this case, the decision maker has to make a selection based on the preference relations between different solutions. The objective function values that are satisfactory or desirable to the decision maker are called aspiration levels and denoted by $z_i', i = 1, \ldots, K$. The vector $z' \in R^K$, consisting of aspiration levels, is called a reference point.

An objective vector minimizing each of the objective functions is called an ideal objective vector. The components z_i' of the ideal objective vector $z' \in R^K$ are obtained by minimizing each of the objective functions individually subject to the constraints, i.e. $x \in S$.

A common approach for the solution of general mutiple objective programming problems is to transform the original multiple problem into a series of scalarized, single criterion subproblems which are then solved using classical methods, like the traditional simplex method, see the paper given in [7] or for the integer case, see [8].

As it is known, the simplex method solves linear programming problems by visiting extreme points, on the boundary of the feasible set, each time improving the cost. In the mid 1980's new algorithms for linear programming were devised that find an optimal solution while moving in the interior of the feasible set, for this reason, they are generally called interior point methods. The field of these methods has its origins in the work described in [4]. This is the paper that introduced the first interior point algorithm with polynomial time complexity. In practice, the interior point methods are competitive with the simplex method, especially for large and sparse problem, they often outperform the simplex method. Details of these interior point methods can be seen in the books given in [9] and [10].

The most computationally expensive step of an interior point method is to find a solution of a linear system of equation, the so-called Newton equation system. All general purpose interior point method codes use a direct approach or iterative methods to solve the Newton equation system. There are two competitive direct approaches for solving the Newton equations: the augmented system approach and the normal equations approach. The former requires factorization of a symmetric indefinite matrix, the latter works with a smaller positive definite matrix.

The most efficient interior point method is the infeasible - primal - dual algorithm. The algorithm generates iterates which are positive, i.e. are interior with respect to the inequality constraints but do not necessarily satisfy the equality constraints. Other difficulty is the choice of a good initial solution.

Most implementations of primal-dual methods are based on the system of normal equations. They use direct Cholesky decomposition of the associated matrix. Iterative methods also could be used to solve the normal equations, but a good and computationally cheap preconditioned matrix could be chosen in order to accelerate the method to obtain the solution of the mentioned linear system.

Some works related to the MLPP can be found in different paper, for example, the paper given in [2] uses a variant of Karmarkar's interior-point method known as the affine-scaling primal algorithm for solving any multi-objective linear programming. The study written in [3] uses the primal- dual interior for solving the above network problems. The associated linear system of normal equations is solved by a direct method, using the Cholesky decomposition approach. This normal system is also solved by using the iterative method like the pre-conditioned conjugate gradient. In this work, the MLPP is implemented in the MATLAB code for a specific traffic problem in a transportation network.

The remainder of the paper is organized as follows: The interior point method is analysed in section 2. Section 3 presents the multiple objective network flow problem and the implementation of the interior point method for finding the solution of this multiple objective problem. Section 4 presents the computational results for networks of different dimensions with two objective functions. The paper is concluded with some remarks made in section 5.

2. The Primal-Dual Interior-Point Method

This section presents a brief description of the primal-dual interior-point method to solve the linear programming problem (LP) in the primal form. This problem is given by:

$$\begin{aligned} \text{minimize} \quad & c^T x & & \qquad (1) \\ \text{subject to:} \quad & Ex = b, & & \qquad (2) \\ & x \geq 0, & & \qquad (3) \end{aligned}$$

being $x \in R^n$ is the decision vector, $c \in R^n$, $b \in R^m$ and E is a matrix, $E \in R^{mxn}$, $m < n$ of full rank. The dual of the linear problem (1) - (3)

has the form:

$$\begin{aligned} \text{maximize} \quad b^T y & & & (4)\\ \text{subject to} : E^t y + z &= c, & & (5)\\ z &\geq 0, & & (6) \end{aligned}$$

being $y \in R^m$ is the dual variables and $z \in R^n$ is the vector of dual slack variable.
The first order optimality conditions, also known as the Karush-Kuhn-Tucker (KKT) conditions, for the linear problems (1)-(3) and (4) - (6) are:

$$\begin{aligned} Ex &= b, \quad x \geq 0, & (7)\\ E^T y + z &= c, \quad z \geq 0, & (8)\\ XZe &= 0, & (9) \end{aligned}$$

where X and Z are diagonal matrices defined as $X = diag(x_1, \ldots, x_q)$, $Z = diag(z_1, \ldots, z_q)$, and e is the q-vector of all ones, that is: $e = (1, \ldots, 1, \ldots, 1) \in R^q$.

To apply the primal-dual interior-point method to solve the LP problem, it is solved the following perturbed KKT conditions :

$$\begin{aligned} Ex &= b, \quad x \geq 0, & (10)\\ E^T y + z &= c, \quad z \geq 0, & (11)\\ XZe &= \mu e, & (12) \end{aligned}$$

where $\mu > 0$ is called the barrier parameter. These modifications (10)-(12) are equivalent to the first order KKT conditions (7)-(9), except that the third condition is perturbed by μ .

Let us notice that if $\mu = 0$ and $x \geq 0$, $z \geq 0$, the KKT conditions (10) - (12) coincide with the KKT conditions (7)-(9). For this reason, the choice of the parameter μ plays an important role in the interior-point method. In the interior point method, at each iteration, the parameter

$\mu > 0$ is reduced by a certain factor. As the sequence of barrier parameters μ converging to zero, the solution $(x(\mu), y(\mu), z(\mu))$ converges to an optimal solution of the LP problem. The system (10)-(12) is solved using Newton's method. Let $dw = (dx, dy, dw)^T$ denote the Newton's direction, obtained by the linearization of system (10)-(12) and determined by the solution of the system of linear equations:

$$\begin{pmatrix} E & 0 & 0 \\ 0 & E^T & I \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \xi_b \\ \xi_c \\ \xi_\mu \end{pmatrix} \tag{13}$$

where

$$\xi_b = b - Ex, \quad \xi_c = c - E^T y - z, \quad \xi_\mu = \mu e - XZe$$

If the third equation of the linear system (13) is eliminated, that is, $dz = X^{-1}(\xi_\mu - Zdx)$, it is obtained the following indefinite symmetric system, also called an augmented system:

$$\begin{pmatrix} -X^{-1}Z & E^T \\ E & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \xi_c - X^{-1}\xi_\mu \\ \xi_b \end{pmatrix}$$

$$dz = \xi_c - E^T dy$$

and making a further substitution, if dx is eliminated from above system, the following linear system, named normal equations, is obtained:

$$(EZ^{-1}XE^T)dy = EZ^{-1}X(\xi_c - X^{-1}\xi_\mu) + \xi_b$$

and the others variables dz and dx can be determined as following:

$$\begin{aligned} dz &= \xi_c - E^T dy \\ dx &= Z^{-1}(\xi_\mu - Xdz) \end{aligned}$$

To summarize an iteration of the infeasible primal-dual interior-point method, let at the j-th iteration, $dw_j = (dx_j, dy_j, dz_j)^T$ denote the solution obtained from the system (13). In the next iteration, a new interior point $w_{j+1} = (x_{j+1}, y_{j+1}, z_{j+1})^T$ is determined using the following rules:

$$\begin{aligned} x_{j+1} &= x_j + \beta\alpha_j dx_j, \\ y_{j+1} &= y_j + \beta\alpha_j dy_j, \\ z_{j+1} &= z_j + \beta\alpha_j dz_j, \end{aligned}$$

α_j being the step length, determined by a suitable line search procedure and $\beta \in (0, 1)$ and near 1.

With this new point w_{j+1}, the barrier parameter μ is updated according to certain rules and a new linear system (13) is formed and solved by any solution method and the iterative procedure follows until a stopping rule is satisfied. Implementation of this interior point method can be found in the work given in [1].

3. The Multiple Objective Network Flow Problem

Let $G = (V, E)$ be a directed graph representing a network, where V is a set of nodes or vertices and E represents a set of arcs or edges. Let m represents the number of vertices in V and n represents the number of arcs in E. For each node $i \in V$, b_i denotes the flow produced or consumed at node i. If $b_i > 0$, node i is called a supply node. If $b_i < 0$, node i is called a demand node. If $b_i = 0$, node i is called a transshipment node. An arc $(i, j) \in E$ represents the link between two nodes i and j. The quantity of commodity shipped from nodes i to node j, called the flow on arc (i,j), is represented by x_{ij}. Let K be the number of objective functions and is assumed to be of minimization type. For each objective function, a cost coefficient c_{ij}^k , $k = 1, \ldots, K, K \geq 2$, is associated to each arc (i, j), representing the cost of shipping one unit of the commodity on arc (i, j). For some problems, x_{ij} is restricted to be within its lower bound l_{ij} and upper bound u_{ij}.

The multiple objective network flow problem can be stated as:

$$\text{minimize} \sum_{(i,j)\in E} c_{ij}^k x_{ij} \quad for \quad k = 1,\dots,K \tag{14}$$

$$\text{subject to}: \sum_{(ij)\in E} x_{ij} - \sum_{(ji)\in E} x_{ji} \quad = \quad b_i, \;\; for\; i \in V, \tag{15}$$

$$l_{ij} \le x_{ij} \le u_{ij}, \quad for \quad (i,j) \in E. \tag{16}$$

The word "*minimize*" means that all the objective functions is minimized simultaneously. Constraints of type (15) are referred to as the conservation of flow equations. Constraints of type (16) are the flow capacity constraints. In the remainder of this work, it is assumed, without loss of generality, that a lower bound $l_{ij} = 0$ and upper bound $u_{ij} = \infty$ for all arcs $(i,j) \in E$.

Formally, in compact form, the multiple objective network flow problem can be stated as:

$$\begin{aligned} \text{“minimize”} \;\; & Cx \\ \text{subject to}: Ax &= b, \\ x &\ge 0, \end{aligned}$$

where $A \in R^{mxn}$ denotes the node-arc incidence matrix of graph $G = (V,E)$ and assumed of full row rank, $b \in R^m$ denotes the vector of node requirements and $x \in R^n$ denotes the vector of flows. The matrix $C \in R^{Kxn}$ denotes the matrix of cost coefficients.

According to the works given in [3], [5] and [7], the augmented weighted Tchebycheff network program, associated to the above problem, can be formulated in the following way:

$$\begin{aligned} \text{minimize} \quad & \alpha + \rho \sum_{k=1}^{K} \sum_{(i,j)\in E} c_{ij}^{k} x_{ij} - \rho \sum_{k=1}^{K} \pi_k \\ \text{subject to :} \quad \sum_{(ij)\in E} x_{ij} - \sum_{(ji)\in E} x_{ji} &= b_i, \quad for \;\; i \in V, \\ \sum_{(i,j)\in E} c_{ij}^{k} x_{ij} - (1/\lambda_k)\alpha + s_k &= \pi_k, \quad k = 1, \dots, K \\ x_{ij} \geq 0, \quad for \quad (i,j) \in E, \quad s \geq 0, \quad \alpha \geq 0. \end{aligned}$$

The above problem can also be written as:

$$\begin{aligned} \text{minimize} \quad & \alpha + \rho \sum_{k=1}^{K} c^k x - \rho \sum_{k=1}^{K} \pi_k \\ \text{subject to :} \; Ax &= b, \\ (c^k)^T x - (1/\lambda_k)\alpha + s_k &= \pi_k, \quad k = 1, \dots, K \\ x \geq 0, \quad s \geq 0, \quad \alpha \geq 0. \end{aligned}$$

where s_k is the slack variable. It is defined the weighting vector space, associated with each of the k objectives functions, as:

$$\lambda_k > 0, \quad for \;\; k = 1, \dots, K \quad such \;\; that \quad \sum_{k=1}^{K} \lambda_k = 1.$$

The vector $\pi = (\pi_1, \dots, \pi_n)^T$ represents the reference point considered in the objective space, and it is assumed that this point is the ideal one, and in this case, $\alpha \geq 0$. Also, $\rho > 0$ is an arbitrary small scalar, $0 < \rho << 1$.

The corresponding dual problem is given by:

$$
\begin{aligned}
\text{maximize} \quad & b^T y - \pi^T v \\
\text{subject to} : A^T y - C^T v + z &= \rho \sum_{k=1}^{K} c^k, \\
(1/\lambda)^T v + z_\alpha &= 1, \\
v \geq 0, z \geq 0, z_\alpha \geq 0.
\end{aligned}
$$

Solving the above linear network flow problem, a non-dominated solution is obtained provides that the decision maker gives an acceptable aspiration level, called a reference point.

The infeasible primal - dual interior point method is used to determine an optimal solution vector x of the above linear problem. In this case, the following optimality conditions are satisfied:

$$
\begin{aligned}
Ax = b, \\
A^T y - C^T v + z = \rho \sum_{k=1}^{K} c^k \\
Cx - (1/\lambda)\alpha + s = \pi, \\
(1/\lambda)^T v + z_\alpha = 1, \\
x^T z = 0, \quad s^T v = 0, \quad \alpha z_\alpha = 0, \\
x \geq 0, \quad s \geq 0, \quad v \geq 0, \\
z \geq 0, \quad \alpha \geq 0, \quad z_\alpha \geq 0.
\end{aligned}
$$

In order to implement the infeasible interior point method, consider the following residuals:

$$\begin{aligned} f_1 &= -A^T y + C^T v - z + \rho \sum_{k=1}^{K} c^k, \\ f_2 &= 1 - (1/\lambda)^T v - z_\alpha, \\ g_1 &= b - Ax, \\ g_2 &= \pi - Cx + (1/\lambda)\alpha - s, \\ h_1 &= \mu e - XZe, \\ h_2 &= \mu e - \alpha z_\alpha, \\ h_3 &= \mu e - SVe, \end{aligned}$$

where

$$\mu = \beta \frac{x^T z + \alpha z_\alpha + s^T v}{2n + K}$$

for a given arbitrary point $(x, z, \alpha, z_\alpha, s, v)$ and $0 < \beta < 1$.
Any infeasible primal-dual interior-point algorithm has to solve a linear system corresponding to the so called Newtons systems given by:

$$Adx = g_1, \tag{17}$$

$$Cdx - (1/\lambda)d\alpha + ds = g_2, \tag{18}$$

$$A^T dy - C^T dv + dz = f_1, \tag{19}$$

$$(1/\lambda)^T dv + dz_\alpha = f_2, \tag{20}$$

$$Zdx + Xdz = h_1, \tag{21}$$

$$z_\alpha d\alpha + \alpha dz_\alpha = h_2, \tag{22}$$

$$Vds + Sdv = h_3, \tag{23}$$

where X is a diagonal matrix defined as X $= diag(x_1, \ldots, x_n)$. A similar definition is for Z, V and S matrices.
From equations (19) and (21), it is obtained:

$$XA^T dy - XC^T dv = Xf_1 - Xdz = Xf_1 + Zdx - h_1,$$

so that:

$$Z^{-1}XA^Tdy - Z^{-1}XC^Tdv = Z^{-1}Xf_1 - dx - Z^{-1}h_1, \qquad (24)$$

and by using relation (17), finally it has:

$$(A\Theta A^T)dy - (A\Theta C^T)dv = A\Theta f_1 + g_1 - AZ^{-1}h_1, \qquad (25)$$

where $\Theta = Z^{-1}X$ is the scaling matriz.
From (18), (20) and (24), it is obtained:

$$\begin{aligned}(C\Theta A^T)dy - C\Theta C^T + V^{-1}S + (1/\lambda)(\alpha/z_\alpha)(1/\lambda)^T dv = \\ -CZ^{-1}h_1 + C\Theta f_1 + g_2 + (1/\lambda)(1/z_\alpha)h_2 - (1/\lambda)(\alpha/z_\alpha)f_2 - V^{-1}h_3\end{aligned} \quad (26)$$

Making $dv' = -dv$, it is solved the system given by (25) and (26) and it is determined the variables dy and dv. With these known variables, it is possible to obtain the remaining variables, that is:

$$\begin{aligned} dz_\alpha &= f_2 - \lambda^T dv, \\ d\alpha &= (1/z_\alpha)(h_2 - \alpha dz_\alpha), \\ dz &= f_1 - A^T dy + C^T dv, \\ dx &= Z^{-1}(h_1 - Xdz), \\ ds &= g_2 - Cdx + \alpha d\alpha. \end{aligned}$$

The next step of the interior point algorithm is to choose a primal step length α_P and a dual step length α_D,

$$\alpha_P = \delta_P max\{\delta : x + \delta dx \geq 0, \alpha + \delta d\alpha \geq 0, s + \delta ds \geq 0\},$$
$$\alpha_D = \delta_D max\{\delta : z + \delta dz \geq 0, z_\alpha + \delta dz_\alpha \geq 0, v + \delta dv \geq 0\},$$

and a new iterate is formed:

$$\begin{aligned} x &= x + \alpha_P dx, \\ \alpha &= \alpha + \alpha_P d\alpha \\ s &= s + \alpha_P ds, \\ y &= y + \alpha_D dy, \\ z_\alpha &= z_\alpha + \alpha_D dz_\alpha, \\ v &= v + \alpha_D dv. \end{aligned}$$

The primal-dual interior-point algorithm applied to multiple objective linear network flow problem stops if an appropriate stopping criterion is satisfied. Otherwise, a new value μ is determined and the algorithm continues until the stopping criterion is satisfied.

4. Computational Results

In this section, it is reported the computational experience to test the computational efficiency of the interior point algorithm for solving mutiple objective problems, and in this case with two objectives. This algorithm is programmed in MATLAB code. All the experiments were obtained on a 2.53 GHz Personal Computer with 4 GB of RAM. The set of test problems were generated based on the basic network of the paper given in [6]. This network consists of 20 nodes and 28 arcs. It was used to solve the traffic problem in a transportation network. This basic network was afterwards extended to generate large-scale networks. A specific MATLAB program was implemented to determine the dimension of the new network, that is, to determine the number of nodes and arcs, defined by the initial and final nodes. Randomly generated cost coefficients of the objectives functions are obtained by using the rand command. These cost coefficients are in the range $1 \leq c_{ij}^k \leq 5$ and integers in all test problems. The value of the weighting vector is $\lambda_1 = 0.7$ and $\lambda_2 = 0.3$, The reference point is given by $\pi_1 = 5$ and $\pi_2 = 10$. The value of

$\rho = 0.01$ and $\beta = 0.99995$.
To solve the linear system, given by (25) and (26) which defines the Newton direction, is used the chol MATLAB command, i.e., the direct method based on Cholesky factorization.

Following, the computational results is given in table 1 for different network dimensions, where m the number of nodes and n is the number of variables ou links. The flow in, given in this work by 50 units, must be equal to the flow out. This result can be seen in the table 1, in the third line, using the name flow and in this case is near to 50.

Table 1. Computational Results

network	$m: 910$, $n: 1760$	$m: 1040$, $n: 2015$	$m: 1240$, $n: 2410$	$m: 2050$, $n: 4010$
iter	19	19	29	20
flow	49.9982	49.9983	49.9997	49.9989
μ	4.4869e-004	4.7207e-004	3.9008e-004	0.0011
time	17.767	26.497	66.353	202.327

Also, from this table 1, it can be seen that the number of iterations, iter, is not big; the time, measured in seconds given by the MATLAB code, is acceptable and increases when the network dimension is also increases. Finally, the value of μ is near to zero that is expected for the efficiency of the interior point method.

5. Conclusions

In this paper, the infeasible primal-dual interior point algorithm is used to solve the multiple objective problems for the network flow model for different dimensions. The mentioned algorithm was coded using the MATLAB language and the respective Newton direction is determined based on Cholesky factorization. The experimental results have demonstrated the efficiency of the interior point algorithm for solving the specific traffic problem. In this work, it not was used the structure of the network in order to avoid the storage of the respective matrix associated to the linear system to find the Newton direction. This alternative will be considered in future work.

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Resumen

En este trabajo, se presenta el problema de flujo en redes con múltiples objetivos. Este problema se puede transformar en un problema lineal con un único objetivo, y se puede resolver usando el método primal dual de puntos interiores. La solución del sistema lineal asociado al método de puntos interiores es determinado usando el método de factorización de Cholesky, implementado en el código MATLAB. Redes de diferentes dimensiones son construídas y resultados computacionales muestran la eficiencia del método de puntos interiores para resolver problemas de flujo en redes con múltiples objetivos.

Palabras clave: Problemas lineales con múltiples objetivos; Flujo en redes; método de puntos interiores.

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