A STUDY OF MODIFIED HERMITE POLYNOMIALS

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Abstract

The present paper is a study of modified Hermite polynomials $H_n(x;a)$ which reduces to Hermite polynomials $H_n(x)$ for $a = e$.

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1 Introduction

Hermite polynomials $H_n(x)$ are defined [2] by

$$exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{1.1}$$

The aim of the present paper is to modify the definition (1.1). The paper contains generating functions, recurrence relations, Rodrigues formula, orthogonality conditions, expansion formulae, integral representation and other properties for the modified Hermite polynomials $H_n(x)$.

2 The Definition of $H_n(x; a)$

The modified Hermite polynomials $H_n(x; a)$ are defined by means of the generating relation

$$a^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x;a)t^n}{n!}, \quad a > 0 \tag{2.1}$$

It follows from (2.1) that

$$H_n(x;a) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!(2x)^{n-2k}(loga)^{n-k}}{n!k!} \tag{2.2}$$

For $a = e$ (2.2) reduces to Hermite polynomials $H_n(x)$.

It may be remarked that $H_n(x; a)$ is an even function of x for even n, an odd function of x for odd n.

$$H_n(-x;a) = (-1)^n H_n(x;a)$$

Also,

$$H_{2n}(0;a) = (-1)^n 2^{2n} \left(\frac{1}{2}\right)_n (loga)^n, \qquad H_{2n+1}(0;a) = 0$$

and

$$H_{2n+1}(0;a) = (-1)^n 2^{n+1} \left(\frac{3}{2}\right)_n (loga)^{n+1}, \qquad H_{2n}(0;a) = 0$$

The first few modified Hermite polynomials are listed below:

$$H_0(x;a) = 1,\ H_1(x;a) = 2xloga,$$

$$H_2(x;a) = 4x^2(loga)^2 - 2loga,$$

$$H_3(x;a) = 8x^3(loga)^3 - 12x(loga)^2,$$

$$H_4(x;a) = 16x^4(loga)^4 - 48x^2(loga)^3 + 12(loga)^2,$$

$$H_5(x;a) = 32x^5(loga)^5 - 160x^3(loga)^4 + 120x(loga)^3,$$

$$H_6(x;a) = 64x^6(loga)^6 - 480x^4(loga)^5 + 720x^2(loga)^4 - 120(loga)^3,$$

3 Recurrence Relations

The following recurrence relations hold for $H_n(x;a)$:

$$xH'_n(x;a) = nH'_{n-1}(x;a) + nH_n(x;a) \tag{3.1}$$

$$H'_n(x;a) = 2nlogaH_{n-1}(x;a) \tag{3.2}$$

$$D^s H_n(x;a) = \frac{(2loga)^s n! H_{n-s}(x;a)}{(n-s)!} \;; \qquad D \equiv \frac{d}{dx} \tag{3.3}$$

$$H_n(x;a) = \; 2 \; x \; loga \; H_{n-1}(x;a) - H'_{n-1}(x;a) \tag{3.4}$$

$$H_n(x;a) = 2loga\{xH_{n-1}(x;a) - (n-1)H_{n-2}(x;a)\} \tag{3.5}$$

$$H''_n(x;a) = 4n(n-1)(loga)^2 H_{n-2}(x;a) \tag{3.6}$$

Also the modified Hermite differential equation is

$$H''_n(x;a) - 2xlogaH'_n(x;a) + 2nlogaH_n(x;a) = 0 \tag{3.7}$$

4 Rodrigues Formula for $H_n(x;a)$

The Rodrigues formula for modified Hermite polynomials $H_n(x;a)$ is given by the following relation:

$$H_n(x;a) = (-1)^n a^{x^2} D^n a^{-x^2}, \qquad D \equiv \frac{d}{dx} \tag{4.1}$$

The proof of (4.1) is same as that of Rodrigues formula for $H_n(x)$.

5 Other Generating Function for $H_n(x;a)$

The other generating function for $H_n(x;a)$ is given by

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x;a)t^n}{n!} = (1-2xtloga)^{-c} \left[\begin{array}{cc} \frac{c}{2}, \frac{c}{2}+\frac{1}{2}; & \frac{-4t^2 loga}{(1-2xtloga)^2} \\ -; & \end{array} \right] \tag{5.1}$$

6 Integrals

Some integral representation for $H_n(x;a)$ are as follows:

$$P_n(x) = \frac{2}{n!}\sqrt{\frac{loga}{\pi}} \int_0^{\infty} a^{-t^2} t^n H_n(xt;a)dt \tag{6.1}$$

$$H_n(x;a) = (2loga)^{n+1} a^{x^2} \int_x^{\infty} a^{-t^2} t^{n+1} P_n(x/t)dt \tag{6.2}$$

$$\int_{-\infty}^{+\infty} a^{-x^2} x^n H_{n-2k}(x;a)dx = \frac{2^{-2k} n!}{k!(loga)^k}\sqrt{\frac{\pi}{loga}} \tag{6.3}$$

$$\int_0^{\infty} a^{-x^2} H_{2k}(x;a) H_{2s+1}(x;a)dx = \frac{(-1)^{k+s} 2^{2k+2s} \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_s (loga)^{k+s}}{(2s+1-2k)} \tag{6.4}$$

$$\int_0^{x} a^{-t^2} H_n(t;a)dt = H_{n-1}(x;0) - a^{-x^2} H_{n-1}(x;a) \tag{6.5}$$

$$\int_{0}^{x} H_n(t;a)dt = \frac{1}{2(n+1)loga}\{H_{n+1}(x;a) - H_{n+1}(0;a)\} \tag{6.6}$$

$$\int_{-\infty}^{+\infty} a^{-t^2} H_{2n}(xt;a)dt = \frac{\sqrt{\pi}(2n)!}{n!}(x^2-1)(loga)^{n-\frac{1}{2}} \tag{6.7}$$

$$\int_{-\infty}^{+\infty} a^{-t^2} tH_{2n+1}(xt;a)dt = \frac{\sqrt{\pi}(2n+1)!}{n!}x(x^2-1)^n(loga)^{n-\frac{1}{2}} \tag{6.8}$$

$$\int_{-\infty}^{+\infty} a^{-t^2} t^n H_n(xt;a)dt = \frac{\sqrt{\pi}n!}{\sqrt{loga}}P_n(x) \tag{6.9}$$

$$\int_{-\infty}^{+\infty} a^{-x^2} H_{2n}(\sqrt{2}x;a)dx = \frac{(2n)!\sqrt{\pi}}{n!}(loga)^{n-\frac{1}{2}} \tag{6.10}$$

$$\int_{-\infty}^{+\infty} a^{-x^2} H_{2n+1}(\sqrt{2}x;a)dx = 0 \tag{6.11}$$

The second result (6.11) is trivial and the first (6.10) may be written

$$\int_{-\infty}^{+\infty} a^{-\frac{1}{2}x^2} H_{2n}(x;a)dx = \frac{(2n)!}{n!}\sqrt{\frac{\pi}{2}}(loga)^{n-\frac{1}{2}} \tag{6.12}$$

$$\int_{0}^{+\infty} a^{-t^2}\left[H_n(t;a)\right]^2 cos(\sqrt{2loga}xt)dt = \sqrt{\pi}2^{n-1}n!L_n(x^2)(loga)^{n-\frac{1}{2}} \tag{6.13}$$

$$\Gamma(n+\mu+1)\int_{-1}^{+1}(1-t^2)^{\mu-\frac{1}{2}}H_{2n}(\sqrt{x}t;a)dt$$

$$= (-1)^n \sqrt{\pi}(2n)!\, \Gamma\left(\mu + \frac{1}{2}\right) (loga)^n L_n^{\mu}(xloga), \left(Re(\mu) > -\frac{1}{2}\right) \tag{6.14}$$

$$\Gamma(n+\mu+1) \int_{-\frac{1}{\sqrt{loga}}}^{+\frac{1}{\sqrt{loga}}} (1-t^2 loga)^{\mu-\frac{1}{2}} H_{2n}(\sqrt{x}t; a)dt$$

$$= (-1)^n \sqrt{\pi}(2n)!\, \Gamma\left(\mu + \frac{1}{2}\right) (loga)^{n-\frac{1}{2}} L_n^{\mu}(x), \left(Re(\mu) > -\frac{1}{2}\right) \tag{6.15}$$

where $L_n^{\mu}(x)$ is the generalized Leguerre polynomial.

7 Modified Hermite Polynomials $H_n(x;a)$ as ${}_2F_0$

$$H_n(x;a) = (2xloga)^n \; {}_2F_0\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; & \frac{1}{x^2 loga} \\ -; & \end{matrix}\right] \tag{7.1}$$

8 Orthogonality

The orthogonality conditions for modified Hermite polynomial $H_n(x;a)$ are as follows:

$$\int_{-\infty}^{+\infty} a^{-x^2} H_n(x;a) H_m(x;a)dx = \begin{cases} 0, & for\ m \neq n \\ 2^n n!(loga)^n \sqrt{\frac{\pi}{loga}}, & for\ m = n \end{cases} \tag{8.1}$$

Further the following result hold for $H_n(x;a)$:

Theorem: For the modified Hermite polynomials $H_n(x;a)$

$$(a) \int_{-\infty}^{+\infty} a^{-x^2} x^k H_n(x;a)dx = 0, \qquad k = 0, 1, 2, \cdots, (n-1).$$

(b) The zeros of $H_n(x;a)$ are real and distinct.

$$(c) \sum_{k=0}^{n} \frac{H_n(x;a)H_k(y;a)}{(2loga)^k k!} = \frac{\{H_{n+1}(y;a)H_n(x;a) - H_{n+1}(x;a)H_n(y;a)\}(loga)^n}{2^{n+1}n!(y-x)}$$

9 Expansion of Polynomials in terms of Modified Hermite Polynomials $H_n(x;a)$

In terms of $H_n(x;a)$ Legendre polynomial can be written as

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{{}_2F_0[-k, \frac{1}{2}+n-k; -; \frac{1}{loga}](-1)^k \left(\frac{1}{2}\right)_{n-k} H_{n-2k}(x;a)(loga)^{2k-n}}{k!(n-2k)!} \tag{9.1}$$

Also in terms of Legendre polynomials $P_n(x)$, modified Hermite polynomials $H_n(x;a)$ can be written as

$$H_n(x;a) = \sum_{k=0}^{[\frac{n}{2}]} \frac{{}_1F_1[-k; \frac{3}{2}+n-2k; loga](-1)^k n!(2n-4k+1)P_{n-2k}(x)(loga)^{n-k}}{k! \left(\frac{3}{2}\right)_{n-2k}} \tag{9.2}$$

10 More Generating Functions

Some other generating functions for $H_n(x;a)$ are as follows:

$$\sum_{n=0}^{\infty} \frac{H_{n+k}(x;a)t^n}{n!} = a^{2xt-t^2} H_k(x-t;a) \tag{10.1}$$

$$\sum_{n=0}^{\infty} \frac{H_n(x;a)H_n(y;a)t^n}{n!} = \left(1-4t^2(loga)^2\right)^{-\frac{1}{2}} a^{\left\{y^2-\frac{(y^2-2xtloga)^2}{(1-4t^2(loga)^2)}\right\}} \tag{10.2}$$

Replacing t by $t/2$ in (10.2), we get

$$\sum_{n=0}^{\infty} \frac{(\frac{t}{2})^n}{n!} H_n(x;a)H_n(y;a) = \left(1-t^2(loga)^2\right)^{-\frac{1}{2}} a^{\left\{\frac{2xtloga-(x^2+y^2)t^2(loga)^2}{(1-t^2(loga)^2)}\right\}} \tag{10.3}$$

$$\sum_{n=0}^{\infty} \frac{{}_2F_0[-n,c;-;y]H_n(x;a)t^n}{n!} \cong a^{2xt-t^2}[1+2ty(x-t)loga]^{-c}$$

$$\times {}_2F_0\left[\begin{matrix} \frac{c}{2}, \frac{c}{2}+\frac{1}{2}; & \\ & -; \end{matrix} \; \frac{-4t^2y^2loga}{(1+2xytloga-2t^2yloga)^2}\right] \tag{10.4}$$

The above result for modified Hermite polynomials is similar to the result given by Brafman [1] for Hermite polynomials.

11 Summation Formulae

The summation formulae for $H_n(x;a)$ are as given below:

$$\sum_{k=0}^{n}\left\{2^k k!(loga)^k\right\}^{-1}[H_k(x;a)]^2 =$$

$$\left\{2^{n+1}n!(loga)^{n+1}\right\}^{-1}\left\{[H_{n+1}(x;a)]^2 - H_n(x;a)H_{n+2}(x;a)\right\} \quad (11.1)$$

$$\sum_{k=0}^{min(m,n)}(-2loga)^k k!\begin{pmatrix} m \\ k \end{pmatrix}\begin{pmatrix} n \\ k \end{pmatrix}H_{m-k}(x;a)H_{n-k}(x;a) = H_{m+n}(x;a) \quad (11.2)$$

$$\sum_{k=0}^{min(m,n)}(2loga)^k k!\begin{pmatrix} m \\ k \end{pmatrix}\begin{pmatrix} n \\ k \end{pmatrix}H_{m+n-2k}(x;a) = H_m(x;a)H_n(x;a) \quad (11.3)$$

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}H_k(\sqrt{2}x;a)H_{n-k}(\sqrt{2}y;a) = 2^{\frac{n}{2}}H_n(x+y;a) \quad (11.4)$$

$$\sum_{k=0}^{n}\begin{pmatrix} 2n \\ 2k \end{pmatrix}H_{2k}(\sqrt{2}x;a)H_{2n-2k}(\sqrt{2}y;a) =$$

$$2^{n-1}\left\{H_{2n}(x+y;a) + H_{2n}(x-y;a)\right\} \quad (11.5)$$

$$\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}H_{2k}(x;a)H_{2n-2k}(y;a) = (-1)^n n!(loga)^n L_n\left\{(x^2+y^2)loga\right\} \quad (11.6)$$

12 Gauss Transforms

The Gauss transform of a function $F(t)$ is defined by

$$g_x^{\alpha}\{F(t)\} = \frac{1}{\sqrt{2\pi\alpha}} \int_{-\infty}^{+\infty} F(t) e^{-\frac{(x-t)^2}{2\alpha}} dt,$$

α being parameter. The following results hold when we apply Gauss transform on modified Hermite polynomials.

$$g_x^{\alpha}\{H_n(t;a)\} = (loga)^{\frac{n-1}{2}} \{1-2\alpha loga\}^{\frac{n}{2}} H_n\left[\{1-2\alpha loga\}^{-\frac{1}{2}} \frac{x}{\sqrt{loga}}\right] \tag{12.1}$$

$$g_x^{\frac{1}{2}}\{H_n(t;a)\} = (2x)^n \tag{12.2}$$

$$g_{x\sqrt{loga}}^{\frac{1}{2}}\{t^n\} = \left(2i\sqrt{loga}\right)^{-n} H_n(ix;a) \tag{12.3}$$

13 Addition Theorem

The following addition theorems hold for $H_n(x;a)$:

$$\frac{(\lambda^2+\mu^2)^{\frac{n}{2}}}{n!} H_n\left\{\frac{\lambda z_1 + \mu z_2}{(\lambda^2+\mu^2)^{\frac{1}{2}}}; a\right\} = \sum_{r+s=n} \frac{\lambda^r \mu^s}{r!s!} H_r(z_1;a) H_s(z_2;a) \tag{13.1}$$

$$\left(\sum_{r=1}^{m}\lambda_r^2\right)^{\frac{n}{2}} H_n\left\{\frac{\sum\limits_{r=1}^{m}\lambda_r z_r}{\left(\sum\limits_{r=1}^{m}\lambda_r^2\right)^{\frac{1}{2}}};a\right\} = n!\sum_{\sum p_r=n}\left\{\prod_{r=1}^{m}\frac{\lambda_r^{p_r}}{(p_r)!}H_{p_r}(z_r;a)\right\} \tag{13.2}$$

Putting $\lambda = \mu$ in (13.1)

$$2^{\frac{n}{2}} H_n\left\{\frac{z_1+z_2}{\sqrt{2}};a\right\} = \sum_{r=0}^{n}\binom{n}{r} H_r(z_1;a)H_{n-r}(z_2;a) \tag{13.3}$$

Further with $z_1 = z_2 = x$, we have

$$2^{\frac{n}{2}} H_n(\sqrt{2}x;a) = \sum_{r=0}^{n}\binom{n}{r} H_r(x;a)H_{n-r}(x;a) \tag{13.4}$$

14 Limiting Relationships

The following relationships between Gegenbaur and Laguerre polynomials with $H_n(x;a)$ exist:

$$H_n(x;a) = n!(loga)^{\frac{n}{2}} \lim_{|\nu|\to\infty}\left\{\nu^{-\frac{n}{2}} C_n^{\nu}\left(\frac{x\sqrt{loga}}{\sqrt{\nu}}\right)\right\} \tag{14.1}$$

$$H_n(\frac{x}{\sqrt{2}};a) = (-1)^n 2^{\frac{n}{2}} n!(loga)^{\frac{n}{2}} \lim_{|\alpha|\to\infty}\left\{\alpha^{-\frac{n}{2}} L_n^{(\alpha)}\left(\alpha + x\sqrt{\alpha loga}\right)\right\} \tag{14.2}$$

15 Other Results For $H_n(x;a)$

Using the identity

$$a^{2xt-t^2} = a^{(2xt-t^2x^2)}.a^{[t^2(x^2-1)]}$$

we get the following result for $H_n(x;a)$

$$H_n(x;a) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n! H_{n-2k}(x;a) x^{n-2k} (x^2-1)^k (loga)^k}{k!(n-2k)!} \tag{15.1}$$

Also we have

$$\sum_{n=0}^{\infty} \frac{H_{2n}(x;a)t^n}{n!} = a^{-t} \ {}_0F_1\left[\begin{array}{l} -; \\ \frac{1}{2}; \end{array} x^2 t (loga)^2\right] \tag{15.2}$$

and

$$\sum_{n=0}^{\infty} \frac{H_{2n+1}(x;a)t^n}{(2n+1)!} = \ 2 \ x \ loga \ a^{-t} \ {}_0F_1\left[\begin{array}{l} -; \\ \frac{3}{2}; \end{array} x^2 t (loga)^2\right] \tag{15.3}$$

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Resumen

El presente trabajo es un estudio de los polinomios de Hermite modificados $H_n(x;a)$ que se reducen a polinomios de Hermite $H_n(x)$ para $a = e$.

Palabras Clave: Funciones generadoras, relaciones de recurrencia, fórmula de Rogrigues, Integrales, Transformadoras de Gauss y Ortogonalidad.

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