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MEROMORPHIC FUNCTIONS ON GERMS OF SURFACES ALONG A RATIONAL CURVE

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Abstract

We give examples of germs of surfaces containing a rational curve with positive self-intersection and with zero, one or two independent meromorphic functions.

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1. Introduction

Let X be a complex surface, not necessarily compact, and $Y \subset X$ a smooth rational curve, we denote by (X, Y) the germ of neighborhood. In these notes we study the field $\mathbb{C}(X, Y)$ of germs of meromorphic functions defined on small open sets of X containing Y. Denote by $d = Y^2$ the self-intersection of Y, that is, the degree of the normal bundle $N_{Y|X}$. We say that the pair (X, Y) is a d-neighborhood. In the case $d < 0$ the germ of neighborhood is linearizable and when $d = 0$ is isomorphic to $(\mathbb{C}, 0) \times \mathbb{P}^1$ (see [6] and [11] respectively), thus we shall focus on the case $d > 0$. In this case, following Andreotti [1] we know that $\mathbb{C}(X, Y)$ has transcendance degree bounded by $2 = \dim X$.

In [4] the authors proved the existence of germs of smooth surfaces X along a smooth rational curve Y with arbitrary positive self-intersection without meromorphic functions other than constants, that is such that $tr \deg_{\mathbb{C}} \mathbb{C}(X, Y) = 0$. Actually, it is proven that there are germs of surfaces with no foliations containing such rational curves.

On the other hand, in [9] S. Lvovski gives examples of non-algebrizable germs of surfaces X containing a smooth rational curve with any positive self-intersection such that $tr \deg_{\mathbb{C}} \mathbb{C}(X, Y) = 2$.

Recently, in [5, Theorem A] it is proven that for any complex projective manifold Y of dimension n and any pair of natural numbers (l, m) with $m \geq 2n$ and $l \leq m$, there exists a germ of manifold X of dimension m containing Y such that the normal bundle of Y in X is ample and $tr \deg_{\mathbb{C}} \mathbb{C}(X, Y) = l$. The construction here is simpler than the one given in [4] and one of the main ingredients is the construction of smooth manifolds transverse to holomorphic foliations.

By a specialization of the method in [5] to the case of $Y \cong \mathbb{P}^1$ we produce examples of germs of surfaces (X, Y) with self-intersection $d \geq 2$ and such that the trascendence degree of $\mathbb{C}(X, Y)$ over $\mathbb{C}$ is arbitrary between 0 and 2. Moreover, with some extra work we can prove the following generalization of [4].

Theorem A. *For any $d \geq 1$ and $l \in \{0, 1, 2\}$ there exists a pair (X, Y), where $Y \cong \mathbb{P}^1$ and X is a germ of surface containing Y such that $Y^2 = d$ and $tr \deg_{\mathbb{C}} \mathbb{C}(X, Y) = l$. Moreover, in the case $l = 0$ we can find X such that it does not admit any web.*

2. Some preliminaries

We give here some definitions and results that will be used in the proof of our theorem.

Given a field extension $K \subseteq L$, a transcendence basis is a maximal algebraically independent subset S of L over K, that is the elements of S do not satisfy any non-trivial polynomial equation with coefficients in K. Note that if S is a transcendence basis then the extension $K(S) \subseteq L$ is algebraic. All transcendence bases of a field extension have the same cardinality, called the transcendence degree of the extension and denoted by $tr \deg_K L$. If we have two extensions $K \subseteq L \subseteq M$ then $tr \deg_K M = tr \deg_L M + tr \deg_K L$. In particular, if $L \subseteq M$ is an algebraic extension then $tr \deg_K L = tr \deg_K M$. We will use these concepts in the context of germs of neighborhoods and ramified covergins.

Let $\widehat{X}$, X be complex manifolds and $\varpi : \widehat{X} \to X$ a ramified covering of order d with ramification locus $\widehat{Y}$ and branching locus Y, that is:

- $\widehat{Y}$ is the set of points $x \in \widehat{X}$ such that ϖ is not a local biholomorphism at x,
- $Y = \varpi(\widehat{Y})$ and
- The restriction $\varpi : \widehat{X} \setminus \widehat{Y} \to X \setminus Y$ defines an unbranching covering of order d.

We will denote this by $\varpi : (\widehat{X}, \widehat{Y}) \to (X, Y)$. The group

$$Deck(\varpi) = \left\{ \phi \in Aut(\widehat{X}) / \varpi \circ \phi = \varpi \right\}$$

of all automorphisms of the branched covering ϖ is called the deck transformation group. The ramified covering $\varpi : (\widehat{X}, \widehat{Y}) \to (X, Y)$ is said to be Galois if $Deck(\varpi)$ acts transitively on each fiber of ϖ.

Proposition 2.1. *Let $\varpi : (\widehat{X}, \widehat{Y}) \to (X, Y)$ be a Galois ramified covering of order d, then ϖ induces an algebraic extension $\mathbb{C}(X, Y) \subseteq \mathbb{C}(\widehat{X}, \widehat{Y})$. In particular*

$$tr \deg_{\mathbb{C}} \mathbb{C}(\widehat{X}, \widehat{Y}) = tr \deg_{\mathbb{C}} \mathbb{C}(X, Y).$$

Proof. Observe first that ϖ induces, by composition, a natural isomorphism

$$\mathbb{C}^{\varpi}(\widehat{X}, \widehat{Y}) := \{ f \in \mathbb{C}(\widehat{X}, \widehat{Y}) : f \circ \phi = f, \ \forall \phi \in Deck(\varpi) \} \simeq \mathbb{C}(X, Y).$$

Take now $h \in \mathbb{C}(\widehat{X}, \widehat{Y})$ and consider the set of functions $\{h \circ \phi \,/\, \phi \in Deck(\varpi)\} = \{h_1 = h, h_2, \ldots, h_d\}$. Then, if $s_j(x_1, \ldots, x_d)$ stands for the symmetric functions on $(x_1, \ldots, x_d)$ we clearly have that $s_j(h_1, \ldots, h_d) \in \mathbb{C}^{\varpi}(\widehat{X}, \widehat{Y})$. So the polynomial $p(T) := (T - h_1) \ldots (T - h_d)$ belongs to $\mathbb{C}^{\phi}(\widehat{X}, \widehat{Y})[T]$ and vanishes at $T = h$, then h is algebraic over $\mathbb{C}(X, Y)$ as we wanted. $\square$

For the sake of completeness we end this section recalling the method used in [5] applied to our particular case $Y \cong \mathbb{P}^1$.

Given a holomorphic foliation $\mathcal{F}$ on a complex manifold W, we say that a submanifold $Z \subset W \setminus \mathrm{sing}(\mathcal{F})$ is weakly transverse to $\mathcal{F}$ if $T_z Z \cap T_z \mathcal{F} = \{0\}$ for every $z \in Z$. For the case $W = \mathbb{P}^N$, the existence of projective manifolds of big codimension weakly transverse to a foliation by curves is achieved by applying Kleinman's transversality of a general translate [8, Theorem 2].

Lemma 2.2 (Lemma 3.5 of [5]). *Let $\mathcal{F}$ be a foliation by curves on $\mathbb{P}^{m+1}$ with isolated singularities. Let $Y \subset \mathbb{P}^{m+1}$ be a projective submanifold of dimension n. If $m \geq 2n$ then Y is weakly transverse to $g^*\mathcal{F}$ for any general $g \in Aut(\mathbb{P}^{m+1})$.*

Take now $Z = \mathbb{P}^1$ linearly embedded on $\mathbb{P}^3$ and $\mathcal{F}$ a foliation by curves weakly transverse to Y. Then for a sufficiently small tubular neighborhood U of Z in $\mathbb{P}^3$, the leaf space $X = U/\mathcal{F}$ of $\mathcal{F}|_U$ is a complex surface and the quotient morphism $\pi : U \to X$ has the following properties (see Proposition 3.1 and Lemma 3.2 of [5])

1. the leaves of $\mathcal{F}|_U$ coincide with the fibers of π; and
2. the morphism π maps Z isomorphically to a submanifold Y of X with normal bundle isomorphic to the quotient of $N_{Z/\mathbb{P}^3}$ by the image of $T_{\mathcal{F}}|_Z$ inside it; and
3. the field of germs of meromorphic functions $\mathbb{C}(X, Y)$ is isomorphic by π^* to the field of germs of meromorphic first integrals of $\mathcal{F}$, $\mathbb{C}(\mathcal{F})$.

3. Proof of Theorem A

Consider the foliation $\mathcal{F}$ on $\mathbb{P}^3$ defined by the vector field

$$v = \sum_{i=1}^{3} \lambda_i x_i \frac{\partial}{\partial x_i}.$$

with $\lambda_i \in \mathbb{C}$. Then the Zariski closure of the general leaf of $\mathcal{F}$ has dimension equal to the $\mathbb{Q}$-vector subspace of $\mathbb{C}$ generated by $\lambda_1, \ldots, \lambda_3$. Acording to [2], there exists a unique foliation by algebraic leaves $\mathcal{G}$ containing $\mathcal{F}$ and such that $\mathbb{C}(\mathcal{F}) = \mathbb{C}(\mathcal{G})$, in particular cod $\mathcal{G} = tr\deg_{\mathbb{C}} \mathbb{C}(\mathcal{F})$. Thus choosing appropriately the λ's we have that $\mathcal{G}$ has dimension $1, 2$ or 3 and so $\mathbb{C}(\mathcal{F})$ has any transcendence degree between

0 and 2. We conclude that for any $k \geq 1$ and any $l \in \{0,1,2\}$ there is a foliation $\mathcal{F}$ on $\mathbb{P}^3$ of degree k with $\mathbb{C}(\mathcal{F})$ of transcendence degree l.

We apply the previous construction to $\mathcal{F}$ and $Z = \mathbb{P}^1$ weakly transversal to $\mathcal{F}$ in order to obtain the germ of surface (X,Y) such that $Y \cong \mathbb{P}^1$ and $tr\deg_{\mathbb{C}} \mathbb{C}(X,Y) = l$. On the other hand, since $N_Z = \mathcal{O}_Z(1) \oplus \mathcal{O}_Z(1)$ and $T_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^3}(1-k)$ we have that $Y^2 = \deg N_{Y|X} = 2-(1-k) = k+1$. This proves the theorem for $Y^2 = d \geq 2$.

Take now a 2-neighborhood (X,Y) with $tr\deg_{\mathbb{C}} \mathbb{C}(X,Y) = l$ and consider the ramified covering

$$\varpi : (\widehat{X}, \widehat{Y}) \to (X,Y)$$

of order 2 totally ramifying over Y and inducing a cyclic covering of order 2 over $X - Y$. Then $\widehat{Y}^2 = 1$ and the pair $(\widehat{X}, \widehat{Y})$ is a 1-neighborhood. By Proposition 2.1 we have that $tr\deg_{\mathbb{C}} \mathbb{C}(\widehat{X}, \widehat{Y}) = l$.

Finally, for the case $l = 0$ we will give examples of neighborhoods with no singular webs (in particular without foliations). First observe that any web on X gives place by pull-back to a web on U tangent to $\mathbb{F}|_U$ and we can use [10, Theorem 3.2] to extend this web to $\mathbb{P}^3$ tangent to $\mathbb{F}$. On the other hand it follows from [7, Proposition 3.3] and [3] that a generic foliation $\mathbb{F}$ of degree $k \geq 2$ on $\mathbb{P}^3$ does not admit a tangent web. Therefore we can construct our desired neighborhood (X,Y) for $Y^2 = k+1 \geq 3$.

Take now a 3-neighborhood (X,Y) with no webs and consider the ramified covering

$$\varpi : (\widehat{X}, \widehat{Y}) \to (X,Y)$$

of order 3 totally ramifying over Y and inducing a cyclic covering of order 3 over $X - Y$ with Galois automorphism ϕ. Then $\widehat{Y}^2 = 1$ and $(\widehat{X}, \widehat{Y})$ does not admit any web. In fact, if $\widehat{\mathbb{W}}$ is a web on $\widehat{X}$ then $\widehat{\mathbb{W}} \boxtimes \phi^*(\widehat{\mathbb{W}}) \boxtimes (\phi^{\circ 2})^*(\widehat{\mathbb{W}})$ would be a web invariant by ϕ therefore inducing by push-forward a web on X, which is imposible.

By a similar argument, we can take a $2:1$ ramified covering of a $+4$-neighborhood (X,Y) in order to establish the case of self-intersection two.

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Resumen

Damos ejemplos de germenes de superficies conteniendo un curva racional con autointersección positiva y con cero, una o dos funciones meromorfas independientes.

Palabras clave: Germen de superficie. Función meromorfa. Foliación holomorfa.

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