Pro Mathematica, **33**, *65 (2024), 7-31*
DOI*: 10.18800/promathematica.202401.001*

MULTIPLY HÖLDER FUNCTIONS[a]

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Noviembre, 2024

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Abstract

In this article we show several properties about multiply Hölder functions. We study the Hölder class of a composition of multiply Hölder functions and prove that a map and its inverse belong — under certain hypotheses — to the same Hölder class. We also prove some extension properties of multiply Hölder functions; for example, we show that a multiply Hölder functions always extends, in the same Hölder class, to "exceptional" sets that are codimension one manifolds.

MSC(2020): 26A16 (primary), 2635 (secondary).

Keywords: *Lipschitz (Hölder) classes; Special properties of functions of several variables, Hölder conditions, etc.*

a. The author was supported by the Vicerrectorado the Investigación de la PUCP

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1. Introduction

Hölder functions are a classic topic in analysis with many applications in other areas, as the theory of partial differential equations and dynamical systems. However, besides the restricted treatment given in books of partial differential equations –see for example [2, 4]– a systematic study of the subject is still incomplete; the best reference in this sense is the book of R. Fiorenza [3].

Let $\alpha \in (0,1]$ and $U \subset \mathbb{R}^m$. A function $f\colon U \to \mathbb{R}^n$ is called α-Hölder if the set

$$\left\{ \frac{|f(x)-f(y)|}{|x-y|^\alpha} \colon x,y \in U,\ x \neq y \right\}$$

is bounded. In this case the function f is uniformly continuous, so it has a unique continuous extension to $\overline{U}$. Observe that the 1-Hölder functions are the Lipschitz functions and it is important to note that these functions are also α-Hölder for any $\alpha \in (0,1]$.

If U is open, if all partial derivatives of f up to order $k \geq 0$ exist and if the partial derivatives of order k of f are α-Hölder, then f is said to be of class $C^{k,\alpha}$: *these functions are called multiply Hölder functions.* If $U' \subset U$ and $f|_{U'}$ is of class $C^{k,\alpha}$, we say that f is of class $C^{k,\alpha}$ on U' and write $f \in C^{k,\alpha}(U')$. Clearly the functions of class $C^{0,\alpha}$ are the α-Hölder functions and a function of class $C^{k,\alpha}$ is necessarily of class C^k.

In this work we show several properties about multiply Hölder functions. In Section 2 we introduce the special class of subconvex open sets, which are specially adapted to the study of $C^{k,\alpha}$ functions (see [6, 1]). In Section 3, given a subconvex open set U, we define the inner distance to a boundary point of U. The main result of the section — Theorem 3.1 — provides a condition for a differentiable function on U to be α-Hölder continuous. In Section 4 we prove some general properties of $C^{k,\alpha}$ functions. Among other basic facts, we study the Hölder classes of compositions and inverses of $C^{k,\alpha}$ functions. Finally, in Section 5, we study some extension properties of $C^{k,\alpha}$ functions. For example, Theorem 5.2 shows that a $C^{k,\alpha}$ function defined around a C^1 manifold of codimension ≥ 2 always extends to the manifold.

2. Subconvex Sets

Let $U \subset \mathbb{R}^n$ be an open connected set. Given $x, y \in U$, we define $\text{dist}_U(x, y)$ as the infimum of the lengths of the rectifiable curves in U connecting x with y. The function

$$\text{dist}_U : U \times U \to [0, +\infty)$$

defines a metric called the inner metric of U. This metric is the intrinsic metric induced by the euclidean metric in U — for more details see [5]. The set U will be said subconvex if the inner metric dist_U is strongly equivalent to the euclidean metric: this is equivalent to the existence of a constant $d > 0$ such that

$$\text{dist}_U(x, y) < d|x - y|, \quad x, y \in U.$$

In this case we say that U is subconvex of constant $d > 0$.

The following proposition establish some basic facts about subconvex sets. In particular, the proposition gives several examples of subconvex sets that are not convex sets.

Proposition 2.1.

1. *A convex set is subconvex.*

2. *If $U \subset \mathbb{R}^m$ is subconvex and $M \subset U$ is a smooth manifold of codimension ≥ 2, then $U \backslash M$ is subconvex.*

3. *If $U \subset \mathbb{R}^m$ is open, bounded and connected, and its boundary ∂U is a smooth hypersurface, then U is subconvex.*

4. *Let $V \subset \mathbb{R}^m$ be open and $K \subset V$ be compact and connected. Then there exists $U \subset V$ open, bounded and subconvex such that $K \subset U$.*

5. *If $U \subset \mathbb{R}^m$ is subconvex and $f\colon U \to \mathbb{R}^n$ is bilipschitz, then $f(U)$ is subconvex.*

6. *Let $U \subset \mathbb{R}^m$ be subconvex and let $f\colon U \to \mathbb{R}^n$ be differentiable such that $|df|$ is bounded on U. Then f is Lipschitz.*

Proof. The first assertion is obvious.

Let $U \subset \mathbb{R}^m$ be subconvex of constant d and let $M \subset U$ be a smooth manifold of codimension ≥ 2. Let $x, y \in U$ be distinct. Then

$$\mathrm{dist}_U(x, y) < d|x - y|$$

and, from the definition of $\mathrm{dist}_U(x, y)$, we can find a curve γ in U connecting x with y, such that

$$\ell(\gamma) < d|x - y|.$$

Since M has codimension ≥ 2, the curve γ can be deformed into a curve γ' in U avoiding the manifold M and being close to γ such that

$$\ell(\gamma') < d|x - y|.$$

Therefore assertion (2) is proved.

Let $U \subset \mathbb{R}^m$ be open, bounded and connected, and suppose that ∂U is a smooth hypersurface. If S is a connected component of ∂U, let

$$d_S\colon S \times S \to [0, +\infty)$$

be the geodesic metric in S. Since S is a compact hypersurface, the geodesic metric d_S is equivalent to the euclidean metric, so there exists $c \geq 1$ such that

$$d_S(x, y) \leq c|x - y|, \quad x, y \in S. \tag{2.1}$$

Since ∂U has finitely many connected components, we can assume that the constant c is independent of the component S. Let $x, y \in U$ be distinct. Deforming the euclidean segment from x to y we find a smooth simple curve γ connecting x with y, transverse to ∂U and such that

$$\ell(\gamma) < 2|x-y|. \tag{2.2}$$

Regarded as a set, γ has a total order induced by its orientation such that $x < y$. Thus, given $p, q \in \gamma$ with $p < q$, we denote by $[p, q]$ the compact segment of γ from p to q, and by (p, q) the open segment $[p, q]\backslash\{p, q\}$. Recall that each connected component of ∂U, since it is a compact hypersurface, separates the space $\mathbb{R}^m$ in two connected components. Therefore, for some $k \in \mathbb{N}$ we find points

$$x = p_1 < p_2 < \ldots < p_{2k} = y$$

in γ with the following properties:

- $(p_{2j-1}, p_{2j}) \subset U$, $j = 1, \ldots, k$.
- $(p_{2j}, p_{2j+1}) \subset \mathbb{R}^m\backslash U$, $j = 1, \ldots, k-1$.
- For each $j = 1, \ldots, k-1$, the points p_{2j} and p_{2j+1} belong to the same component of ∂U.

It follows from the last property and (2.1) that

$$d_S(p_{2j}, p_{2j+1}) \leq c|p_{2j} - p_{2j+1}|, \quad j = 1, \ldots, k-1.$$

Thus if α_j is the minimal geodesic in ∂U connecting p_{2j} with p_{2j+1}, we have

$$\ell(\alpha_j) \leq c|p_{2j} - p_{2j+1}|, \quad j = 1, \ldots, k-1.$$

Let γ' be the curve obtained from γ by replacing each $[p_{2j}, p_{2j+1}]$ by α_j.

Then

$$\begin{aligned}\ell(\gamma') &= \sum_{j=1}^{k} \ell([p_{2j-1}, p_{2j}]) + \sum_{j=1}^{k-1} \ell(\alpha_j)\\ &\leq \sum_{j=1}^{k} \ell([p_{2j-1}, p_{2j}]) + \sum_{j=1}^{k-1} c|p_{2j} - p_{2j+1}|\\ &\leq \sum_{j=1}^{k} \ell([p_{2j-1}, p_{2j}]) + c\sum_{j=1}^{k-1} \ell([p_{2j}, p_{2j+1}])\\ &\leq c\ell(\gamma),\end{aligned}$$

so from (2.2),

$$\ell(\gamma') < 2c|x-y|.$$

Then we can deform $\gamma' \subset \overline{U}$ into a curve $\gamma'' \subset U$ connecting x with y, such that we also have

$$\ell(\gamma'') < 2c|x-y|,$$

which finishes the proof of assertion (3).

Assertion (4) is a consequence of assertion (3): we only have to take U being open, bounded, connected, with

$$K \subset U \subset V,$$

such that ∂U is a smooth hypersurface. So (4) holds.

Now, let $U \subset \mathbb{R}^m$ be subconvex and let $f\colon U \to \mathbb{R}^n$ be a bilipschitz map. Then there are constants $c_1, c_2 > 0$ such that

$$c_1|x-y| \leq |f(x) - f(y)| \leq c_2|x-y|, \quad x, y \in U. \tag{2.3}$$

This guarantees that f is a homeomorphism, whence $f(U)$ is open and connected. Let $u, v \in f(U)$ be distinct. Since U is subconvex there exists $d > 0$ such that

$$\operatorname{dist}_U(x,y) < d|x-y|, \quad x, y \in U.$$

Then, since $f^{-1}(u), f^{-1}(v) \in U$, from the definition of dist_U we can find a rectifiable curve γ in U, connecting $f^{-1}(u)$ with $f^{-1}(v)$, such that

$$\ell(\gamma) < d\left|f^{-1}(u) - f^{-1}(v)\right|. \tag{2.4}$$

On the other hand, since f is Lipschitz of constant c_2, the curve $f(\gamma)$, which connects u with v, is such that

$$\ell(f(\gamma)) \leq c_2\ell(\gamma).$$

From this, (2.4)and (2.3) we obtain

$$\begin{aligned}\ell(f(\gamma)) &\leq c_2\ell(\gamma) \leq c_2 d\left|f^{-1}(u) - f^{-1}(v)\right| \\ &\leq c_2 d(1/c_1)|u-v|,\end{aligned}$$

which proves that $f(U)$ is subconvex.

Finally, assume the hypotheses of assertion (6). Then there exists a constant $C > 0$ such that

$$|df(x)| \leq C, \quad x \in U.$$

Given $x, y \in U$, since U is subconvex we can find a smooth curve $\gamma\colon [0,1] \to U$ with $\gamma(0) = x$ and $\gamma(1) = y$, such that

$$\ell(\gamma) < d|x-y|,$$

where $d > 0$ is a constant depending only on U. Then

$$\begin{aligned}|f(y) - f(x)| &= \left|\int_0^1 (f(\gamma(t)))'\,dt\right| \leq \int_0^1 |df(\gamma(t))|\,|\gamma'(t)|\,dt \\ &\leq C\int_0^1 |\gamma'(t)|\,dt = C\ell(\gamma) \\ &\leq Cd|x-y|,\end{aligned}$$

so assertion (6) is proved. □

3. Differentiability and the Hölder condition

Let $U \subset \mathbb{R}^n$ be a subconvex open set and consider a continuous function $f\colon U \to \mathbb{R}^n$. It is well known that, if f is differentiable, the Hölder property for f is strongly related with the geometry of ∂U and the behavior of f in relation with ∂U. In this section we prove that, if f extends continuously to a compact set $K \subset \partial U$, if $f|_K$ is α-Hölder and $df(x)$ is bounded in terms of the "inner" distance from x to K, then f α-Hölder.

Inner distance to the boundary

Let U be an open subconvex proper subset of $\mathbb{R}^m$. Given $x_0 \in U$ and $x_\infty \in \partial U$, we define the inner distance from x_0 to x_∞ as the infimum of the lengths of the continuous rectifiable curves

$$\gamma\colon [0,1) \to U, \quad \gamma(0) = x_0, \quad \lim_{t\to 1} \gamma(t) = x_\infty. \tag{3.1}$$

Let us see that this infimum is finite. Given

$$r > |x_0 - x_\infty|,$$

we can take a sequence of points $x_1, x_2, \ldots$ in U with $x_j \to x_\infty$ as $j \to \infty$, such that

$$|x_0 - x_1| + |x_1 - x_2| + |x_2 - x_3| + \ldots < r.$$

If U is subconvex of constant $d > 0$, for each $j \in \mathbb{N}$ we can take a continuous rectifiable curve γ_j connecting x_{j-1} with x_j, such that

$$\ell(\gamma_j) < d|x_{j-1} - x_j|.$$

Then the infinite juxtaposition

$$\gamma\colon = \gamma_1 * \gamma_2 * \cdots$$

defines a curve as in (3.1) such that

$$\ell(\gamma) < dr.$$

Therefore the infimum of the lengths of the curves in (3.1) is a finite number. We keep the notation $\mathrm{dist}_U(x_0, x_\infty)$ for the inner distance between $x_0 \in U$ and $x_\infty \in \partial U$. We note that, since the number $r > |x_0 - x_\infty|$ above is arbitrarily chosen, we have

$$\mathrm{dist}_U(x_0, x_\infty) \leq d|x_0 - x_\infty|.$$

Now, let $K \subset \partial U$ be nonempty. Given $x \in U$, we define the inner distance from x to K as

$$\delta(x) \colon = \inf\{\mathrm{dist}_U(x, x_\infty) \colon x_\infty \in K\}.$$

Theorem 3.1. *Let U be an open subconvex proper subset of $\mathbb{R}^m$ and consider $K \subset \partial U$ nonempty. Let $f \colon U \cup K \to \mathbb{R}^n$ be continuous, such that $f|_K$ is α-Hölder for some $\alpha \in (0,1]$. Given $x \in U$, let $\delta(x)$ denote the inner distance in U from x to K. Suppose that f is differentiable in U and such that, for some $c > 0$,*

$$|df(x)| \leq c\delta(x)^{\alpha-1}, \quad x \in U. \tag{3.2}$$

Then f is α-Hölder.

We begin with an elementary lemma.

Lemma 3.2. *Let $g \colon [0,1] \to \mathbb{R}^n$ be continuous and such that, for some $c > 0$ and $\alpha \in (0,1]$, we have $|g'(t)| \leq ct^{\alpha-1}$ for all $t \in (0,1)$. Then*

$$|g(u) - g(v)| \leq \frac{c}{\alpha}|u - v|^\alpha, \quad u, v \in [0,1].$$

Proof. Suppose that $1 > u \geq v > 0$. Given $n \in \mathbb{N}$, let $v = t_0 < t_1 < \ldots < t_n = u$ be the regular partition of norm $(u - v)/n$. By the Mean Value Theorem, there exists numbers $s_j \in [t_{j-1}, t_j]$ such that $g(t_j) - g(t_{j-1}) = g'(s_j)(u-v)/n$. Then

$$\begin{aligned} |g(u)-g(v)| \leq \sum_{j=1}^{n} |g(t_j)-g(t_{j-1})| &\leq \sum_{j=1}^{n} |g'(s_j)|(u-v)/n \\ &\leq \sum_{j=1}^{n} c s_j^{\alpha-1}(u-v)/n. \end{aligned}$$

Thus, if $n \to \infty$, we obtains that

$$|g(u)-g(v)| \leq \int_{v}^{u} ct^{\alpha-1} dt = \frac{c}{\alpha}\left(u^\alpha - v^\alpha\right) \leq \frac{c}{\alpha}(u-v)^\alpha,$$

so the result easily follows. □

Proof of Theorem 3.1. It is enough to show that $f|_U$ is α-Hölder. Let $x, y \in U$ be arbitrary. Suppose that U is subconvex of constant $d > 0$.

Case 1: Suppose that $\delta(x) < (d+1)|x-y|$ and $\delta(y) < (d+1)|x-y|$. Since

$$\delta(x) < (d+1)|x-y|,$$

there exist $x_\infty \in K$ and a smooth rectifiable curve

$$\gamma\colon (0,1] \to U, \quad \gamma(1) = x, \quad \lim_{t\to 0} \gamma(t) = x_\infty$$

such that

$$\ell(\gamma) < (d+1)|x-y|.$$

Clearly we can continuously extend γ to $[0,1]$ defining $\gamma(0) = x_\infty$. Moreover we can assume that γ has constant velocity, that is

$$|\gamma'(t)| = \ell(\gamma), \quad t \in (0,1].$$

Define the function

$$g(t) = f(\gamma(t)), \quad t \in [0,1].$$

Then, if $t \in (0,1]$,

$$|g'(t)| = |df(\gamma(t))||\gamma'(t)| \leq c[\delta(\gamma(t))]^{\alpha-1}\ell(\gamma). \tag{3.3}$$

Since the curve $\gamma|_{(0,t]}$ connects $x_\infty \in K$ with $\gamma(t) \in U$, we have

$$\delta(\gamma(t)) \leq \ell\left(\gamma|_{(0,t]}\right) = t\ell(\gamma),$$

so from (3.3) we obtain

$$|g'(t)| \leq c\ell(\gamma)^\alpha t^{\alpha-1}. \tag{3.4}$$

Then it follows from Lemma 3.2 that

$$|g(u) - g(v)| \leq \frac{c}{\alpha}\ell(\gamma)^\alpha|u-v|^\alpha, \quad u, v \in [0,1].$$

In particular,

$$|f(x) - f(x_\infty)| = |g(1) - g(0)| \leq \frac{c}{\alpha}\ell(\gamma)^\alpha$$

and so, since $\ell(\gamma) < (d+1)|x-y|$,

$$|f(x) - f(x_\infty)| \leq (d+1)^\alpha\frac{c}{\alpha}|x-y|^\alpha.$$

Analogously, we find $y_\infty \in K$ such that

$$|f(y) - f(y_\infty)| \leq (d+1)^\alpha\frac{c}{\alpha}|x-y|^\alpha.$$

On the other hand, since $f|_{\overline{K}}$ is α–Hölder, there exists a constant $c_\infty > 0$ depending only on f such that

$$|f(x_\infty) - f(y_\infty)| \leq c_\infty|x_\infty - y_\infty|^\alpha.$$

Using the last three inequalities together we finally obtain

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f(x_\infty)| + |f(y) - f(y_\infty)| + |f(x_\infty) - f(y_\infty)| \\
&\leq 2(d+1)^\alpha\frac{c}{\alpha}|x-y|^\alpha + c_\infty|x_\infty - y_\infty|^\alpha \\
&\leq 2(d+1)^\alpha\frac{c}{\alpha}|x-y|^\alpha + c_\infty\Big(\delta(x) + |x-y| + \delta(y)\Big)^\alpha \\
&\leq 2(d+1)^\alpha\frac{c}{\alpha}|x-y|^\alpha + c_\infty\Big((2d+3)|x-y|\Big)^\alpha \\
&\leq \Big(2(d+1)^\alpha\frac{c}{\alpha} + c_\infty(2d+3)^\alpha\Big)|x-y|^\alpha.
\end{aligned}$$

Case 2: If the first case does not happen, without loss of generality we can assume that

$$\delta(x) \le \delta(y) \quad \text{and} \quad (d+1)|x-y| \le \delta(y).$$

Then, since U is subconvex of constant d, we find a smooth curve

$$\gamma\colon [0,1] \to U, \quad \gamma(0) = x, \quad \gamma(1) = y,$$

such that

$$\ell(\gamma) < d|x-y|.$$

We assume that γ has constant velocity:

$$|\gamma'(t)| = \ell(\gamma), \quad t \in [0,1].$$

From the definition of inner distance we see that, for any $t \in [0,1]$,

$$\delta(y) \le \mathrm{dist}_U(y, \gamma(t)) + \delta(\gamma(t)).$$

Thus, since $\mathrm{dist}_U(y, \gamma(t)) \le \ell(\gamma)$,

$$\delta(y) \le \ell(\gamma) + \delta(\gamma(t)),$$

whence

$$\begin{aligned} \delta(\gamma(t)) &\ge \delta(y) - \ell(\gamma) \ge (d+1)|x-y| - \ell(\gamma) \\ &\ge (d+1)|x-y| - d|x-y| \\ &\ge |x-y|. \end{aligned}$$

Therefore

$$\begin{aligned} |f(y) - f(x)| &= \left| \int_0^1 df\big(\gamma(t)\big)\gamma'(t)dt \right| \le \int_0^1 c\,[\delta(\gamma(t))]^{\alpha-1}\,|\gamma'(t)|dt \\ &\le \int_0^1 c|x-y|^{\alpha-1}\ell(\gamma)dt \le \int_0^1 cd|x-y|^{\alpha}dt \\ &\le cd|x-y|^{\alpha}. \end{aligned}$$

□

4. Some general properties

In this section we prove some general properties of $C^{k,\alpha}$ functions. For example, Theorem 4.2 gives the Hölder class of a composition of multiply Hölder functions, and Theorem 4.3 gives conditions for the inverse of a $C^{k,\alpha}$ function to be a $C^{k,\alpha}$ function. These theorems complement some known similar results; see Proposition 1.2.7 and Theorem 1.3.4 in [3]. We start with the following property summarizing some elementary properties of multiply Hölder functions. Although these properties can be founded in [3], we include a complete proof here for the convenience of the reader.

Theorem 4.1. *Let $U \subset \mathbb{R}^m$ be open, bounded and subconvex, and consider $f\colon U \to \mathbb{R}^n$, $g\colon U \to \mathbb{R}^p$, $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0,1]$.*

1. *If $f \in C^{k,\alpha}(U)$, then f is bounded.*
2. *If $f \in C^{k+1,\alpha}(U)$, then $f \in C^{k,1}(U)$ — in particular $f \in C^{k,\alpha}(U)$.*
3. *If $f \in C^{k,\alpha}(U)$, then f extends to $\overline{U}$ as an α-Hölder function.*
4. *If $n = p$ and $f, g \in C^{k,\alpha}(U)$, then $f \pm g \in C^{k,\alpha}(U)$.*
5. *If $H\colon \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$ is bilinear and $f, g \in C^{k,\alpha}(U)$, then $H(f,g)$ is of class $C^{k,\alpha}$.*
6. *If f extends as a C^{k+1} function on a neighborhood of $\overline{U}$, then $f \in C^{k,\alpha}(U)$.*

Proof. Suppose that $f \in C^{0,\alpha}(U)$. Then there exists $c > 0$ such that

$$|f(y) - f(x)| \leq c|y - x|^{\alpha}, \quad x, y \in U.$$

Fix $a \in U$ and let $x \in U$ be arbitrary. Then, since $|f(x) - f(a)| \leq c|x - a|^{\alpha}$,

$$|f(x)| \leq |f(a)| + c|x - a|^{\alpha} \leq |f(a)| + c(\mathrm{Diam} U)^{\alpha},$$

which proves item (1) for $k = 0$. Then, item (1) will be proved if we prove item (2): if $f \in C^{k,\alpha}(U)$, by successively applications of item (2) we have $f \in C^{0,\alpha}(U)$, whence — by item (1) for $k = 0$ — the function f is bounded. Let us prove item (2). Suppose that $f \in C^{1,\alpha}(U)$. Then any partial derivative $\frac{\partial f}{\partial x_i}$ belongs to $C^{0,\alpha}(U)$. Thus, by item (1) for $k = 0$ the functions $\frac{\partial f}{\partial x_i}$ are bounded. Then it follows from (6) of Proposition 2.1 that f is Lipschitz, which proves item (2) for $k = 0$. Suppose now that $f \in C^{k+1,\alpha}(U)$. Then any partial derivative of order k of f belongs to $C^{1,\alpha}(U)$. So, by item (2) for $k = 0$, any partial derivative of order k of f belongs to $C^{0,1}(U)$, which means that $f \in C^{k,1}(U)$; item (2) is proved.

If $f \in C^{k,\alpha}(U)$, it follows from item (2) that $f \in C^{0,\alpha}(U)$, so f extends to $\overline{U}$ as an α-Hölder function and item (3) is proved.

It suffices to prove item (4) for the sum of functions; the other case is similar. Suppose that $f, g \in C^{0,\alpha}(U)$. Then we can find $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^{\alpha}, \quad |g(x) - g(y)| \leq c|x - y|^{\alpha}, \quad x, y \in U.$$

Therefore

$$\begin{aligned} |(f(x) + g(x)) - (f(y) + g(y))| &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &\leq 2c|x - y|^{\alpha}, \end{aligned}$$

which proves item (4) for $k = 0$. Suppose that item (4) is true for $k = l$, and let $f, g \in C^{l+1,\alpha}(U)$. Then the partial derivatives

$$\frac{\partial f}{\partial x_i}, \quad \frac{\partial g}{\partial x_i}, \quad i = 1, \ldots, m$$

belong to $\in C^{l,\alpha}(U)$. Thus, by the inductive hypothesis, the partial derivatives

$$\frac{\partial (f + g)}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}, \quad i = 1, \ldots, m$$

belong to $C^{l,\alpha}(U)$. Therefore $f + g \in C^{l+1,\alpha}(U)$, so item (4) is proved.

Now, assume the hypotheses of item (5). Since H is bilinear there is a constant $c_H > 0$ such that

$$|H(u,v)| \leq c_H |u||v|, \quad u \in \mathbb{R}^n, v \in \mathbb{R}^p.$$

Suppose first that $k = 0$. Then there exists $c > 0$ such that

$$|f(x) - f(y)| \leq c|x-y|^\alpha, \quad |g(x) - g(y)| \leq c|x-y|^\alpha, \quad x, y \in U.$$

Moreover, by item (1) there exist $C > 0$ such that

$$|f(x)| \leq C, \quad |g(x)| \leq C, \quad x \in U.$$

Thus, if $x, y \in U$,

$$\begin{aligned}
|H(f(x), g(x)) - H(f(y), g(y))| \\
&\leq |H(f(x), g(x)) - H(f(x), g(y))| \\
&\qquad + |H(f(x), g(y)) - H(f(y), g(y))| \\
&\leq |H(f(x), g(x) - g(y))| + |H(f(x) - f(y), g(y))| \\
&\leq c_H |f(x)||g(x) - g(y)| + c_H |f(x) - f(y)||g(y)| \\
&\leq c_H C \left(c|x-y|^\alpha\right) + c_H \left(c|x-y|^\alpha\right) C \\
&\leq 2 c_H C c |x-y|^\alpha,
\end{aligned}$$

which proves item (5) for $k = 0$. Suppose item (5) is true for $k = l$, and let $f, g \in C^{l+1,\alpha}(U)$. Since $f, g \in C^{l+1,\alpha}(U)$, for any $i \in \{1, \ldots, m\}$ we have $\frac{\partial f}{x_i}, \frac{\partial g}{\partial x_i} \in C^{l,\alpha}(U)$ and — by item (2) — we also have $f, g \in C^{l,\alpha}(U)$. Then, by the inductive hypothesis, $H\left(\frac{\partial f}{x_i}, g\right)$ and $H\left(f, \frac{\partial g}{\partial x_i}\right)$ belong to $C^{l,\alpha}(U)$, whence — by item (4) — the partial derivative

$$\frac{\partial H(f,g)}{\partial x_i} = H\left(\frac{\partial f}{x_i}, g\right) + H\left(f, \frac{\partial g}{\partial x_i}\right)$$

belongs to $C^{l+1,\alpha}(U)$. Therefore item (5) is proved.

Finally, assume that f extends as a C^{k+1} function on a neighborhood of $\overline{U}$. Let $\mathfrak{f}$ be a partial derivative of order k of f. Then, since f extends in the class C^{k+1} to a neighborhood of the compact set $\overline{U}$, each partial derivative of $\mathfrak{f}$ extends continuously to that neighborhood of $\overline{U}$, so the partial derivatives of $\mathfrak{f}$ are bounded on U. It follows from (6) of Proposition 2.1 that $\mathfrak{f}$ is Lipschitz. That is, $\mathfrak{f} \in C^{0,\alpha}(U)$, which means that $f \in C^{k,\alpha}(U)$. □

Theorem 4.2. *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. Let $f\colon U \to V$ be of class $C^{k,\alpha}$ and $g\colon V \to \mathbb{R}^p$ of class $C^{k,\beta}$, where $k \in \mathbb{Z}_{\geq 0}$ and $\alpha, \beta \in (0,1]$. Then, if U is bounded and subconvex, $g \circ f$ is of class $C^{k,\alpha\beta}$.*

Proof. Suppose first that $f \in C^{0,\alpha}(U)$ and $g \in C^{0,\beta}(V)$. Thus there are constants $c_f, c_g > 0$ such that

$$|f(x) - f(y)| \leq c_f |x - y|^\alpha, \quad x, y \in U$$

and

$$|g(x) - g(y)| \leq c_g |x - y|^\beta, \quad x, y \in V.$$

Then, if $x, y \in U$,

$$\begin{aligned} |g \circ f(x) - g \circ f(y)| &\leq c_g |f(x) - f(y)|^\beta \leq c_g \left(c_f |x - y|^\alpha\right)^\beta \\ &\leq c_g c_f^\beta |x - y|^{\alpha\beta}, \end{aligned}$$

which proves the proposition for $k = 0$. Suppose as inductive hypothesis that the proposition holds for some $k \in \mathbb{Z}_{\geq 0}$. Let $f \in C^{k+1,\alpha}(U)$ and $g \in C^{k+1,\beta}(V)$. Then $df \in C^{k,\alpha}(U)$ and $dg \in C^{k,\beta}(V)$ and, by (2) of Proposition 4.1, we also have $f \in C^{k,\alpha}(U)$. It follows from the inductive hypothesis that $dg(f) \in C^{k,\alpha\beta}(U)$. Then, since

$$df \in C^{k,\alpha}(U) \subset C^{k,\alpha\beta}(U),$$

from (5) of Proposition 4.1 we see that

$$d(g \circ f) = dg(f) \cdot df$$

belongs to $C^{k,\alpha\beta}(U)$, which means that $g \circ f \in C^{k+1,\alpha\beta}(U)$. □

Corollary 4.3. *Let $U \subset \mathbb{R}^m$ be open, bounded and subconvex, and let $f\colon U \to \mathbb{R}$ be of class $C^{k,\alpha}$, where $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0,1]$. If $|f| > \epsilon$ for some $\epsilon > 0$, then $1/f$ is of class $C^{k,\alpha}$.*

Proof. Since $|f| > \epsilon$ and $f(U)$ is connected, without loss of generality we can assume that

$$f(U) \subset (\epsilon, +\infty).$$

Furthermore, from (1) of Proposition 4.1 we find $c > 0$ such that

$$f(U) \subset V \colon = (\epsilon, c).$$

Since (6) of Proposition 4.1 guarantees that $g\colon V \to \mathbb{R}$ defined by $g(x) = 1/x$ is of class $C^{k,1}$, the corollary follows from Proposition 4.2 □

Theorem 4.4. *Let $U \subset \mathbb{R}^m$ be a bounded subconvex open set and let $f\colon U \to \mathbb{R}^m$ be of class $C^{k,\alpha}$ for $k \geq 1$ and $\alpha \in (0,1]$. From Proposition 4.1 we see that $f\colon U \to \mathbb{R}^m$ and $df\colon U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ extend continuously to $\overline{U}$. Suppose that the extension of f to $\overline{U}$ is univalent and the extension of df to $\overline{U}$ takes values in $GL(m,\mathbb{R})$. Then $f(U)$ is a bounded subconvex open set and $f^{-1}\colon f(U) \to U$ is of class $C^{k,\alpha}$.*

Proof. We still denote by f and df the extensions of f and df to $\overline{U}$. Since $k \geq 1$, from (1) and (2) of Proposition 4.1 the partial derivatives of f are bounded. Then, it follows from (6) of Proposition 2.1 that f is Lipschitz. Let us prove that $f^{-1}\colon f(U) \to U$ is also Lipschitz. Otherwise, there are points $x_n, y_n \in U$, $n \in \mathbb{N}$ such that

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.1}$$

Since U is bounded we can assume that, for some $a, b \in \overline{U}$,

$$x_n \to a, \quad y_n \to b \quad \text{as} \quad n \to \infty.$$

Thus, if $a \neq b$ we $f(a) \neq f(b)$ and therefore

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} \to \frac{|f(a) - f(b)|}{|a - b|} \neq 0, \tag{4.2}$$

which contradicts (4.1). So we assume $a = b$. Since $df(a) \in \mathrm{GL}(n, \mathbb{R})$, there exists a constant $c_a > 0$ such that

$$|df(a) \cdot v| \geq 2c_a|v|, \quad v \in \mathbb{R}^n \backslash \{0\}.$$

On the other hand, suppose that U is subconvex of constant d. Then, since df is continuous on $\overline{U}$, there is a neighborhood Ω of a in $\overline{U}$ such that

$$|df(x) - df(a)| \leq c_a/d, \quad x \in \Omega. \tag{4.3}$$

Since U is subconvex of constant d, we can find a smooth curve $\gamma\colon [0,1] \to U$ such that $\gamma(0) = x_n$, $\gamma(1) = y_n$ and

$$\ell(\gamma) < d|x_n - y_n|.$$

We can assume x_n and y_n to be so close to a such that the curve γ is contained in Ω. Then it follows from (4.3) that

$$|df(\gamma(t)) - df(a)| \leq c_a/d, \quad t \in [0,1]. \tag{4.4}$$

Thus, if we define

$$g(t) = f(\gamma(t)), \quad t \in [0,1],$$

we can write

$$g(t) = df(a) \cdot \gamma(t) + \delta(t),$$

where we have

$$\begin{aligned} |\delta'(t)| &= |df(\gamma(t)) \cdot \gamma'(t) - df(a) \cdot \gamma'(t)| \\ &= |\,[df(\gamma(t)) - df(a)] \cdot \gamma'(t)| \\ &\leq |df(\gamma(t)) - df(a)| \cdot |\gamma'(t)| \\ &\leq (c_a/d)|\gamma'(t)|. \end{aligned}$$

Therefore

$$\begin{aligned}
|f(y_n) - f(x_n)| &= \left| \int_0^1 g'(t)dt \right| = \left| \int_0^1 (df(a) \cdot \gamma'(t) + \delta'(t))dt \right| \\
&\geq \left| \int_0^1 df(a) \cdot \gamma'(t)dt \right| - \left| \int_0^1 \delta'(t)dt \right| \\
&\geq \left| df(a) \cdot (y_n - x_n) \right| - (c_a/d) \int_0^1 |\gamma'(t)|dt \\
&\geq 2c_a|y_n - x_n| - (c_a/d)\ell(\gamma) \\
&\geq 2c_a|y_n - x_n| - (c_a/d)(d|y_n - x_n|) \\
&\geq c_a|y_n - x_n|,
\end{aligned}$$

whence we conclude that

$$\frac{|f(y_n) - f(x_n)|}{|y_n - x_n|} \geq c_a$$

for n large enough, which contradicts (4.1).

Now, the fact of f being bilipschitz implies that $f(U)$, like U, is open, bounded and subconvex.

Since df extends continuously to $\overline{U}$ and takes values in $\mathrm{GL}(n, \mathbb{R})$, the image of df is a compact connected set $K \subset \mathrm{GL}(n, \mathbb{R})$. Thus, from (4) of Proposition 2.1 we can find an open bounded subconvex set $\mathcal{U}$ in the space of $n \times n$ matrices, such that

$$K \subset \mathcal{U} \quad \text{and} \quad \overline{\mathcal{U}} \subset \mathrm{GL}(n, \mathbb{R}).$$

It follows from (6) of Proposition 4.1 that the function

$$\begin{aligned}
\mathcal{I} \colon \mathcal{U} &\to \mathrm{GL}(n, \mathbb{R}) \\
M &\mapsto M^{-1}
\end{aligned}$$

is of class $C^{l,1}$ for all $l \geq 0$. Since $k \geq 1$, the derivative of f^{-1} can be expressed as

$$df^{-1} = \mathcal{I} \circ df \circ f^{-1}. \tag{4.5}$$

Suppose first that $k = 1$. Then $df\colon U \to \mathrm{GL}(n, \mathbb{R})$ is of class $C^{0,\alpha}$. Thus, since f^{-1} is of class $C^{0,1}$, it follows from Theorem 4.2 that

$$df \circ f^{-1}\colon f(U) \to \mathcal{U}$$

is of class $C^{0,\alpha}$. Therefore, since $\mathcal{I} \in C^{0,1}$, it follows from (4.5) and Theorem 4.2 that df^{-1} is of class $C^{0,\alpha}$. Then f^{-1} is of class $C^{1,\alpha}$, so Theorem 4.4 holds true for $k = 1$. Suppose that Theorem 4.4 holds true for $k = l \geq 1$ and let f be satisfying the hypotheses of Theorem 4.4 for $k = l + 1$. Since f is of class $C^{l+1,\alpha}$, by (2) of Proposition 4.1 we have $f \in C^{l,1}(U)$ and therefore, by the inductive hypothesis, f^{-1} is also of class $C^{l,1}$. Then, since df is of class $C^{l,\alpha}$, it follows from Theorem 4.2 that $df \circ f^{-1}$ is of class $C^{l,\alpha}$. Therefore, since $\mathcal{I} \in C^{l,1}$, it follows from (4.5) and Theorem 4.2 that df^{-1} is of class $C^{l,\alpha}$, which means that f^{-1} is of class $C^{l+1,\alpha}$. □

5. Extension properties of multiply Hölder functions

In this section we prove a couple of results about the extension properties of $C^{k,\alpha}$ functions.

Proposition 5.1. *Let $U \subset \mathbb{R}^m$ be a subconvex open set, let $U_1, U_2 \subset U$ be such that $U \subset \overline{U_1} \cup \overline{U_2}$, let $f\colon U \to \mathbb{R}^n$ be continuous, and consider $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0, 1]$. Then the following properties hold:*

1. *If $f|_{U_1}$ and $f|_{U_2}$ are α-Hölder, then f is α-Hölder.*
2. *If $f \in C^k$, if the sets U_1 and U_2 are open, and if $f|_{U_1}$ and $f|_{U_2}$ are of class $C^{k,\alpha}$, then f is of class $C^{k,\alpha}$.*

Proof. Suppose that $f|_{U_1}$ and $f|_{U_2}$ are α-Hölder. Then we find $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^{\alpha} \quad \text{whenever} \quad x, y \in \overline{U_1} \text{ or } x, y \in \overline{U_2}. \tag{5.1}$$

Then, since $U \subset \overline{U_1} \cup \overline{U_2}$, it suffices to find an inequality like above when $x \in \overline{U_1}$, $y \in \overline{U_2}$ and when $x \in \overline{U_2}$, $y \in \overline{U_1}$. We assume $x \in \overline{U_1}$, $y \in \overline{U_2}$ — the other case is similar. Since U is subconvex, we find a smooth curve γ in U connecting x with y such that

$$\ell(\gamma) < d|x - y|, \tag{5.2}$$

where the constant $d > 0$ depends only on U. Since γ is connected, $\gamma \subset \overline{U_1} \cup \overline{U_2}$ and γ meets both sets $\overline{U_1}$ and $\overline{U_2}$, we can find $z \in \gamma$ such that

$$z \in \overline{U_1} \cap \overline{U_2}.$$

Thus, from (5.1) we have

$$|f(x) - f(z)| \leq c|x - z|^{\alpha} \quad \text{and} \quad |f(z) - f(y)| \leq c|z - y|^{\alpha},$$

whence

$$|f(x) - f(y)| \leq c|x - z|^{\alpha} + c|z - y|^{\alpha} \leq c\ell(\gamma)^{\alpha} + c\ell(\gamma)^{\alpha} \leq 2c\ell(\gamma)^{\alpha},$$

and from (5.2),

$$|f(x) - f(y)| \leq 2cd^{\alpha}|x - y|^{\alpha}.$$

Assertion (1) is proved.

Suppose now that U_1 and U_2 are open, $f \in C^k$ and $f|_{U_1}$ and $f|_{U_2}$ are of class $C^{k,\alpha}$. Let g be any partial derivative of order k of f. Since $f|_{U_1}$ and $f|_{U_2}$ are of class $C^{k,\alpha}$, we have that $g|_{U_1}$ and $g|_{U_2}$ are α-Hölder. Then, since $f \in C^k$ means that g is continuous, it follows from assertion (1) that g is α-Hölder, which proves assertion (2). □

Theorem 5.2. *Let U be an open subset of $\mathbb{R}^m$ and let $M \subset U$ be a proper embedded C^1 manifold of codimension ≥ 2. Let $f\colon U \backslash M \to \mathbb{R}^n$ be of class $C^{k,\alpha}$ for some $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0, 1]$. Then f extends to U in the class $C^{k,\alpha}$.*

We need the following lemma.

Lemma 5.3. *Let U be an open subset of $\mathbb{R}^m$ and let $M \subset U$ be a proper embedded C^1 manifold of codimension ≥ 1. Let $f\colon U\backslash M \to \mathbb{R}^n$ be of class C^1. Suppose that f and its partial derivatives extend continuously to U. Then f extends to U in the class C^1.*

Proof. Let $\bar{f}\colon U \to \mathbb{R}^n$ be the extension of f to U. In view of the hypotheses, it is enough to prove that, given $p \in M$, we can find a C^1 coordinate system $(x_1, \ldots, x_m)$ around p such that the partial derivatives of $\bar{f}$ at p exist and we have

$$\frac{\partial \bar{f}}{\partial x_j}(p) = \lim_{x \to p} \frac{\partial f}{\partial x_j}(x), \quad j = 1, \ldots, m. \tag{5.3}$$

Fix $p \in M$. Consider affine coordinates such that the canonical unitary vectors $e_1, \ldots, e_m$ are transverse to M at p, and fix $j \in \{1, \ldots, m\}$. Since e_j is transverse to M at p, for $t \in \mathbb{R}^*$ small enough the euclidean segment $[p, p + te_j]$ intersects M only at p. Thus, if we set $f = (f_1, \ldots, f_m)$ and $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_m)$, by the Mean Value Theorem we have

$$\begin{aligned}
\frac{\bar{f}(p + te_j) - \bar{f}(p)}{t} &= \left(\frac{\bar{f}_1(p + te_j) - \bar{f}_1(p)}{t}, \ldots, \frac{\bar{f}_m(p + te_j) - \bar{f}_m(p)}{t} \right) \\
&= \left(\frac{\partial f_1}{\partial x_j}(w_1), \ldots, \frac{\partial f_m}{\partial x_j}(w_m) \right),
\end{aligned}$$

where $w_1, \ldots, w_m$ are points in the open segment

$$(p, p + te_j) \subset U\backslash M.$$

Then, since the points $w_1, \ldots, w_m$ tend to p as t tends to 0, we have

$$\begin{aligned}
\lim_{t \to 0} \frac{\bar{f}(p + te_j) - \bar{f}(p)}{t} &= \left(\lim_{x \to p} \frac{\partial f_1}{\partial x_j}(x), \ldots, \lim_{x \to p} \frac{\partial f_m}{\partial x_j}(x) \right) \\
&= \lim_{x \to p} \frac{\partial f}{\partial x_j}(x).
\end{aligned}$$

□

Proof of Theorem 5.2. Suppose first that $k = 0$. Then $f\colon U \setminus M \to \mathbb{R}^n$ is α-Hölder and therefore f extends as an α-Hölder function to

$$\overline{U\backslash M} \supset U,$$

which proves the proposition for $k = 0$. Suppose now that the proposition holds true for $k = l \in \mathbb{Z}_{\geq 0}$ and let $f\colon U\backslash M \to \mathbb{R}^n$ be of class $C^{l+1,\alpha}$. Our intention is to apply Lemma 5.3 to f. Thus, let us show that f and its partial derivatives extend continuously to U. To do so it is enough to show that each $p \in M$ has a neighborhood Ω in U such that the restrictions of f and its derivatives to $\Omega\backslash M$ are uniformly continuous. Fix $p \in M$ and let $\Omega \subset U$ be an open ball centered at p. Since Ω is convex, by (1) and (2) of Proposition 2.1 we have that $\Omega\backslash M$ is subconvex. Thus, since f is of class $C^{l+1,\alpha}$, it follows from (2) of Proposition 4.1 that f and its derivatives are of class $C^{0,\alpha}$ on $\Omega\backslash M$, so they are uniformly continuous on $\Omega\backslash M$. Therefore, by Lemma 5.3, the function f is the restriction to $U\backslash M$ of a C^1 function $\bar{f}\colon U \to \mathbb{R}^n$. To complete the induction, we shall prove that the partial derivatives of $\bar{f}$ belong to $C^{l,\alpha}(U)$. Let g be a partial derivative of $\bar{f}$. Since $g|_{U\backslash M}$ is a partial derivative of f, we have that $g|_{U\backslash M}$ is of class $C^{l,\alpha}$. Therefore, by the inductive hypothesis, $g|_{U\backslash M}$ extends to U in the class $C^{l,\alpha}$, which means that g is of class $C^{l,\alpha}$. □

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Resumen

En este artículo mostramos varias propiedades de las funciones Hölder multiples. Estudiamos la clase Hölder de una composición de funciones Hölder multiples y demostramos que una función y su inversa pertenecen – bajo ciertas hipótesis – a la misma clase Hölder. También demostramos algunas propiedades de extensión de las funciones Hölder multiples; por ejemplo, demostramos que una función Hölder multiple siempre extiende, en la misma clase Hölder, a «conjuntos excepcionales» que son variedades de codimensión uno.

Palabras clave: Clases de Lipschitz (Hölder); Propiedades especiales de funciones de varias variables, condiciones de Hölder.

Presentado: noviembre del 2024.
Aceptado: febrero del 2025.

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