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EXISTENCE OF OBSERVABLE SETS FOR CONTROL SYSTEMS ON LIE GROUPS

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Abstract

In this paper, we introduce the concept of observable sets for control systems. For this reason, we review general control systems on Lie groups with observation (output) functions and give the solution of the affine control system on a connected Lie group. Then, we study the existence of observable sets for linear control systems on examples.

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1. Introduction

The main purpose of this paper is to introduce a new approach to observability by searching the existence of observable sets. Observability problem is one of the classical fundamental problem in control theory and it is important from the application points of view. We approach from differential geometric point of view.

In 1972, the utility of Lie groups and coset spaces in modelling system dynamics has been first highlighted by Brokett, through group manifolds, [6]. Later, in 1977, observability of nonlinear control systems has been studied from differential geometric point of views by Hermann and Krener, [8]. In 1990, [7] by Cheng, Dayawansa, and Martin, is the first key paper directly addressing observability on Lie groups and the authors have studied local and global observabilities using Lie algebraic methods, leveraging the structure of Lie groups and coset spaces.

For a linear control system on $\mathbb{R}^n$, $\Sigma = (\mathbb{R}^n, D, h, \mathbb{R}^s)$ is given by the following data:

$$\dot{x} = Ax + Bu$$
$$h(x) = Cx,$$

where $x \in \mathbb{R}^n$, A, B and C are matrices of appropriate orders and $h : \mathbb{R}^n \to \mathbb{R}^s$ is a linear function. The well-known observability rank condition by Kalman is

$$\Sigma \; is\; observable \quad \Longleftrightarrow \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$

In general, observability of control systems on Lie groups has been studied on the characterization of the set of indistinguishable elements of the state space in order to find conditions for local and global observabilities, [3], [4] and [5]. In our work, we focus on the complements of this kind of sets of indistinguishable elements of the state spaces, which we will call them observable sets.

Our work consists of five sections. In the second section, we review shortly control systems on Lie groups with observation. In the third section, we introduce observable sets by giving some definitions on observability. Moreover, we give the solution for the affine control system on a Lie group G with some output function and the observability is induced by the drift vector field of the system. In the fourth and fifth sections, we study the existence of observable sets for linear control systems both on Euclidean spaces and on Heisenberg group of dimension 3 on examples. Lastly, we write our conclusion of this work.

2. Control Systems on Lie groups with Observation

A control system $\Sigma = (G, \mathcal{D}, h, V)$ on a Lie group G with an observation (output) function is a four-tuple, where G and V are finite dimensional Lie groups, $\mathcal{D}$ is the dynamic and h is a smooth function between G and V. We assume that the state space G is connected. Here, V is called observation (output) space and h is called observation (output) function. The dynamic $\mathcal{D}$ is a set of smooth vector fields on G which are parametrized by the controls u and $u = (u_1, u_2, \dots, u_k) \in \mathcal{U}$, where $\mathcal{U}$ is a family of piecewise constant real-valued functions.

Control system Σ induces a pseudo group of diffeomorphisms

$$G_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} | Z^j \in \mathcal{D}, t_j \in \mathbb{R}, j \in \mathbb{N}\}$$

and a pseudo-semi group

$$S_\Sigma = \{Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} | Z^j \in \mathcal{D}, t_j \geq 0, j \in \mathbb{N}\}.$$

Besides,

$$G_\Sigma(g) = \{\varphi(g) | g \in G, \forall \varphi \in G_\Sigma\}$$

is the orbit, and

$$S_\Sigma(g) = \{\varphi(g) | g \in G, \forall \varphi \in S_\Sigma\}$$

is the positive orbit of the system at the state $g \in G$.

Denote by $L(G)$ the Lie algebra of G and consider $X, Y^1, \ldots, Y^d \in L(G)$ with right invariant directions. Besides, consider the elements $D, D^1, D^2, \ldots, D^n$ of the derivation algebra $Der(L(G))$ of Lie group G. The derivation algebra $Der(L(G))$ is a Lie algebra consisting of endomorphisms D on $L(G)$ satisfying

$$D[X, Y] = [D(X), Y] + [X, D(Y)], \forall X, Y \in L(G).$$

Let χ be a linear vector field on G. Then, by [1], there is a derivation D associated to χ defined by

$$DY = -[\chi, Y], \forall Y \in L(G)$$

and the flow Φ_t of χ is related to D by

$$(d\Phi_t)_e = e^{tD}, \ for\ any\ t \in \mathbb{R}$$

and it follows that

$$\Phi_t(expY) = exp(e^{tD}Y) \ for\ any\ t \in \mathbb{R},\ Y \in L(G).$$

If D is an inner derivation, then $D = ad(X)$ for some $X \in L(G)$. This is a special case and it happens when the Lie group is semisimple.

Any linear vector field χ is an infinitesimal automorphism which means that its flow is a 1-parameter subgroup of automorphisms on G, [1], and $\chi + X$ forms an affine vector field on G, so are $(\chi^j + Y^j)$ for $j = 1, \ldots, d$. The dynamic of affine control system Σ on G is defined by the following differential equations together with an observation (output) function h:

$$\begin{cases} \dot{g}(t) = (\chi + X)(g(t)) + \sum_{j=1}^{d} (\chi^j + Y^j)(g(t))u(t) \\ \\ h(g) = v \in V. \end{cases} \tag{2.1}$$

Affine control systems represent a wide class of control systems, [9]. Indeed, if we consider, linear vector field χ is 0 in 2.1, then the system turns into an invariant control system, and if we consider, for each j, Y^j and X are 0 in 2.1, then the system turns into a bilinear control system. Moreover, if we consider this 2.1 system on an abelian Lie group, then it turns into a linear control system.

3. The Observable Set

In this section, we introduce the concept of observable sets. The idea of observable sets comes from the control sets presented in [2]. By this inspiration we search the existence of possible observable parts of a given system. Control sets are the maximal controllable parts of the systems and they are important to understand the controllability behaviour of the system. Similarly, observable sets can serve as a source for observing some behavior of a system. For this aim, we characterize observable sets and their properties which lead us to work on two examples in the next section.

We would like to review some of the fundamental definitions related to the observability of control systems. For the following three definitions, we consider a control system $\Sigma = (G, \mathcal{D}, h, V)$ on a connected Lie group G with an observation (output) function h which is a smooth function from G to V. Each vector field $Z^j \in \mathcal{D}$ parametrized by the control u (piecewise constant admissible control) induces a 1-parameter group of automorphisms (flow) $Z^j_{t_j}$, $t \in \mathbb{R}$, $j \in \mathbb{N}$, on G.

Definition 3.1 (Indistinguishable)**.** *Let g and l be two different elements of a Lie group G. Then these two elements are called indistinguishable, if the following equation holds*

$$h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(g)) = h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(l)), \forall t_j \geq 0, j \in \mathbb{N}.$$

In another words, g and l are called indistinguishable, if the following equation holds

$$h \circ S_\Sigma(g) = h \circ S_\Sigma(l).$$

We use $g \sim l$, when g and l are indistinguishable, and denote by I_g the set of all indistinguishable elements from g. We know that vector fields Z^j of the dynamic $\mathcal{D}$ are complete, [1], then the relationship $\sim$ is an equivalence relationship. Besides,

$$\tilde{g} = I_g = \{l \in G | g \sim l\} \iff I = \bigcup_g I_g,$$

where I is the set of all indistinguishable elements in G.

Definition 3.2. *For any given state g of a control system Σ, if g has a neighbourhood U such that all points in U are distinguishable from g, then the control system Σ is called locally observable at g. If the system Σ is locally observable at every $g \in G$, then it is called locally observable.*

Definition 3.3. *If, for all $l \in G$ different than g and $\forall t_j \geq 0, j \in \mathbb{N}$,*

$$h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(g)) \neq h(Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k}(l)),$$

then the control system Σ is called globally observable at g. If the control system Σ is globally observable at every $g \in G$, then it is called globally observable.

In another words, if there are no two different states forming the same curve in the observation (output) space, then it is said that the control system Σ is globally observable. Moreover, a system is globally observable if its internal state can be inferred from external outputs.

If we consider a linear control system $\Sigma = (\mathbb{R}^n, \mathcal{D}, h, \mathbb{R}^s)$ with an observation (output) function h which is a linear transformation, then local observability and global observability are the same and there is a well-known observability rank condition for this system. In general, for a control system $\Sigma = (G, \mathcal{D}, h, V)$ on a Lie group G, local observability and global observability are studied separately and global observability implies local observability.

In [3], the authors consider a linear control system Σ on a connected Lie group G with a Lie group homomorphism from the state space to the observation (output) space and give the solution for every admissible control. By using a similar approach, we have the following result.

Theorem 3.4. *Consider an affine control system* $\Sigma = (G, \mathcal{D}, h, V)$ *given by* (2.1)

$$\dot{g}(t) = (\chi + X)(g(t)) + \sum_{j=1}^{d} (\chi^j + Y^j)(g(t))u(t)$$

on a connected Lie group G *with the initial condition* $\gamma(0) = g$*, where* $g \in G$*,* $X, Y^1, \ldots, Y^d \in L(G)$*, the Lie algebra of* G*, and for* $j = 1, \ldots, d$*,* χ *and* χ^j *are linear vector fields on* G*, then the solution is*

$$\gamma(t) = (\chi + X)_t(\beta(t) \cdot g), t \in \mathbb{R},$$

where $(\chi+X)_t$ *is a one-parameter group of automorphisms on* G *induced by the drift vector field* $(\chi+X)$ *and* $\beta(t)$ *satisfies the following differential equation*

$$\dot{\beta(t)} = ((\chi + X)_{-t})_\star \circ \sum_{j=1}^{d} u_j(\chi^j + Y^j) \circ (\chi + X)(\beta(t)).$$

Proof. The dynamic $\mathcal{D}$ of the affine control system is

$$\mathcal{D} = \left\{ \begin{array}{ll} Z = (\chi + X) + u(\chi^j + Y^j): & X, Y^j \in L(G)\ \forall,\ j = 1, \ldots d; \\ & \chi \text{ and } \chi^j \text{ are linear vector fields} \end{array} \right\}$$

If we differentiate $\gamma(t)$, then we have the following

$$\dot{\gamma(t)} = ((\chi + X)_t(\beta(t)\cdot g))_\star = ((\chi + X)(\beta(t)\cdot g))(\beta(t)\cdot g)_\star$$

$$= (\chi + X)(\beta(t)\cdot g)[(\chi + X)_{-t})_\star \circ \sum_{j=1}^{d} u_j(\chi^j + Y^j)\circ(\chi + X)(\beta(t)\cdot g)]$$

$$= [((\chi + X)_t)_\star \circ (\chi + X)_{-t})_\star \circ \sum_{j=1}^{d} u_j(\chi^j + Y^j)\circ(\chi + X)](\beta(t)\cdot g)$$

$$= [\sum_{j=1}^{d} u_j(\chi^j + Y^j)\circ(\chi + X)](\beta(t)\cdot g),$$

where, $(\ )_\star$ denotes the derivative. Thus, we get the form of the dynamic. □

Corollary 3.5. *Let Σ be an affine control system on a connected Lie group G with an observation (output) function h which is a Lie group homomorphism. Then, by the special form of the solution $\gamma(t)$, local and global observabilities of the system Σ depend on the drift vector field $(\chi + X)$.*

Proof. Each Z^i in the dynamic $\mathcal{D}$ induces a one-parameter group $Z^i_{t_i}$ of diffeomorphisms defined on G, and for each $i = 1, 2, \ldots, d$, there exists a differentiable curve $\beta_i : \mathbb{R} \to G$ such that for each $g \in G$,

$$Z^i_{t_i}(g) = (\chi + X)_{t_i}(\beta_i(t)\cdot g),$$

where $Z^1_{t_1} \circ Z^2_{t_2} \circ \cdots \circ Z^k_{t_k} \in G_\Sigma$.

Since h is a homomorphism, $\forall g, l \in G$, we have the following

$$g \sim l \iff h(Z^i_{t_i}(g)) = h(Z^i_{t_i}(l)), \forall Z^i_{t_i} i \in S_\Sigma(g).$$

Thus

$$\begin{aligned}
&\iff h((\chi + X)_{t_i}(\beta_i(t) \cdot g)) = h((\chi + X)_{t_i}(\beta_i(t) \cdot l)), \forall t_i \geq 0;\\
&\iff h((\chi + X)_{t_i}(g)(\beta_i(t) \cdot e)) = h((\chi + X)_{t_i}(l)(\beta_i(t) \cdot e)), \forall t_i \geq 0;\\
&\iff h(\chi + X)_{t_i}(g) = h(\chi + X)_{t_i}(l), \forall t_i \geq 0;
\end{aligned}$$

where e denotes the identity element of G. □

Moreover, we know that the vector fields of the control system on a Lie group G are complete and therefore indistinguishability is an equivalence relationship. Hence, the set of indistinguishable elements from e is

$$\tilde{e} = I_e = \{g \in G | g \sim e\}$$

and for each $g \in G$, $I_g = \tilde{g} = g\tilde{e} = gI_e$. Indeed,

$$g \sim l \iff l \in gI_e.$$

Definition 3.6. *$\Sigma = (G, \mathcal{D}, h, V)$ be a control system on a connected Lie group G with observation (output). A nonempty subset $\hat{B} \subset G$ is called an observable set of the control system Σ if*

i) *it is S_Σ-invariant, i.e.*

$$h \circ S_\Sigma(g) \neq h \circ S_\Sigma(l) \iff g \nsim l, \forall g, l \in B$$

ii)

$$\hat{B} = G - \tilde{e}$$

iii) *maximal with respect to i) and ii).*

Remark 3.7. $\tilde{e}$ is an equivalence class and therefore it is a closed subset of Lie group G. Hence, $\hat{B}$ is open and has a differentiable manifold structure induced by G, [11].

In order to characterize observability of control systems on Lie groups, classical approach is to study the set of indistinguishable elements of the state space and then to find conditions for local and global observability, [3], [4] and [5]. In our work, we focus on the complement of the set of indistinguishable elements of the state spaces, i.e. our observable sets.

4. Existence of Observable Sets for Linear Control Systems on $\mathbb{R}^n$

A linear control system on $\mathbb{R}^n$ is a four-tuple denoted by $\Sigma = (\mathbb{R}^n, D, h, \mathbb{R}^s)$ and their dynamic is determined by the following differential equations:

$$\dot{x} = Ax + Bu$$
$$h(x) = Cx,$$

where $x \in \mathbb{R}^n$, A, B and C are matrices of appropriate orders and $h : \mathbb{R}^n \to \mathbb{R}^s$ is a linear function. Linear control systems on $\mathbb{R}^n$ are well-known and are important from their application points of view.

For any initial condition $x = x_0$, solution of the system has the following form:

$$\gamma(x_0, t, u) = e^{tA}\{x_0 + \int_0^t e^{-\tau A} Bu(\tau) d\tau\}.$$

In particular, if we consider the solution at the origin (i.e, identity element), then we get

$$\gamma(0, t, u) = e^{tA}\{\int_0^t e^{-\tau A} Bu(\tau) d\tau\}.$$

Thus, $S_\Sigma(0) = \{\gamma(0, t, u) : u \in \mathcal{U}, t \geq 0\}$ and the indistinguishability relation can be written by

$$x_1 \sim x_2 \iff C(e^{tA}(x_1)) = C(e^{tA}(x_2)), \ \forall t \geq 0$$

which is independent of the controlled vector field B.

Moreover, indistinguishable elements of this kind of systems from the identity element are characterized by

$$\tilde{0} = \bigcap_{i=0}^{n-1} Ker(CA^i).$$

Example 4.1. *We consider the following control system which is evolving according to Newton's law with the observation (output) function C given in [10]:*

$$\dot{x} = y$$
$$\dot{y} = u$$
$$h(x, y) = C(x, y)^T.$$

For any initial condition $(x, y) = (x_0, y_0)$, solution of the system has the following form:

$$\gamma((x_0, y_0), t, u) = (u\frac{t^2}{2} + y_0 t + x_0, ut + y_0)$$

and, in particular, we have the following at the identity:

$$\gamma((0, 0), t, u) = (u\frac{t^2}{2}, ut).$$

This system is globally observable, if for any two distinct points (x_1, y_1) and (x_2, y_2) of $\mathbb{R}^2$,

$$C(u\frac{t^2}{2} + y_1 t + x_1, ut + y_1) \neq C(u\frac{t^2}{2} + y_2 t + x_2, ut + y_2).$$

(a) *If we observe the position of the system, i.e., the output map is $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$, then by the observability rank condition this system is globally observable. Thus, observable set is the whole state space.*

Indeed, for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $(x_1, y_1) \neq (x_2, y_2)$, we have

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_1 t + x_1 \\ ut + y_1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_2 t + x_2 \\ ut + y_2 \end{pmatrix}.$$

(*b*) *If we observe the velocity of the system, i.e., the output map is* $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, *then by the observability rank condition this system is not globally observable.*

Indeed, $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ *and* $(x_1, y_1) \neq (x_2, y_2)$, *we have*

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_1 t + x_1 \\ ut + y_1 \end{pmatrix} = ut + y_1$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} u\frac{t^2}{2} + y_2 t + x_2 \\ ut + y_2 \end{pmatrix} = ut + y_2.$$

Here, we do not have any information of the position of the system in the output space even if there is information on the velocity of the system.

Example 4.2. *Consider* $\Sigma = (\mathbb{R}^3, D, h, \mathbb{R}^2)$, *where*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then, the rank of $\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = 2 \neq 3$. *Therefore, this system is not globally observable.*

The set of indistinguishable elements from the neutral element is

$$\tilde{0} = \{(0, 0, k) | k \in \mathbb{R}\}$$

and the observable set is

$$\hat{B} = \mathbb{R}^3 - \{(0, 0, k) | k \in \mathbb{R}\}.$$

Existence of observable sets depends on the choices of observation (output) functions first and drift vector fields after. One can work on the same system by changing output functions or drift vector fields. Therefore, existence and optimization of the observable set is up to both output function and drift vector field.

5. Existence of Observable Sets for 3-dimensional Linear Control Systems on Heisenberg Lie groups

In this section, we consider a linear control system on a Heisenberg Lie group of dimension 3. A linear control system on a connected Lie group G is a four-tuple denoted by $\Sigma = (G, D, h, V)$ and their dynamic is determined by the following differential equations:

$$\dot{g}(t) = \chi^i(g(t)) + \sum_{j=1}^{d} Y^j(g(t))u(t)$$

$$h(g) = v \in V,$$

where $g \in G$, $\chi^1, \chi^2, \ldots, \chi^n$ are infinitesimal automorphisms, i.e.; each χ_t^i induces a 1-parameter group of automorphisms on G for each $t \in \mathbb{R}$, $Y^1, \ldots, Y^d \in L(G)$, the Lie algebra of G, and $h : G \to V$ is a differentiable function.

In [3], authors have given an algorithm where one can calculate indistinguishable points of linear controls systems on connected Lie groups with Lie group homomorphisms as observation (output) functions. We use this algorithm for 3-dimensional linear control systems on a Heisenberg Lie group in the example below and want to give its brief explanation. In the first and the second steps of the algorithm, the kernel of the observation (output) function and the basis of the Lie algebra of the kernel of observation (output) function are calculated, respectively. Later, the dual basis of the basis of the Lie algebra of the kernel of observation (output) function is found to determine dual of all indistinguishable points.

Example 5.1. *Consider Heisenberg Lie group of dimension 3*

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$

and its Lie algebra is

$$L(H)=\left\{\begin{pmatrix}0 & x & z\\ 0 & 0 & y\\ 0 & 0 & 0\end{pmatrix} \mid x,y,z\in\mathbb{R}\right\},$$

where

$$span\left\{X=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, Y=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, Z=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\},$$

$$[X,Y]=Z.$$

Here,

$$[X,Y]=XY-YX.$$

(a) *We consider the following linear control system on H with observation (output) function p_d,*

$$\begin{cases}\Sigma=(H,\mathcal{D},\pi,H/exp(\mathbb{R}Z))\\ \\ p_d: H\rightarrow H/exp(\mathbb{R}Z)\end{cases}$$

Then, the kernel of the observation (output) function is

$$Ker(p_d)=\left\{\begin{pmatrix}1 & 0 & t\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \mid t\in\mathbb{R}\right\},$$

and the basis of the Lie algebra of $Ker(p_d)$ is $\mathcal{B}_{K_p}=Z$. Besides, the dual is

$$\mathcal{B}^{\perp}_{K_p}=\{X^{\perp},Y^{\perp}\}.$$

The adjoint action of the Lie algebra element X on Y, $ad_X(Y)$, is equal to $[X,Y]=Z$. Then, $ad_X(X)=ad_X(Z)=0$. For the ordered basis (X,Y,Z) for ad_X, we have the following matrix

$$R=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}.$$

It follows that, the co-adjoint actions of X, L_X, on $X^{\perp}$ and $Y^{\perp}$ are

$$L_X(X^{\perp}) = X^{\perp}R = 0$$
$$L_X(Y^{\perp}) = Y^{\perp}R = 0$$

and then the co-adjoint action of X, L_X, on $\mathcal{B}_{K_p}^{\perp}$ gives the following set

$$ad(X)\mathcal{B}_{K_p}^{\perp} = \{X^{\perp}, Y^{\perp}\} = \mathcal{I}^{\perp}.$$
$$\mathcal{I} = span\{Z\}.$$

Here, $\mathcal{I}$ is the Lie algebra of the Lie group I, the set of indistinguishable elements from the neutral element of H. Thus, Σ is not globally observable and the observable set $\hat{B} = H - exp(tZ)$.

(b) We consider the following linear control system on H with observation (output) function π, [3]

$$\begin{cases} \Sigma = (H, \mathcal{D}, \pi, H/exp(\mathbb{R}Y)) \\ \\ \pi : H \to H/exp(\mathbb{R}Y) \end{cases}$$

Then, the kernel of the observation (output) function is

$$Ker(\pi) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

and the basis of the Lie algebra of $Ker(\pi)$ is $\mathcal{B}_K = Y$. Besides, the dual is

$$\mathcal{B}_K^{\perp} = \{X^{\perp}, Z^{\perp}\}.$$

It follows that, the co-adjoint action of X, L_X, on $Z^{\perp}$ is

$$L_X(Z^{\perp}) = Z^{\perp}R = Y^{\perp}$$

and then the co-adjoint action of X, L_X, on each element of $\mathcal{B}_K^{\perp}$ gives the following set

$$ad(X)\mathcal{B}_K^{\perp} = \{X^{\perp}, Y^{\perp}, Z^{\perp}\} = \mathcal{I}^{\perp}.$$

Thus,

$$\mathcal{I} = 0.$$

Thus, Σ is locally observable, since the Lie algebra of the all indistinguishable elements is null. This system is also globally observable as given in [3]. Thus, the observable set $\hat{B} = H$.

Conclusion:

In this work, we introduced the concept of observable sets. For this aim, we reviewed general control systems on Lie groups and gave the solution of the affine control system on a connected Lie group which represents a wide range of control systems on Lie groups. In the last two sections, we studied existence of observable sets for linear control systems on examples.

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Resumen

En este artículo, presentamos el concepto de conjuntos observables para sistemas de control. Por esta razón, revisamos sistemas de control generales en grupos de Lie con funciones de observación y damos la solución del sistema de control afín en un grupo de Lie conectado. Luego, estudiamos la existencia de conjuntos observables para sistemas de control lineal en ejemplos.

Palabras clave: conjunto observable, observabilidad, sistemas de control afín, sistemas de control lineal.

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