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THE GROEBNER BASIS AND SOLUTION SET OF A POLYNOMIAL SYSTEM RELATED TO THE JACOBIAN CONJECTURE[a]

Christian Valqui[1*]; Valeria Ramírez[2]

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Abstract

We compute the Groebner basis of a system of polynomial equations related to the Jacobian conjecture, and describe completely the solution set.

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1. *Pontificia Universidad Católica del Perú (PUCP) Lima, Perú* & *IMCA Perú*

(*). *Autor de correspondencia*
2. *Pontificia Universidad Católica del Perú (PUCP) Lima, Perú*

1. Introduction

Let K be a field of characteristic zero. The two–dimensional Jacobian Conjecture (JC), formulated by Keller in [7], asserts that any pair of polynomials $P, Q \in R := K[x, y]$ with

$$[P, Q] := \partial_x P\, \partial_y Q - \partial_x Q\, \partial_y P \in K^{\times}$$

defines an automorphism of R.

In [8], T. T. Moh investigated possible counterexamples (P, Q) of total degree below 101, identifying four exceptional pairs $(m, n) = (48, 64)$, $(50, 75)$, $(56, 84)$ and $(66, 99)$, where $(n, m) = (\deg P, \deg Q)$. He then ruled out these cases by explicitly solving certain ad-hoc systems of equations for the coefficients of the potential counterexamples.

Motivated by Moh's approach, in [5] the authors introduce a family of polynomial systems

$$\mathrm{St}(n, m, (\lambda_i), F_{1-n})$$

consisting of $m+n-2$ equations in $m+n-2$ variables with coefficients in a commutative K-algebra D. Here $(\lambda_i)_{0\le i\le m+n-2} \subset K$ and $F_{1-n} \in D$. Among other results, they prove that a specific instance of this system (with $D = K[y]$ and $F_{1-n} = y$) has a solution in D^{m+n-2} if and only if there exists a counterexample (P, Q) to JC with $(n, m) = (\deg P, \deg Q)$. The argument relies on an equivalent formulation of JC due to Abhyankar in [1], which states that JC holds provided that for every Jacobian pair (P, Q) either $\deg P \mid \deg Q$ or $\deg Q \mid \deg P$. They also show that, when D is an integral domain, the set of solutions of $\mathrm{St}(n, m, (\lambda_i), F_{1-n})$ is finite. Furthermore, they examine in detail the "homogeneous" case $\lambda_i = 0$ for $i > 0$, giving an explicit description of its solutions. The usefulness of this method is shown in the last section of [5], where the method is illustrated with the case $(n, m) = (50, 75)$, showing—via a degree-reduction technique as in [4]—that no counterexample arises.

At the moment, there is no other method to discard small possible counterexamples arising from the lists of families of possible counterexamples given in [6] (see also [4]).

An advantage of this formulation is that the system of equations remains canonical, even under the modifications needed for computations as in [5]. This feature makes it suitable for algorithmic implementation and, potentially, for discarding infinite families of possible counterexamples rather than isolated cases.

In order to understand better the system, it could be helpful to understand a Groebner basis of the system. In [9], an explicit Groebner basis for the system $\mathrm{St}(2, m, (0), F_{1-n})$ is found. In the present paper we will analyze the system $\mathrm{St}(3, m, (0), F_{1-n})$. We first find a Groebner basis for a partial system, and then we manage to give a detailed description of the solution set.

2. The Jacobian conjecture as a system of equations

Let K be a characteristic zero field and let $K[y]((x^{-1}))$ be the algebra of Laurent series in x^{-1} with coefficients in $K[y]$. We will start from the following theorem, proved in [5, Theorem 1.9].

Theorem 2.1. *The Jacobian conjecture in dimension two is false if and only if there exist*

- $P, Q \in K[x, y]$ *and* $C, F \in K[y]((x^{-1}))$,
- $n, m \in \mathbb{N}$ *such that* $n \nmid m$ *and* $m \nmid n$,
- $\nu_i \in K$ *(*$i = 0, \dots, m + n - 2$*) with* $\nu_0 = 1$,

such that

- C *has the form*

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \qquad \text{with each } C_{-i} \in K[y],$$

- $gr(C) = 1$ *and* $gr(F) = 2 - n$, *where* gr *is the total degree,*

- $F_+ = x^{1-n}y$, *where* F_+ *is the term of maximal degree in* x *of* F,

- $C^n = P$ *and* $Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F$.

Furthermore, under these conditions (P, Q) *is a counterexample to the Jacobian conjecture.*

In [5], the authors consider the following slightly more general situation. Let D be a K-algebra (for example, in Theorem 2.1 we have $D = K[y]$), n, m positive integers such that $n \nmid m$ and $m \nmid n$, $(\nu_i)_{1 \leq i \leq n+m-2}$ a family of elements in K with $\nu_0 = 1$ and $F_{1-n} \in D$ (in Theorem 2.1 we take $F_{1-n} = y$). A Laurent series in x^{-1} of the form

$$C = x + C_{-1}x^{-1} + C_{-2}x^{-2} + \cdots \qquad \text{with } C_{-i} \in D,$$

is a *solution of the system* $S(n, m, (\nu_i), F_{1-n})$, if there exist $P, Q \in D[x]$ and $F \in D[[x^{-1}]]$, such that

$$F = F_{1-n}x^{1-n} + F_{-n}x^{-n} + \cdots, \quad \text{with } F_{1-n}, F_{-n}, \ldots \text{ in } D,$$

$$P = C^n \qquad \text{and} \qquad Q = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F.$$

For example, if $n = 3$, then

$$\begin{aligned} P(\mathbf{x}) = C^3 = {} & \mathbf{x}^3 + 3C_{-1}\ \mathbf{x} + 3C_{-2} + (3C_{-1}^2 + 3C_{-3})\ \mathbf{x}^{-1} \\ & + (6C_{-1}C_{-2} + 4C_{-4})\ \mathbf{x}^{-2} \\ & + (C_{-1}^3 + 3C_{-2}^2 + 6c_{-2}C_{-3} + 3C_{-5})\ \mathbf{x}^{-3} \\ & + (3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6})\ \mathbf{x}^{-4} \\ & + (3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} \\ & \quad + 6C_{-7})\ \mathbf{x}^{-5} \\ & + \ldots \end{aligned}$$

and the condition $C^3 \in K[x]$ translates into the following conditions on C_{-k}:

$$\begin{aligned}
0 = (C^3)_{-1} &= 3C_{-1}^2 + 3C_{-3},\\
0 = (C^3)_{-2} &= 6C_{-1}C_{-2} + 4C_{-4},\\
0 = (C^3)_{-3} &= C_{-1}^3 + 3C_{-2}^2 + 6c_{-2}C_{-3} + 3C_{-5},\\
0 = (C^3)_{-4} &= 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 6C_{-6},\\
0 = (C^3)_{-5} &= 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5}\\
&\quad + 6C_{-7},\\
&\vdots
\end{aligned}$$

In the general case, the condition $P(x) = C^n \in K[x]$ yields the equations $(C^n)_{-k} = 0$, whereas the condition $Q(x) = \sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F \in K[x]$ gives us the equations $\left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i} + F\right)_{-k} = 0$, where we note that $F_{-k} = 0$ for $k = 1, \dots, n-2$.

It is easy to see (e.g. [5, Remark 1.13]) that the first $m+n-2$ coefficients determine the others, i.e., the coefficients $C_{-1}, \dots, C_{-m-n+2}$ determine univocally the coefficients C_{-k} for $k > m+n-2$. Moreover, the F_{-k} for $k > n-1$ depend only on F_{1-n} and C. Consequently, having a solution C to the system $S(n, m, (\nu_i), F_{1-n})$ is the same as having a solution $(C_{-1}, \dots, C_{-m-n+2})$ to the system

$$\begin{aligned}
&E_k := (C^n)_{-k} = 0, && \text{for } k = 1, \dots, m-1,\\
&E_{m-1+k} := \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i}\right)_{-k} = 0, && \text{for } k = 1, \dots, n-2,\\
&E_{m+n-2} := \left(\sum_{i=0}^{m+n-2} \nu_i C^{m-i}\right)_{1-n} + F_{1-n} = 0,
\end{aligned} \tag{2.1}$$

with $m+n-2$ equations $E_k = 0$ and $m+n-2$ unknowns C_{-k}.

In order to understand the solution set of this system, it would be very helpful to find a Groebner basis for the ideal generated by the polynomials E_k in $D[C_{-1}, \ldots, C_{m+n-2}]$. In this paper we compute such a Groebner basis of (2.1) in a very particular case: we assume $n = 3$, $m = 3r + 1$ or $m = 3r + 2$ for some integer $r > 0$, and $\nu_i = 0$ for $i > 0$. Moreover we consider $D = \mathbb{C}[y]$ and $F_{1-n} = y$, as in Theorem 2.1.

3. Computation of a Groebner basis for I_{m-1}

Assume $n = 3$, $3 \nmid m > 3$ and $\nu_i = 0$ for $i > 0$. Set also $D = \mathbb{C}[y]$ and $F_{1-n} = y$.

Then the system (2.1) reads

$$E_i \quad = \quad \begin{cases} (C^3)_{-i}, & i = 1, \ldots, m-1, \\ (C^m)_{-1}, & i = m, \\ (C^m)_{-2} + y, & i = m+1, \end{cases} \tag{3.1}$$

where $(C^2)_{-i}$ denotes the coefficient of x^{-i} in the Laurent series C^3. Explicitly, the polynomials E_i are given by

$$\begin{aligned} E_1 &= 3C_{-1}^2 + 3C_{-3}, \\ E_2 &= 6C_{-1}C_{-2} + 3C_{-4}, \\ E_3 &= C_{-1}^3 + 3C_{-2}^2 + 6C_{-1}C_{-3} + 3C_{-5}, \\ E_4 &= 3C_{-1}^2C_{-2} + 6C_{-2}C_{-3} + 6C_{-1}C_{-4} + 3C_{-6}, \\ E_5 &= 3C_{-1}C_{-2}^2 + 3C_{-1}^2C_{-3} + 3C_{-3}^2 + 6C_{-2}C_{-4} + 6C_{-1}C_{-5} + 3C_{-7}, \\ &\vdots \\ E_{m-1} &= (C^3)_{1-m}, \\ E_m &= (C^m)_{-1}, \\ E_{m+1} &= (C^m)_{-2} + y. \end{aligned} \tag{3.2}$$

Each E_i is a polynomial in the ring $\mathbb{C}[C_{-1}, C_{-2}, \dots, C_{m+1}, y]$, and the $m+1$ polynomials yield the ideal

$$I \quad = \quad \langle E_1, \dots, E_m, E_{m+1} \rangle.$$

Our goal is to find a Groebner basis for the ideal I, but we find it nearly explicit only for $I_{m-1} := \langle E_1, E_2, \dots, E_{m-2}, E_{m-1} \rangle$. For this we note that the equations are homogeneous, for the weight obtained by setting

$$w(C_{-i}) = i+1, \quad \text{and} \quad w(y) = m+n-1 = m+2.$$

We consider y as a variable, so the equations remain homogeneous. Then

$$w(E_k) = k{+}3, \text{ for } k = 1, \dots, m-1, \quad w(E_m) = m{+}1 \quad w(E_{m+1}) = m{+}2.$$

Note that for $k = 1 \dots, m-1$ we have

$$\begin{aligned} E_k :=& 3\Big(\sum_{\substack{i=-1\\ 3i\neq k}}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)}\Big) + 6\Big(\sum_{\substack{0<i<j\\ i+j=k+1}} C_{-i}C_{-j}\Big) \\ &+ 6\Big(\sum_{\substack{0<i<j<l\\ i+j+l=k}} C_{-i}C_{-j}C_{-l}\Big) + \varepsilon (C_{-\frac{k}{3}})^3, \end{aligned} \tag{3.3}$$

where

$$\varepsilon = \begin{cases} 1, & 3|k \\ 0, & 3\nmid k \end{cases}.$$

Note that $C_1 = 1$ and $C_0 = 0$, and so

$$3\sum_{i=-1}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)} = 3C_{k+2} + 3\sum_{i=1}^{[[\frac{k+1}{2}]]} C_{-i}^2 C_{-(k-2i)}. \tag{3.4}$$

In order to compute a Groebner basis we will consider the degree reverse lexicographic monomial order, but for the degree given by the above mentioned weight. This means that the monomial order is given by the matrix

$$\text{wmat} = \begin{pmatrix} m+2 & m+1 & m & \dots & 4 & 3 & 2 & m+2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix},$$

on the variables $C_{-(m+1)}, C_{-m}, C_{-(m-1)}, \dots, C_{-3}, C_{-2}, C_{-1}, y$. We first compute the reduced Groebner basis $(\widetilde{E}_1, \widetilde{E}_2, \dots, \widetilde{E}_{m-1})$ for the ideal $I_{m-1} := \langle E_1, E_2, \dots, E_{m-2}, E_{m-1} \rangle$.

Proposition 3.1. *The set $\{E_1, \dots, E_{m-1}\}$ is a Groebner basis of I_{m-1}. The reduced Groebner basis of I_{m-1} is given by polynomials $\widetilde{E}_k$ for $k = 1, \dots, m-1$, each of the form*

$$\widetilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2}),$$

where $R_k(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ is an homogeneous polynomial in the variables C_{-1} and C_{-2} of weight $w(\widetilde{E}_k) = w(E_k) = k+3$.

Proof. By (3.3) and (3.4) we know that E_k is of the form

$$E_k = 3C_{-k-2} + T(C_{-1}, \dots, C_{-k}), \quad \text{for } k = 1, \dots, m-1,$$

where T is a polynomial in the variables $C_{-1}, \dots, C_{-k}$. Then by Proposition 2.9.4 of [2], since

$$\begin{aligned} LCM(LT(E_i)/3, LT(E_j)/3) &= LCM(C_{-i-2}, C_{-j-2}) = C_{-i-2}C_{-j-2} \\ &= (LT(E_i)/3)(LT(E_j)/3) \end{aligned}$$

we have $S(E_i, E_j) \longrightarrow_G 0$, and so, by Theorem 2.9.3 of [2], the set $G = \{E_1/3, \ldots, E_{m-1}/3\}$ is a Groebner basis of I_{m-1}. One verifies directly that it is a minimal Groebner basis, according to Definition 2.7.4 of [2]. If we apply the process described in the proof of [2, Proposition 2.7.6] to the Groebner basis $G = \{E_1/3, \ldots, E_{m-1}/3\}$ we obtain that

$$\widetilde{E}_1 = \overline{E_1/3}^{G\setminus\{\frac{E_1}{3}\}} = E_1/3 \quad \text{and} \quad \widetilde{E}_1 = \overline{E_2/3}^{G\setminus\{\frac{E_2}{3}\}} = E_2/3.$$

Moreover, for $k = 3, \ldots, m-1$, set $G_k = \{\widetilde{E_1}, \ldots, \widetilde{E_{k-1}}, E_k, \ldots, E_{m-1}\}$ and then

$$\widetilde{E}_k = \overline{E_k}^{G_k \setminus E_k}.$$

Clearly the remainder can have only the variables C_{-1} and C_{-2}, hence $\widetilde{E}_k$ is of the form

$$\widetilde{E}_k = C_{-(k+2)} + R_k(C_{-1}, C_{-2}),$$

as desired. □

Although we have no explicit formula for $R_k(C_{-1}, C_{-2})$, we can compute it for small k.

$$\begin{aligned} \tilde{E}_1 &= C_{-3} + C_{-1}^2, \\ \tilde{E}_2 &= C_{-4} + 2C_{-1}C_{-2}, \\ \tilde{E}_3 &= C_{-5} + C_{-2}^2 - \frac{5}{3}C_{-1}^3, \\ \tilde{E}_4 &= C_{-6} - 5C_{-1}^2C_{-2}, \\ \tilde{E}_5 &= C_{-7} + \frac{10}{3}C_{-1}^4 - 5C_{-1}C_{-2}^2. \end{aligned}$$

Dividing the polynomials E_m and E_{m+1} by the polynomials

$$\{\widetilde{E}_{m-1}, \dots, \widetilde{E}_2, \widetilde{E}_1\}$$

with respect to the given order, we obtain

$$\overline{\frac{E_m}{3}}^{G_m \setminus \{\frac{E_m}{3}\}} = \widetilde{E}_m = R_m(C_{-1}, C_{-2})$$

and

$$\overline{\frac{E_{m+1}}{3}}^{G_m \setminus \{\frac{E_{m+1}}{3}\}} = \widetilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2}),$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ are homogeneous polynomials such that $w(\widetilde{E}_m) = w(E_m) = m + 1$ and $w(\widetilde{E}_{m+1}) = w(E_{m+1}) = m + 2$.

Although we don't give an explicit description of the Groebner Basis of the whole system, in the next section we show how to determine the solution set of the polynomial system, using that

$$I = \langle E_1, E_2, \dots, E_m, E_{m+1} \rangle = \langle \widetilde{E}_1, \widetilde{E}_2, \dots, \widetilde{E}_m, \widetilde{E}_{m+1} \rangle.$$

4. The solution set of the system of polynomial equations

In this section we analyze the solutions of the system of equations. Note that the partial system I_{m-1} shows that the values of C_{-1} and C_{-2} determine univocally the values of C_{-k} for $k > 2$. Moreover, C_{-1} and C_{-2} can be computed using the following two equations:

$$\widetilde{E}_m = R_m(C_{-1}, C_{-2}) = 0 \tag{4.1}$$

and

$$\widetilde{E}_{m+1} = y + R_{m+1}(C_{-1}, C_{-2}) = 0, \tag{4.2}$$

where $R_m(C_{-1}, C_{-2}), R_{m+1}(C_{-1}, C_{-2}) \in \mathbb{Q}[C_{-1}, C_{-2}]$ are homogeneous polynomials with respect to the weight considered before, i.e. $w(C_{-1}) =$

2, $w(C_{-2}) = 3$. Moreover $w(\widetilde{E}_m) = m+n-2 = m+1$ and $w(\widetilde{E}_{m+1}) = m+n-1 = m+2$. Then (4.1) and (4.2) read

$$\widetilde{E}_m = \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j \tag{4.3}$$

and

$$\widetilde{E}_{m+1} = y + \sum_{2i+3j=m+2} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j, \tag{4.4}$$

for some constants $\lambda_m^{ij}, \lambda_{m+1}^{ij} \in K$. By (4.4) the two variables cannot be zero at the same time. We compute first the solutions in the cases where one of the variables is zero.

FIRST CASE: $C_{-1} = 0$ and $C_{-2} \neq 0$.

In this case the only term surviving in (4.3) is

$$0 = \widetilde{E}_m = \lambda_m^{0j} C_{-2}^j,$$

with $3j = m+1$. So necessarily

$$\lambda_m^{0,(m+1)/3} = 0 \quad \text{if} \quad 3|m+1. \tag{4.5}$$

Similarly, the only term surviving in the sum (4.4) has $i = 0$, and so we obtain

$$0 = \widetilde{E}_{m+1} = y + \lambda_{m+1}^{0j} C_{-2}^j \quad \text{with } 3j = m+2.$$

Since $y \neq 0$, necessarily $\lambda_{m+1}^{0j} \neq 0$ for $3j = m+2$, and so $3|m+2$, i.e. $m \equiv 1 \mod 3$. This shows that the condition (4.7) is trivially satisfied.

Lemma 4.1. *If* $3|m+2$, *and* $C_{-1} = 0$, *then* $\lambda_{m+1}^{0j} \neq 0$ *for* $3j = m+2$.

Proof. It is easy to check that $P = x^3 + 3C_{-2}$, and then, by Newtons binomial theorem we have

$$C^m = P^{m/3} = \sum_{k=0}^{\infty} \binom{m/3}{k} (3C_{-2})^k (x^3)^{\frac{m}{3}-k}. \tag{4.6}$$

Thus $\lambda^{0j}_{m+1}C^j_{-2} = (C^m)_{-2}$ is the coefficient of $x^{-2} = (x^3)^{\frac{m}{3}-j}$, since $m = 3j - 2$. Then

$$\lambda^{0j}_{m+1} = \binom{m/3}{j} 3^j \neq 0,$$

as desired. □

Thus we have proved the following proposition.

Proposition 4.2. *If* $(C_{-1}, C_{-2}, \ldots, C_{-(m+1)})$ *is a solution of the system* (3.1)*, with* $C_{-1} = 0$ *and* $C_{-2} \neq 0$*, then*

- $m \equiv 1 \mod 3$*,*
- $\lambda^{0j}_{m+1} \neq 0$ *for* $j := \frac{m+2}{3}$*,*
- *There are* j *solutions of the system* (3.1) *in* $K[y^{1/j}]$*, given by*

$$C_{-1} = 0, \quad C_{-2} = \left(\frac{-y}{\lambda^{0j}_{m+1}}\right)^{\frac{1}{j}} \quad \text{and} \quad C_{-k} = -R_{k-2}(C_{-1}, C_{-2})$$

for $3 \leq k \leq m+1$*.*

SECOND CASE: $C_{-1} \neq 0$ and $C_{-2} = 0$.

In this case the only term surviving in (4.3) is

$$0 = \widetilde{E}_m = \lambda^{i0}_m C^i_{-1},$$

with $2i = m + 1$. So necessarily

$$\lambda^{(m+1)/2,0}_m = 0 \quad \text{if} \quad 2|m+1. \tag{4.7}$$

Similarly, the only term surviving in the sum (4.4) has $j = 0$, and so we obtain

$$0 = \widetilde{E}_{m+1} = y + \lambda^{i0}_{m+1}C^i_{-1} \quad \text{with } 2i = m + 2.$$

Since $y \neq 0$, necessarily $\lambda^{i0}_{m+1} \neq 0$ for $2i = m + 2$, and so $2|m + 2$, i.e. m is even. This shows that the condition (4.7) is trivially satisfied.

Lemma 4.3. *If $2|m$ and $C_{-2}=0$, then $\lambda^{i0}_{m+1}\neq 0$ for $2i=m+2$.*

Proof. It is easy to check that $P=x^3+3xC_{-1}$, and then, by Newtons binomial theorem we have

$$C^m=P^{m/3}=\sum_{k=0}^{\infty}\binom{m/3}{k}(3xC_{-1})^k(x^3)^{\frac{m}{3}-k}. \tag{4.8}$$

Thus $\lambda^{i0}_{m+1}C^i_{-1}=(C^m)_{-2}$ is the coefficient of $x^{-2}=(x)^i(x^3)^{\frac{m}{3}-i}$, since $m=2i-2$. Then

$$\lambda^{i0}_{m+1}=\binom{m/3}{i}3^i\neq 0,$$

as desired. □

Thus we have proved the following proposition.

Proposition 4.4. *If $(C_{-1},C_{-2},\dots,C_{-(m+1)})$ is a solution of the system* (3.1)*, with $C_{-1}\neq 0$ and $C_{-2}=0$, then*

- $m\equiv 1 \mod 3$,
- $\lambda^{i0}_{m+1}\neq 0$ *for* $i:=\frac{m+2}{2}$,
- *There are i solutions of the system* (3.1) *in $K[y^{1/i}]$, given by*

$$C_{-1}=\left(\frac{-y}{\lambda^{i0}_{m+1}}\right)^{\frac{1}{i}},\quad C_{-2}=0\quad\text{and}\quad C_{-k}=-R_{k-2}(C_{-1},C_{-2})$$

for $3\leq k\leq m+1$.

THIRD CASE: $C_{-1}\neq 0$, $C_{-2}\neq 0$ and m even.
In this case we introduce a new auxiliary variable t satisfying $C^2_{-2}=tC^3_{-1}$. The equality (4.3) now reads

$$\begin{aligned}\widetilde{E}_m&=\sum_{2i+3j=m+1}\lambda^{ij}_mC^i_{-1}C^j_{-2}=\sum_{2i+6r+3=m+1}\lambda^{i,2r+1}_mC^i_{-1}C^{2r+1}_{-2}\\&=\sum_{2i+6r+2=m}\lambda^{i,2r+1}_mC^{i+3r}_{-1}C_{-2}t^r,\end{aligned}$$

since m even implies that the weight $2i+3j=m+1$ is odd, so j is odd and can be written as $2r+1$. Moreover, for the terms in the sum we have $i+3r=\frac{m-2}{2}$, and so we arrive at

$$0 = C_{-1}^{\frac{m-2}{2}} C_{-2} \sum_{\substack{2i+6r=m-2\\ j=2r+1}} \lambda_m^{ij} t^r.$$

Thus t is a root of the polynomial

$$f(t) = \sum_{r=0}^{\lfloor \frac{m-2}{6} \rfloor} a_r t^r, \quad \text{where } a_r = \lambda_m^{\frac{m-2-6r}{2}, 2r+1}. \tag{4.9}$$

Let $\{t_1, \ldots, t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality (4.4) the power j has to be even, since m is even and $2i+3j=m+2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in (4.4), we obtain

$$\widetilde{E}_{m+1} = y + \sum_{\substack{2i+3j=m+2\\ j=2r}} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j = y + \sum_{2i+6r=m+2} \lambda_{m+1}^{i,2r} C_{-1}^{i+3r} t_l^r.$$

Note that for each of the terms in the last sum we have $i+3r=\frac{m+2}{2}$, and so

$$0 = y + C_{-1}^{\frac{m+2}{2}} g(t_l), \quad \text{where } g(t) = \sum_{r=0}^{\lfloor \frac{m+2}{6} \rfloor} b_r t^r,$$

with $b_r = \lambda_{m+1}^{\frac{m+2-6r}{2}, 2r}$. It follows that

$$C_{-1} = \left(\frac{-y}{g(t_l)} \right)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 4.5. *If* $(C_{-1}, C_{-2}, \ldots, C_{-(m+1)})$ *is a solution of the system* (3.1)*, with* $C_{-1} \neq 0$*,* $C_{-2} \neq 0$ *and* m *even, then the system has at most* $s \cdot (m+2)$ *solutions, where* s *is the number of roots of*

$f(t)$ defined in (4.9). Moreover, for every choice of a root t_l of f, the solutions are given by

$$\begin{aligned}
&C_{-1} = \left(\frac{-y}{g(t_l)}\right)^{\frac{2}{m+2}}, && \frac{m+2}{2} \quad \textit{choices,}\\
&C_{-2} = \left(t_l C_{-1}^3\right)^{\frac{1}{2}}, && 2 \quad \textit{choices,}\\
&C_{-k} = -R_{k-2}(C_{-1}, C_{-2}) \quad \textit{for} \quad 3 \le k \le m+1.
\end{aligned}$$

FOURTH CASE: $C_{-1} \neq 0$, $C_{-2} \neq 0$ and m odd.
In this case we introduce a new auxiliary variable t satisfying $C_{-2}^2 = tC_{-1}^3$. The equality (4.3) now reads

$$\begin{aligned}
\tilde{E}_m &= \sum_{2i+3j=m+1} \lambda_m^{ij} C_{-1}^i C_{-2}^j = \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^i C_{-2}^{2r}\\
&= \sum_{2i+6r=m+1} \lambda_m^{i,2r} C_{-1}^{i+3r} t^r,
\end{aligned}$$

since m odd implies that the weight $2i+3j = m+1$ is even, so j is even and can be written as $2r$. Moreover, for the terms in the sum we have $i + 3r = \frac{m+1}{2}$, and so we arrive at

$$0 = C_{-1}^{\frac{m+1}{2}} \sum_{\substack{2i+6r=m+1\\ j=2r}} \lambda_m^{ij} t^r.$$

Thus t is a root of the polynomial

$$f(t) = \sum_{r=0}^{\lfloor \frac{m+1}{6} \rfloor} a_r t^r, \quad \text{where } a_r = \lambda_m^{\frac{m+1-6r}{2},2r}. \tag{4.10}$$

Let $\{t_1, \dots, t_s\}$ be the roots of the polynomial $f(t)$. Note that in the equality (4.4) the power j has to be odd, since m is odd and $2i + 3j = m + 2$. Hence, if we replace C_{-2}^2 by $t_l C_{-1}^3$ in (4.4), we obtain

$$\widetilde{E}_{m+1} = y + \sum_{\substack{2i+3j=m+2\\ j=2r+1}} \lambda_{m+1}^{ij} C_{-1}^i C_{-2}^j = y + \sum_{2i+6r+3=m+2} \lambda_{m+1}^{i,2r+1} C_{-1}^{i+3r} C_{-2} t_l^r.$$

Note that for each of the terms in the last sum we have $i + 3r = \frac{m-1}{2}$, and so

$$0 = y + C_{-1}^{\frac{m-1}{2}} C_{-2} g(t_l), \quad \text{where } g(t) = \sum_{r=0}^{\lfloor \frac{m-1}{6} \rfloor} b_r t^r,$$

with $b_r = \lambda_{m+1}^{\frac{m-1-6r}{2}, 2r+1}$. We also replace C_{-2} by $\big(t_l C_{-1}^3\big)^{\frac{1}{2}}$. It follows that

$$0 = y + C_{-1}^{\frac{m+2}{2}} (t_l)^{\frac{1}{2}} g(t_l),$$

and so

$$C_{-1} = \Big(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)}\Big)^{\frac{2}{m+2}}.$$

Thus we have arrived at the following result.

Proposition 4.6. *If* $\big(C_{-1}, C_{-2}, \ldots, C_{-(m+1)}\big)$ *is a solution of the system* (3.1)*, with* $C_{-1} \neq 0$*,* $C_{-2} \neq 0$ *and* m *odd, then the system has at most* $2 \cdot s \cdot (m+2)$ *solutions, where* s *is the number of roots of* $f(t)$ *defined in* (4.10)*. Moreover, for every choice of a root* t_l *of* f*, we first choose a square root of* t_l *and then the solutions are given by*

$$C_{-1} = \Big(\frac{-y}{(t_l)^{\frac{1}{2}} g(t_l)}\Big)^{\frac{2}{m+2}}, \qquad \frac{m+2}{2} \ \textit{choices},$$

$$C_{-2} = \big(t_l C_{-1}^3\big)^{\frac{1}{2}}, \qquad 2 \ \textit{choices},$$

$$C_{-k} = -R_{k-2}(C_{-1}, C_{-2}) \quad \textit{for} \quad 3 \leq k \leq m+1.$$

Final Remark.

The solution sets arising in the four cases reveal that no solution exists in $K[y]$, whereas all solutions lie in $K\big[y^{1/(m+2)}\big]$. This, in turn, implies that there is no counterexample (P, Q) to the Jacobian Conjecture with $\deg(P) = 3$ and $3 \nmid \deg(Q)$. Although this fact is already known—for instance, because no counterexample can occur when $\gcd\big(\deg(P), \deg(Q)\big) = 1$—a more detailed analysis of the corresponding Gröbner bases in broader settings may still yield new insights toward a proof or disproof of the Jacobian Conjecture.

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Resumen

Calculamos la base de Groebner de un sistema de ecuaciones polinomiales relacionadas con la conjetura jacobiana y describimos completamente el conjunto de soluciones.

Palabras clave: Conjetura jacobiana. Base de Groebner

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Christian Valqui,
https://orcid.org/0000-0003-2716-3830
Pontificia Universidad Católica del Perú.
Av Universitaria, 1801.
Lima, Peru.
Email: cvalqui@pucp.pe

Instituto de Matemática y Ciencias Afines (IMCA)
Calle Los Biólogos 245. Urb San César. La Molina,
Lima 12, Perú.
Email: cvalqui@pucp.pe

Valeria Ramírez,
https://orcid.org/0009-0006-4938-7113
Pontificia Universidad Católica del Perú.
Av Universitaria, 1801.
Lima, Peru.
Email: valeria.ramireza@pucp.edu.pe