*Pro Mathematica: Vol. V, Nos. 9 - 10, 1991*

# ON THE ALGEBRAIC MULTIPLICITIES OF THE EIGEN VALUES OF THE INTEGRAL

**Julio ALCANTARA BODE***

*It is proven that all non-cero eigenvalues of the integral operator on*

$$L^2(0,1), \ [A_\rho f](\theta) = \int_0^1 \rho(\theta/x)\, f(x)\, dx \,,$$

*where* $\rho$ *is the fractionary part function, have algebraic multiplicity equal to one.*

*Profesor Asociado de la PUCP and UPCH, Lima Perú

## 1. Introduction

In a previous work [1] we reformulated the Riemann Hypothesis as a problem of Functional Analysis by means of the following Theorem.

**Theorem.** Let $[A_\rho f](\theta) = \int_0^1 \rho(\frac{\theta}{x}) f(x)\,dx$, where $\rho(x) = x - [x]$ , $[x] \le x < [x] + 1$ , $[x] \in \mathbb{Z}$ , be considered as an operator on $L^2(0,1)$ . Then, the Riemann Hypothesis holds if and only if Ker $A_\rho = \{0\}$ or if and only if $h \notin$ Ran $A_\rho$, where $h(x) = x$ , $\forall x \in [0,1]$.

Among others things we also proved that:

i) $A_\rho$ is Hilbert-Schmidt but not nuclear.

ii) $\lambda \neq 0$ is an eigenvalue of $A_\rho$ if and only if $T(\lambda^{-1}) = 0$, where

$$T(\mu) = 1- \mu+ \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{(r+1)!\ (r+1)} \prod_{1=1}^{r} \zeta\ (1+1)\ \mu^{r+1}$$

is an entire function of order one and type one; moreover each non-zero eigenvalue $\lambda$ has geometric multiplicity one and the corresponding eigenfunction

$\varphi_\lambda$ is $\varphi_\lambda(x) = x\ T'(x/\lambda)$.

iii) If $\{\lambda_n\}_{n\ge1}$ is the sequence of non-zero eigenvalues of $A_\rho$, where each eigenvalue is repeated a number of times equal to its algebraic multiplicity, and the ordering is such that $|\lambda_n| \geq |\lambda_{n+1}|$ $\forall n \geq 1$, then $|\lambda_n| \leq e/n$ $\forall n \geq 1$ and $\sum_{n=1}^{\infty} |\lambda_n| = \infty$ .

iv) If $D^*$ is the modified Fredholm determinant of $A_\rho$ then $D^*(\mu) = e^\mu\ T(\mu)$ $\forall \mu \in \mathbb{C}$ .

v) If $\varphi \in ]0, \frac{\pi}{2}[$ then, in the complement of each of the wedges

$$W_\varphi = \{\lambda \mid |\mathrm{Im}\ \lambda| \leq \mathrm{tg}\ \varphi\ \mathrm{Re}\ \lambda\} \quad \text{and}$$

$$-W_\varphi = \{\lambda \mid -\lambda \in W_\varphi\},\ A_\varphi$$

has an infinite number of eigenvalues.

vi) The largest eigenvalue $\lambda_1$ of $A_\rho$ is positive and has algebraic multiplicity one.

The last result was proven as an application of the Krein-Rutman Theorem. The purpose of this note is to prove that all non-zero eigenvalues of $A_\rho$ have algebraic multiplicity one or equivalently that the entire functions $D^*$ and T have only simple zeros, [3], p.349 Theorem 2.

## 2. Proof of result

**Theorem.** All non-zero eigenvalues of $A_\rho$ have algebraic multiplicity one.

**Proof:** First we prove that if $T^{(l)}(\lambda_1^{-1}) = 0$ for $0 \le l \le k - 1$ and $T^{(k)}(\lambda_j^{-1}) \ne 0$, then the functions $\xi_{j;l}(x) = x^l\ T^{(l)}(x/\lambda_j)$ , $1 \le l \le k$, obey the equations

$$(A_\rho - \lambda_j)^l\ \xi_{j;l} = 0 ,$$

$$(A - \lambda_j)^{l-1}\ \xi_{j;l} = (-1)^{l-1}\ (l-1)!\ \lambda_j^{2l-2}\ \xi_{j;l}$$

$$1 \le l \le k \qquad (1)$$

In [1] it has ben shown that if $\psi_\mu(x) = \mu x\ T'(\mu x)$ and $h(x) = x$ , then $(A_\rho - \mu^{-1})\ \psi_\mu = T(\mu)\ h$ or

$$A_\rho\ \psi_\mu - \mu^{-1}\ \psi_\mu = T(\mu)\ h \qquad (2)$$

If $\mu_0 \in \mathbb{C}$ we have by Taylor's theorem that

$$\psi_\mu(x) = \sum_{n=o}^{\infty} \frac{(\mu - \mu_o)^n}{n!}\ [\mu_o x^{n+1}\ T^{(n+1)}(\mu_o x) + n\ x^n\ T^{(n)}(\mu_o x)] \qquad (3)$$

$$\mu^{-1}\ \psi_\mu(x) = \sum_{n=o}^{\infty} \frac{(\mu - \mu_o)^n}{n!}\ x^{n+1}\ T^{(n+1)}(\mu_o x) \qquad (4)$$

$$T(\mu)\ h(x) = \sum_{n=o}^{\infty} \frac{(\mu - \mu_o)^n}{n!}\ T^{(n)}(\mu_o)\ x \qquad (5)$$

Replacing equations (3), (4) and (5) in (2) and equating coefficients of the same powers of $(\mu - \mu_o)$ on both sides of the resulting equation we get

$$(A_\rho - \mu_o^{-1})\mu_o\, \psi_{\mu_o;n+1} = T^{(n+1)}(\mu_o)h - n\, A_\rho\, \psi_{\mu_o;n}\,, \qquad n \geq 1 \qquad (6)$$

where $\psi_{\mu,n}(x) = x^n\, T^{(n)}(\mu x)\ \forall\ n \geq 1$ .

From equation (6), we get both equations (1) by induction.

Now in equation (6) we replace $\mu_o$ by $\mu$ , where $\mu^{-1}$ is not in the spectrum of $A_\rho$ . If we apply operator $(A_\rho - \mu^{-1})^{-1}$ to the resulting equation we get

$$n\mu^{-1}\, (A_\rho - \mu^{-1})^{-1}\ \psi_{\mu;n} = \frac{T^{(n)}(\mu)}{T(\mu)}\, \psi_\mu - n\, \psi_{\mu;n} - \mu\, \psi_{\mu;n+1} \qquad (7)$$

Assume now that $\mu_j^{-1}$ is a non-zero eigenvalue of $A_\rho$ of algebraic multiplicity $p_j$ and that $A_\rho$ does not have an eigenvalue in $\Omega_j = \{\mu |\ 0 < |\mu - \mu_j| < \varepsilon\}$ . In $\Omega_j$ is valid the following Laurent expansion for the resolvent $(\mu^{-1} - A_\rho)^{-1}$ of $A_\rho$, [2] Satz 5.14 and Satz 4.12

$$(\mu^{-1} - A_\rho)^{-1} = \sum_{k=1}^{p_j} (\mu^{-1} - \mu_j^{-1})^{-k} (A_\rho - \mu_j^{-1})^{k-1}\, P_j + W_j(\mu) \qquad (8)$$

Where $W_j$ is analytic in $\Omega_j \cup \{\mu_j\}$ and $P_j$ is the proyection into the finite dimensional subspace generated by the principal vectors of $A_\rho$ associated to the eigenvalue $\mu_j^{-1}$ of $A_\rho$, i.e.

$P_j L^2(0,1) = \bigcup_{k=1}^{\infty} \mathrm{Ker}\,(A - \mu_j^{-1})^k$ . It is known that $P_j^2 = P_j$, $A_\rho P_j = P_j A_\rho$ $(A_\rho - \mu_j^{-1})^n P_j \neq 0$ if $0 \leq n \leq p_j - 1$ and $(A_\rho - \mu_j^{-1})^{p_j} P_j = 0$ [2], Satz 4.12 . We have that

$$T(\mu) = (\mu - \mu_j)^{p_j} g(\mu) \ , \ g(\mu_j) \neq 0 \qquad (9)$$

If in (7) we replaced equations (8) and (9),take $1 \leq n \leq p_j$ and then the limit $\mu \longrightarrow \mu_j$ we get that the left hand side of (7) behaves as

$$-n \sum_{k=1}^{\infty} \mu_j^{2k-1} (\mu_j - \mu)^{-k} (A_\rho - \mu_j^{-1})^{k-1} P_j \ \psi_{\mu_j ; n} \qquad (10)$$

and the right hand side behaves as

$$\frac{p_j!}{(p_j - n)!} (\mu - \mu_j)^{-n} \ \psi_{\mu_j} \qquad (11)$$

comparing terms in (10) and (11) we get that

$$(A_\rho - \mu_j^{-1})^{k-1} P_j \ \psi_{\mu_j ; n} = 0 \quad \text{if} \quad n < k \leq p_j \qquad (12)$$

and

$$(-1)^{n-1} n \ \mu_j^{2n-1} (A_\rho - \mu_j^{-1})^{n-1} P_j$$

$$\psi_{\mu_j;n} = \frac{p_j!}{(p_j - n)!} \psi_{\mu_j} \tag{13}$$

By the equations (1) we know that

$$P_j \psi_{\mu_j;n} = \psi_{\mu_j;n} \quad \text{if} \quad 1 \le n \le p_j \tag{14}$$

and

$$(A_\rho - \mu_j^{-1})^{n-1} \psi_{\mu_j;n} = (-1)^{n-1}(n-1)!\, \mu_j^{2-2n} \psi_{\mu_j;1} \tag{15}$$

From (13), (14) and (15) we get

$$\frac{p_j!}{(p_j - n)!\, n} = (n-1)! \quad \text{or} \quad \frac{p_j!}{(p_j - n)!\, n!} = 1$$

Therefore $n = p_j = 1$ , since $1 \le n \le p_j$ .

## Acknowledments

*This work was realized while attending the X ELAM, held in Córdoba, Argentina. I would like to thank Proffessors R. Macias and G. Corach for the invitation to take part in this event. I am also grateful to the organizers of the X ELAM, and the Alexander von Humboldt Stiftung for financial support.*

**References**

[1] *J. Alcántara–Bode* , An integral equation formulation of the Riemann Hypothesis, manuscript submitted to the Journal of Integral Equations and Operator Theory.

[2] *K. Jörgens* , Lineare Integralenoperatoren, Teubner, Stuttgart, 1970.

[3] *A.C.Zaanen* , Linear analysis,North Holland, Amsterdam, 1964.