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BUBBLES SOLITON SOLUTIONS IN THE Φ^6 - FIELD THEORY

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Abstract

We have studied the ϕ^6 - model in the parameter domain $A > 1$. For this case we have found out a new types of soliton solutions such kinks, bubbles, and drops. The investigation of these waves around the stable minimum shows that the sound velocity provide a rigid constraint for this oscillations to be or no to stable: this is the question.

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1. Introduction

It has long been recognized that a remarkable progress has been made in obtaining exact solutions for nonlinear dynamical systems.The "non thermalize" effect is really very exciting problem and plays a great role in developing the understanding of soliton concepts in nature [1]. The role of solitons in the contemporary physics is generally recognized. The esential soliton aspect of natural phenomena has been made evident by the remarkable success in condensed matter and high energy physics. In this paper we will restrict ourselves to study the fi-six model in 1 + 1 dimension specifically. The fi-six model occurs in many branches of physics for example in the theory of ferromagnetism, the 2-body attractive and 3-body repulsive with skyrme forces in nuclear hydrodynamics [2] and in the study of soliton models of hadrons [3]. This theory also was used to describe phenomenologically phase transitions [4] and [5]. In addition of those mentioned works this model is also extensively used in the study of propagation of stionary light beams in the medium with the weakly saturating nonlinearity [6]. It is proved that all the above mentioned problem are governed by the following equation of motion.

$$i\psi_t + \Delta\psi + \alpha\psi + (|\psi|^2 - |\psi|^4)\psi = 0 \qquad (1)$$

or performing the scale transformation

$$\psi = \phi[2/3(2+A)]^{-1/2}$$

$$t' = \frac{4}{3}(A+2)t$$

$$x' = \frac{2}{\sqrt{3}}(A+2)x$$

we arrive at

$$i\phi_{t'}+\phi_{x'x'}-(1+2A)\phi+2(2+A)|\phi|^2\phi - 3|\phi|^4\phi = 0 \qquad (2)$$

This equation is more convenient for studying the excitations near the condensate. The coupling between the parameters is

$$\alpha = -3/4(1+2A)/(A+2)^2 \tag{3}$$

These equations support two types of solitons: a) topological solitons which are very similar to the well known soliton solutions of the cubic nonlinear Schrödinger equation NSE of the repulsive type. In $D = 1$ space these solution correspond to kinks, in $D = 2$ they are named vortices and there is no analogue at $D = 3$[7] The second type of solitons: b) nontopological solitons with specific propierties.These solitions have the form of stationary rarefaction bubbles [13] and exist in different domain of the parameter A.

The main aim of this work is to show that the equation (1) exhibits bubble, kink and drop type of solitions in the range of the parameter $A > 1$ which was not considered earlier, e.g. in [10], [12] and [13].

"One wants to be able to take a realistic view of the world, to talk about the world as if it is really there, even when it is not being observed... our business is to try to find out about it, and the technique for doing that is indeed to make models and to see how far we can go with them in accounting for the rel word" [9].

Pursuing on such a suggestion we will work with the model (1). The physically interesting boundary condition can be both zero and of the condensate type i.e.

$$\phi(x,t) \longrightarrow 0 \tag{4}$$

or

$$\phi(x,t) \longrightarrow \phi_0 \tag{5}$$

Let us consider first the bubble boundary condition (5). In the paper [13] the equation (2) under the condition (5) was considered. The new type of solitons were found namely "Bubbles". These nontopological solitons were assumed to exist at $A \in (0,1)$ not only for the one dimensional case but also they survive passing to arbitrary higher dimension. For this kind of solitons the complete investigation was performed including the analysis of the stability problem for the static and moving bubbles, see also [10]. They found that in the $D = 1$ case there exist a certain critical velocity v_c such that the bubble is stable if it is moving with a velocity v move greter then v_c and unstable otherwise.

The essentially important characteristic of solitons is their stability region which determines the boundaries of the validity of such a description and the phase transition to a new state. This kind of problematics will be presented elsewhere in more details.

2. Sound velocity and small oscillations

Now, for our convenience let us write the energy and particle number integrals of our model

$$E = \int dx\{|\phi_x|^2 + (|\phi|^2 - 1)^2 (|\phi|^2 - A)\} \tag{6}$$

$$N = \int dx\{|\phi|^2 - 1\} \tag{7}$$

The potential part of the energy E

$$U = (|\phi|^2 - 1)^2 (|\phi|^2 - A) \tag{8}$$

exibits a minimum when $A > 1$ in the point

$$|\phi|^2 = \beta = \frac{2A + 1}{3} \tag{9}$$

Note that the earlier condensate type of boundary condition $\phi_0 = 1$, completely studied earlier in [13] is also the minimun of (8) when $A < 1$. Normal mode perturbations with frecuency ω and wave number k are taken proportional to the factor

$$e^{-i\omega t+ikx}$$

. In linear systems the frecuency is subjet to a dispersion relation $\omega = \omega(k)$. The system is linearly unstable,if there exists a possitive growth rate $n = Im(\omega)$ for at least one real value of k. Now let us calculate the dispersion of small oscillations in the vacuum

$$\phi = \sqrt{\beta} + \xi(x,t) = \sqrt{\frac{1 + 2A}{3}} + \xi(x,t) \tag{10}$$

Where

$$\xi(x,t) = \eta(x,t)_1 e^{i(kx-\omega t)} + \eta(x,t)_2 e^{-i(kx-\omega t)}$$

The $\eta(x,t)_i$ satisfy the following equations

$$\omega\eta_1 - k^2\eta_1 + \eta_1(4\beta(2+A) - 9\beta^2 - \beta(1+2A)) + \eta_2(2\beta(2+A-3\beta)) = 0 \tag{11}$$

$$-\omega\eta_2 - k^2\eta_2 + \eta_2(4\beta(2+A) - 9\beta^2 - \beta(1+2A)) + \eta_1(2\beta(2+A-3\beta)) = 0 \tag{12}$$

After some algebra with operations regarding the determinant of the equations written above,one obtains

$$\omega^2 = k^4 + k^2(4/3)(A - 1)(1 + 2A) \tag{13}$$

This is the Bogoliubov dispersion law.From that we easily find the sound velocity

$$v_s = \lim_{k\to 0} \frac{\omega}{k} = \left(\frac{2}{\sqrt{3}}\right) \sqrt{(A-1)(2A+1)} \tag{14}$$

From the last equation we see that the condensate now is considered to be stable when $A \geq 1$.

3. Soliton Excitations of the Condensate

One of the primary requirements of all physical situation related with the problem of energy of the condensate is the fact that is must be always finite. It is the reason that the formula for energy takes the form (6) as it is easily to see varying this equation one can obtain the equation of motion. In order to obtain localized solutions of (3) we employ the formula

$$\phi(x,t) = \sqrt{\rho(x,t)}e^{i\Theta(x,t)} = \sqrt{\rho(\xi)}e^{i\Theta(\xi)}$$

and make

$$\phi(x,t) = \phi(\xi), \tag{15}$$

where

$$\xi = x - vt$$

Then we have

$$\frac{\rho''}{2\rho} - \frac{\rho'^2}{4\rho^2} - \Theta_\xi(\Theta_\xi - v) - 3(\rho-1)(\rho-\beta) = 0 \tag{16}$$

where the function Θ is defined by

$$\Theta_\xi = \frac{v}{2\rho}(\rho-\beta) \tag{17}$$

where

$$\beta = \frac{2A+1}{3}$$

Substituting the second equation (17) in the (16), then integrating and denoting

$$r = \rho-1; \qquad a = \frac{1}{4}(6\beta(\beta-1)-v^2) = \frac{1}{4}(v_s - v)$$

on can find that the above formulas reduce to the conventional form

$$\pm 2(\xi-\xi_0) = \int \frac{1}{r\sqrt{r^2 + (\frac{5A-2}{3})r + a}} dr$$

From this expression we can conclude that the regular localized soliton-like solutions exist if the parameter $a > 0$ or in other words if $v_s^2 - v^2 > 0$. We then have obtained the condition for existing a set of soliton solutions which take interesting forms in diferent regions of A. Inverting the integral written above one has

$$r(\xi) = \frac{-2a}{\sqrt{b^2 - 4a}\, ch(2\sqrt{a}(\xi-\xi_0)) \pm b} \tag{18}$$

where

$$b = \frac{5A - 2}{3} \tag{19}$$

Now if one wants to obtain a static soliton solution for $1 < A < 4$. For avoiding singularities we consider in the denominator of equation (18) only the (+) value of b

$$\phi_b = \sqrt{\beta} e^{i\Theta_0} \frac{ch(y)}{\sqrt{\frac{2A+1}{4-A} + sh^2(y)}} = \tag{20}$$

$$= e^{i\Theta} \sqrt{(4-A)/3} \frac{ch(y)}{\{1 + \frac{4-A}{3\beta} sh^2(y)\}^{1/2}}$$

where

$$y = (v_s/2)(x-x_0)$$

From the above formula we see that the function ϕ_b is an even function of the variable $(x-x_0)$. This function describes a bubble in the condensate. The amplitude of this bubble depends of the parameter $4-A$. For small values of this parameter we obtain a greater rarefaction in the bubble.

For $A > 4$ the soliton at rest take the form

$$\phi_k = \sqrt{\beta} e^{i\Theta_0} \frac{th(y)}{\sqrt{\frac{2A+1}{A-4} + ch^2(y)}} = \tag{21}$$

$$= \sqrt{(A-1)/3}\; e^{i\Theta} \frac{sh(y)}{\sqrt{1 + \frac{A-4}{2A+1} ch^2(y)}}$$

From here we conclude that ϕ_k is an odd function of the variables $(x-x_0)$. This solution has a kink form. The solution which joins the kinks and bubbles above written was found solving the differential equation not only for ρ but also for Θ. After a bit of calculation one gets

$$\phi_{k,b} = e^{i\Theta}\sqrt{2}\, \mathrm{ch}(\xi - i\mu)[(b)(b^2-4a)^{1/2} + ch2\xi]^{-1/2} \tag{22}$$

where

$$\xi = \frac{1}{2} (v_s{}^2 - v^2)^{1/2} (x - vt - x_0)$$

$$\cos 2\mu = \frac{[(A-\mu)\beta + v^2/2]}{\beta\sqrt{b^2 - 4a}}$$

The sign of $A - 4$ determines the solution to be or not to be a kink (bubble) like soliton solution. Fluctuations around the condensate would cause the appearance of kinks and to pass through the unstable vacum to the another stable symmetric one. If the kink is too small, the gain in volume energy by forming a region of true vaccum may not be enough so as to compensate the loss in the surface energy.

When $A = 4$ or $\alpha = -3/16$, we get a solution under the following mixed boundary conditions

$$\phi(x,t) \longrightarrow 0$$

for

$$\alpha \longrightarrow -\infty$$

$$\phi(x,t) \longrightarrow \sqrt{\beta} e^{\mathrm{i}\Theta}$$

for

$$x \longrightarrow +\infty$$

In order to have a finite system energy the solution obeys

$$2(\xi - \xi_0) = \int \frac{d\rho}{(\rho-2)\sqrt{\rho^2 - \frac{v^2}{4}}}$$

The integral of motion: the hole number and the energy for the solution (22) have the forms:

$$N_{k,b} = \int_{-\infty}^{\infty} dx\{\rho - 1\} \tag{23}$$

$$E_{k,b} = \int_{-\infty}^{\infty} dx\{|\phi_x|^2 + (\rho-1)^2(\rho-A)\} \tag{24}$$

where

$$\phi_x = e^{i\Theta}\left[\frac{\rho_x}{2\sqrt{\rho}} + i\Theta_x\sqrt{\rho}\right] \tag{25}$$

When the amplitude of the solutions are small or in other words the velocity of the kinks (bubbles) are closed to the sound velocity.

The modules of the solution are

$$\sqrt{|\phi|^2} = \left\{1-2a\left\{\sqrt{b^2-4a}\ \mathrm{ch}(2\sqrt{a}(\xi-\xi_0)) + b\right\}^{-1}\right\}^{-1/2} \tag{26}$$

Note that for small a we can write

$$\sqrt{b^2 - 4a} \approx b - \frac{2a}{b}$$

if

$$a \ll b = \frac{5A - 2}{3}$$

The coefficient $\sqrt{b^2 - 4a}$ in the second term can have been replaced by $\frac{5A-2}{3}$ and one gets

$$\sqrt{|\phi|^2} = \left\{1 - \frac{2a\left(\frac{5A-2}{3}\right)^{-1}}{1 + \mathrm{ch}(2\sqrt{a}(\xi-\xi_0))}\right\}^{1/2} = \tag{27}$$

$$\left\{1 - \frac{3a(5A-2)^{-1}}{\mathrm{ch}^2(\sqrt{a}(\xi-\xi_0))}\right\}^{1/2} \tag{28}$$

This solution can be found by means of the method which will be developed in our next paper treating small amplitude soliton waves.

Besides the solutions given above, the ϕ^6 model possesses other localized solutions; for more information see [12].

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References

[1] A. Newell, *Soliton and Mathematics* SIAM

[2] V.G. Kartavenko. *Yader. Fiz,* **40** 1984, p.377

[3] R. Friedberg, et.al. *Nuclear Physics,* 1976, **115B** p.32, Phys.Rev. D35 (1987) N°6, p.1550

[4] M.A. Agüero G. In *Procceding of Nonlinear Evolution Equation and Dynamical Systems.* Dubna 1990. Springer-Verlag 1991 p.207
M.A. Agüero G Makhankov V.G. Preprint JINR, 1991

[5] L. Masperi *Phys. Rev.* D, **41**, (1990) 3263

[6] V.E. Zakharov, V.V. Sobolev and V.S.Synasch *Sov. Phys. JETP* **33** 1971, p.77

[7] L.P. Pitayevsky *Sov. Phys JETP* **13** (1961) p.451

[8] V.L. Ginsburg and A.A. Solianin; *Usp. Fiz. Nauk* 120 (1976) 153

[9] J.S. Bell *The Ghost in the Atom* Cambridge University Press, 1986, p.50

[10] Puzinin I.V. et. al. Physica D 34 (1989) p.240

[11] A.S. Kosevich *et. al Sov. J.Low Temp.Phys.***2** 1976 p.913

[12] V.G.Makhankov *Soliton Phenomenology.* Mathematics and Applications **33**, Kluwer Academic Publishers, 1990

[13] I.V. Barashenkov, V.G.Makhankov. *Phys.Lett.*,**A128** (1982) 52.