Pro Mathematica, **33**, *66 (2025), 7-29*
DOI: *10.18800/promathematica.202501.001*

Representability of Chow Groups of 0-Cycles and the Gysin Kernel

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agosto, 2025

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Abstract

The study of algebraic cycles is a highly developed and fascinating subject, which not only occupies a prominent place in Algebraic Geometry, but also shares common ground with, for example, Complex Geometry (e.g., via Hodge Conjecture), Arithmetic Geometry (e.g., via the study of rational points), Number Theory (e.g., via Tate Conjecture), Algebraic K-Theory (e.g., via Motivic Cohomology), and Mathematical Physics. In this work I will discuss the problem of the representability of Chow groups of algebraic 0*-cycles on surfaces through study of the Gysin kernel which allows us to obtain useful results for studying Bloch's conjecture and constant cycles curves (ccc).*

MSC(2020): 14C15 (primary) 14D15 (secondary).

Keywords: *Algebraic cycles; Chow groups; Gysin kernel; Representability; Constant cycle curves.*

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1. Introduction

Let k be an algebraically closed field of characteristic 0. Let X be a smooth projective variety over k (i.e., separated integral scheme of finite type over k) of dimension n. Let $\mathsf{Z}_r(X)$ be the group of algebraic cycles of dimension r, $\mathsf{Z}_r(X)_{\mathsf{rat}}$ the subgroup of $\mathsf{Z}_r(X)$ of r-cycles rationally equivalent to 0, and $\mathsf{CH}_r(X) = \frac{\mathsf{Z}_r X)}{\mathsf{Z}_r(X)_{\mathsf{rat}}}$ the Chow group of cycles of dimension r on X. Let $\mathsf{Z}_r(X)_{\mathsf{alg}}$ be the subgroup of $\mathsf{Z}_r(X)$ of r-cycles algebraically equivalent to 0, and let $\mathsf{A}_r(X) = \frac{\mathsf{Z}_r(X)_{\mathsf{alg}}}{\mathsf{Z}_r(X)_{\mathsf{rat}}} \subset \mathsf{CH}_r(X)$ be the continuous part of the Chow group $\mathsf{CH}_r(X)$ of r-cycles.

Since X is a variety, if we prefer to think in terms of codimension, we can use the following identity: $\mathsf{Z}_r(X) = \mathsf{Z}^{n-r}(X)$, where $\mathsf{Z}^{n-r}(X)$ is the group of cycles of codimension $n-r$.

It is already known that $A^0(X) = \mathsf{A}_n(X) = 0$, and $A^1(X) = \mathsf{A}_{n-1}(X)$ is representable, i.e., it forms in a natural way the set of points of an algebraic variety (in fact, it is an abelian variety). Naively speaking, this means that we can draw a picture of $\mathsf{A}^1(X)$. However, when $i \geq 2$ a fascinating mystery involves $\mathsf{A}^i(X)$ because $\mathsf{A}^i(X)$ is in general not representable, so one encounters for the first time objects which are geometric in content and joyously non-representable; and the following two questions have been intensively studied: *What kind of structure do* $\mathsf{A}^i(X)$ *have?* and *Are there hypotheses in* X *that will ensure that* $\mathsf{A}^i(X)$ *is representable?* (see [4]).

Note that when X is a surface, we have $\mathsf{A}^2(X) = \mathsf{A}_0(X)$. In this case, it is known that $\mathsf{A}_0(X) = \mathsf{CH}_0(X)_{\mathrm{deg}=0} = \mathsf{CH}_0(X)_{\mathrm{hom}}$, where $\mathsf{CH}_0(X)_{\mathrm{deg}=0}$ is the Chow group of 0-cycles of degree zero and $\mathsf{CH}_0(X)_{\mathrm{hom}}$ is the group of r-cycles homologically equivalent to 0 modulo rational equivalence, and there is a hypotheses in X, due to S. Bloch, that will ensure that $\mathsf{A}^2(X)$ is representable known as *Bloch's conjecture.* In this note we present another hypothesis through the study of the Gysin kernel obtained in [13] and [14] that will ensure that $\mathsf{A}^2(X)$ is representable.

In the first section, we give the necessary background. In the second section, we review the known results for surfaces obtained in the papers [13] and [14]. In the third section, we explain the relation of the results obtained in [13] and [14] with the representability of Chow groups and Bloch's conjecture, and in the fourth section, we explain the relation of the results obtained in loc. cit. with the study of constant cycle curves (ccc).

2. Preliminaries

2.1. Algebraic Cycles

Let k be an algebraically closed field of characteristic 0 and let X be a smooth projective k-variety (separated integral scheme of finite type over k).

Definition 2.1 (Algebraic r-cycle)**.** *An* algebraic cycle of dimension r *or simply an* r-cycle *on X is a finite formal linear combination*

$$Z = \sum n_i Z_i,$$

where $n_i \in \mathbb{Z}$, and Z_i are subvarieties of dimension r of X.

Notation The set of r-cycles on X is a group and is denoted by

$$\mathsf{Z}_r(X).$$

Remark 2.2. If X is a variety with $\dim(X) = n$ (so, in particular, it is a purely n-dimensional scheme), then we have

$$\mathsf{Z}_r(X) = \mathsf{Z}^{n-r}(X),$$

where $\mathsf{Z}^{n-r}(X)$ is the group of cycles of codimension $n - r$ on X.

Two important examples are.

Example 2.3. $0-$cycles *on X are finite formal linear combinations of points of X, i.e.,*

$$Z = \sum n_i P_i,$$

where $n_i \in \mathbb{Z}$ and P_i are points of X.

Example 2.4. Cycles of codimension 1 *or* divisors *on X are finite formal linear combinations of subvarieties of codimension 1 of X, i.e.,*

$$Z = \sum n_i Z_i,$$

where $n_i \in \mathbb{Z}$ and Z_i are subvarieties of codimension 1 of X.

2.2. Rational, Algebraic and Homological Equivalence

Let X be a smooth projective variety over an algebraically closed field k of characteristic 0.

Informally, we say that two r-cycles Z and Z' on X are rationally equivalent if there exists a family of r-cycles on X parametrized by $\mathbb{P}^1$ interpolating between them. More precisely, if we restrict ourselves to smooth projective varieties, we have the following definition of rational equivalence.

Definition 2.5 (Rational equivalence)**.** *Two r-cycles Z and Z' on X are* rationally equivalent*, denoted by $Z \sim_{\mathsf{rat}} Z'$, if there exists $W \in \mathsf{Z}_{r+1}(X \times \mathbb{P}^1)$ such that defining by*

$$W_t := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\})), \qquad t \in \mathbb{P}^1$$

we have

$$Z = W_{t_1} \text{ and } Z' = W_{t_2} \text{ for some } t_1, t_2 \in \mathbb{P}^1.$$

Here, $\cdot$ denotes the intersection product.

(see [7, §1.6] or [12, Lemma 1.2.5], [6, §1.2.2], [4, Introduction] and [8, Introduction]).

Notation. The set of r-cycles rationally equivalent to 0 is a subgroup of $\mathsf{Z}_r(X)$ and is denoted by

$$\mathsf{Z}_r(X)_{\mathsf{rat}} = \{Z \in \mathsf{Z}_r(X) : Z \sim_{\mathsf{rat}} 0\}.$$

Definition 2.6 (Chow group)**.** *The group of r-cycles modulo rational equivalence on X, i.e., the group quotient*

$$\mathsf{CH}_r(X) = \frac{\mathsf{Z}_r(X)}{\mathsf{Z}_r(X)_{\mathsf{rat}}},$$

of rational equivalence classes of r-cycles is called the Chow group of r-cycles.

Definition 2.7 (The Chow group of 0-cycles of degree zero.)**.** *Let* deg : $\mathsf{CH}_0(X) \to \mathbb{Z}$, *be the degree homomorphism. The kernel of the degree homomorphism*

$$\mathsf{CH}_0(X)_{\deg=0} = \mathsf{Ker}(\deg : \mathsf{CH}_0(X) \to \mathbb{Z})$$

is the Chow group of 0-cycles of degree zero.

Rather than defining cycles as equivalent only if there is a family parametrized by $\mathbb{P}^1$ we can instead allow families parametrized by curves; this gives rise to the notion of algebraic equivalence.

Definition 2.8 (Algebraic equivalence)**.** *Two r-cycles Z and Z' on X* algebraically equivalent, *denoted by $Z \sim_{\mathsf{alg}} Z'$, if and only if there exists a smooth connected curve C, a cycle $W \in \mathsf{Z}_{r+1}(X \times C)$, such that for any $t \in C$ defining by*

$$W_t := (\mathsf{pr}_X)_*(W \cdot (X \times \{t\})),$$

we have

$$W_{t_1} = Z \text{ and } W_{t_2} = Z'$$

for some $t_1, t_2 \in C$.

Notation. The set of r-cycles that are algebraically equivalent to 0 is a subgroup of $\mathsf{Z}_r(X)$ and is denoted by

$$\mathsf{Z}_r(X)_{\mathsf{alg}} = \{Z \in \mathsf{Z}_r(X) : Z \sim_{\mathsf{alg}} 0\}.$$

Remark 2.9. By definition, it is clear that $\mathsf{Z}_r(X)_{\mathsf{rat}} \subset \mathsf{Z}_r(X)_{\mathsf{alg}}$.

Definition 2.10 (The continuous part of the Chow group)**.** *The group quotient of cycles algebraically equivalent to* 0 *modulo rational equivalence is denoted by*

$$\mathsf{A}_r(X) = \frac{\mathsf{Z}_r(X)_{\mathsf{alg}}}{\mathsf{Z}_r(X)_{\mathsf{rat}}} \subset \mathsf{CH}_r(X).$$

This group should be thought of as the continuous part of the Chow group of r-cycles (see [4, Introduction]).

One way to define a *Weil-cohomology theory* is as a contravariant functor

$$X \to H^*(X)$$

from the category of varieties to the category of augmented, finite dimensional, anti-commutative F-algebras which satisfies some properties, see [10, §1.2.] (here F is a field of characteristic 0 called the coefficient field). In particular, there are group homomorphisms

$$cl_X : \mathsf{Z}^r(X) \to H^{2r}(X)$$

called *cycle maps.*

Fixing a Weil-cohomology theory, we can now define homological equivalence.

Definition 2.11 (Homological equivalence)**.** *Let* $\dim(X) = n$*. A cycle* Z *of dimension* r *on* X *is* homologically equivalent to 0*, denoted by* $Z \sim_{\text{hom}} 0$*, if* $cl_X(Z) = 0$*. Here* $cl_X : \mathsf{Z}_r(X) \to H^{2(n-r)}(X)$ *is the cycle map.*

Notation. The set of cycles of dimension r homologically equivalent to 0 forms a subgroup of $\mathsf{Z}_r(X)$ which is denoted by $\mathsf{Z}_r(X)_{\hom}$.

Remark 2.12. This definition depends on the choice of a Weil cohomology theory ([12, Definition 1.2.16]).

Since $\mathsf{Z}_r(X)_{\mathsf{rat}} \subset \mathsf{Ker}(cl)$ by the fundamental theorem on homomorphisms, the cycle map

$$cl_X : \quad \mathsf{Z}_r(X) \quad \to \quad H^{2n-2r}(X)$$

thus gives the *cycle class map*

$$\mathrm{cl}_X : \mathsf{CH}_r(X) \to H^{2n-2r}(X).$$

Definition 2.13 (The group $\mathsf{CH}_r(X)_{\hom}$)**.** *The kernel of the cycle* class *map* cl_X *is denoted by*

$$\mathsf{CH}_r(X)_{\hom} = \mathsf{Ker}(\mathrm{cl}_X : \mathsf{CH}_r(X) \to H^{2n-2r}(X)).$$

Note that $\mathsf{CH}_r(X)_{\hom}$ is the group of r-cycles homologically equivalent to 0 modulo rational equivalence, i.e.,

$$\mathsf{CH}_r(X)_{\hom} = \frac{\mathsf{Z}_r(X)_{\hom}}{\mathsf{Z}_r(X)_{\mathsf{rat}}}.$$

Proposition 2.14. *Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic* 0*. Then*

$$\mathsf{CH}_0(X)_{\deg=0} = \mathsf{CH}_0(X)_{\hom} = \mathsf{A}_0(X).$$

(See [13], see also [12]).

2.3. Representability of $\mathsf{CH}_0(X)_{\deg=0}$

Let X be a connected smooth projective variety over $\mathbb{C}$ of dimension n.

Definition 2.15 (Symmetric product)**.** *The* d-th symmetric product *of a variety* X*, denoted by* $\mathsf{Sym}^d(X)$*, is the quotient variety*

$$\mathsf{Sym}^d(X) = X^d/\Sigma_d,$$

where X^d *is the self-product of* X *and* Σ_d *is the group of permutations of the factors.*

The d-th symmetric product $\mathsf{Sym}^d(X)$ is a singular variety of dimension nd, where $n = \dim(X)$ (see [15, §10]).

Remark 2.16. As a set $\mathsf{Sym}^d(X)$ coincides with the set of effective 0-cycles of degree d, i.e.,

$$\mathsf{Sym}^d(X) \underset{\text{as set}}{=} \{ \text{ effective 0-cycles of degree } d \text{ on } X\}.$$

Definition 2.17 (Difference map)**.** *The set-theoretic map*

$$\begin{array}{rccc} \theta_d^X : & \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X) & \to & \mathsf{CH}_0(X)_{\deg=0} \\ & (A, B) & \mapsto & [A - B] \end{array}$$

where $[A-B]$ *is the class of the cycle* $A-B$ *modulo rational equivalence, is usually called the* difference map.

Definition 2.18 (Representability of $\mathsf{CH}_0(X)_{\deg=0}$)**.** $\mathsf{CH}_0(X)_{\deg=0}$ *is* representable *if one of the following equivalent facts holds:*

1. *the difference map*

$$\theta_d^X : \mathsf{Sym}^d(X) \times \mathsf{Sym}^d(X) \to \mathsf{CH}_0(X)_{\deg=0}$$

is surjective for sufficiently large d (see[15, Definition 10.6]).

2. *the Albanese map*

$$alb_X : \mathsf{CH}_0(X)_{\deg=0} \to \mathrm{Alb}(X)$$

is an isomorphism. Here $\mathrm{Alb}(X)$ *is the Albanese variety of* X *(see [15, Theorem 10.11]).*

3. $\mathsf{CH}_0(X)_{\deg=0}$ *is finite dimensional (see [15, Proposition 10.10]).*

3. The Gysin Kernel

Before stating the results on the Gysin kernel whose detailed proofs can be found in [13] and [14] let us first recall the following definitions

Definition 3.1 (**c-open**)**.** *Given an integral scheme T, a (Zariski)* c-closed *subset in T is a countable union of irreducible closed subsets in T, and a (Zariski)* c-open *subset in T is the complement of a c-closed subset of T.*

Definition 3.2 (**Very general** notion)**.** *Let T be an integral scheme. We say that a* property Q holds for a very general point on T *if there exists a c-open subset in T such that Q holds for each point in this c-open.*

3.1. A Theorem on The Gysin Kernel

Let S be a smooth projective surface over $\mathbb{C}$, and consider a complete linear system $\Sigma = |D| = |\mathcal{O}_S(D)|$ of a very ample divisor D on S with corresponding very ample invertible sheaf $\mathcal{O}_S(D)$, say of dimension d.

Recall that

$$\Sigma = |D| \cong \mathbb{P}(\Gamma(S, \mathcal{O}_S(D))) \cong \mathbb{P}^{d*},$$

where $\mathbb{P}^{d*}$ is the dual projective space of $\mathbb{P}^d$ parametrizing hyperplanes in $\mathbb{P}^d$, and let $\overline{\eta}$ be its geometric generic point.

For each closed point $t \in \Sigma \cong (\mathbb{P}^d)^*$, let H_t be the corresponding hyperplane in $\mathbb{P}^d$, $C_t = S \cap H_t$ the corresponding hyperplane section of S, and

$$r_t : C_t \hookrightarrow S$$

the closed embedding of the curve C_t into S.

Consider the *discriminant variety of S* or the *discriminant locus of Σ*

$$\Delta_S = \{t \in \Sigma : C_t \text{ is singular } \},$$

i.e., the subset of Σ parameterizing singular hyperplane sections of S, and let

$$U = \Sigma \setminus \Delta_S = \{t \in \Sigma : C_t \text{ is smooth }\}$$

be the complement of Δ_S parametrizing smooth hyperplane sections of S.

For any closed point $t \in U$: Let

$$r_{t*} : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z}),$$

be the Gysin homomorphism on cohomology groups induced by r_t, and let $H^1(C_t, \mathbb{Z})_{van} = \mathsf{Ker}(r_{t*} : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z}))$, called the *vanishing cohomology.*

Let $J_t = J(C_t)$ be the Jacobian of the curve C_t, and B_t the subvariety in the Jacobian J_t of the curve C_t associated or corresponding to the Hodge substructure on $H^1(C_t, \mathbb{Z})_{van}$.

Let

$$\mathsf{CH}_0(C_t)_{\deg=0} = \mathsf{Ker}(deg : \mathsf{CH}_0(C_t) \to \mathbb{Z})$$

be the Chow group of 0-cycles of degree zero on C_t,

$$\mathsf{CH}_0(S)_{\deg=0} = \mathsf{Ker}(deg : \mathsf{CH}_0(S) \to \mathbb{Z})$$

the Chow group of 0-cycles of degree zero on S, and

$$r_{t*} : \mathsf{CH}_0(C_t)_{\deg=0} \to \mathsf{CH}_0(S)_{\deg=0}$$

the Gysin homomorphism on Chow groups induced by r_t whose kernel

$$G_t = \mathsf{Ker}(r_{t*})$$

will be called the *Gysin kernel* (Gk) associated with C_t.

Theorem 3.3 (**A theorem on the Gysin kernel**). *Let S, U, G_t, A_t, B_t and J_t be as above.*

(a) *For every $t \in U$, there is an abelian subvariety $A_t \subset B_t \subset J_t$ such that*

$$G_t = \bigcup_{countable} \textit{translates of } A_t$$

(b) *For a very general $t \in U$ either, $A_t = 0$, or $A_t = B_t$.*

For the above we mean: There exits a c-open subset $U_0 \subset U$ such that either $A_{\overline{\eta}} = 0$, in which case $A_t = 0$ for all $t \in U_0$, or $A_{\overline{\eta}} = B_{\overline{\eta}}$, in which case $A_t = B_t$ and for all $t \in U_0$.

As an application of this theorem, we prove the following theorem

Theorem 3.4 (A theorem on the 0-cycles on surfaces)**.** *If*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$$

is not an isomorphism, then G_t is countable, for a very general $t \in U$.

Its contrapositive form, and hence equivalent form, says

Theorem 3.5 (A theorem on the 0-cycles on surfaces)**.** *If G_t is uncountable, for a very general $t \in U$, then*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$$

is an isomorphism.

This theorem immediately implies the following result

Corollary 3.6. *If $p_g(S) \neq 0$, then G_t is countable for a very general $t \in U$.*

Proof. If $p_g(S) \neq 0$, then by Mumford theorem (see Theorem 4.1) alb_S is not an isomorphism, then by Theorem 3.4 we have that G_t is countable for a very general $t \in U$. $\square$

This corollary provides us with the following example.

Example 3.7. *Let S be a K3 surface or an abelian surface over $\mathbb{C}$, and let Σ be the complete linear system of a very ample divisor on S. It is known that $p_g(S) \neq 0$ (see [1, Chapter VIII]), so by Corollary 3.6 G_t is countable, for a very general $t \in U \subset \Sigma$.*

If in addition S is a surface with irregularity $q(S) := \dim H^1(S, \mathcal{O}_S) = 0$ (so, then $H^3(S, \mathbb{Z}) = 0$ and $\mathrm{Alb}(S) = 0$) the Gysin homomorphism

$$r_{t*} : H^1(C_t, \mathbb{Z}) \to H^3(S, \mathbb{Z}),$$

is the zero map, then $H^1(C_t, \mathbb{Z}) = H^1(C_t, \mathbb{Z})_{van}$, and therefore $J_t = B_t$, for every $t \in U$.

In this case, Theorem 3.3 says

Corollary 3.8. *If in addition $q(S) = 0$. Then*

(a) This item remains the same as in Theorem 3.3.

(b) For very general $t \in U$, either $A_t = 0$ or $r_{t*} = 0$.

For the above, we mean: there exists a c-open subset $U_0 \subset U$ such that for all $t \in U_0$ we have $A_t = 0$ or for all $t \in U_0$ we have $r_{t*} = 0$.

On the other hand, Theorem 3.5 says

Corollary 3.9. *Assume, in addition, that $q(S) = 0$. If $r_{t*} = 0$, for a very general $t \in U$, then*

$$\mathsf{CH}_0(S)_{\deg=0} \cong 0.$$

4. The Gysin kernel and the representability of Chow groups

Let S be a connected smooth projective surface over $\mathbb{C}$. Mumford proved that in general $\mathsf{CH}_0(S)_{\deg=0}$ does not have the size of an algebraic variety; more precisely he proved the following theorem

Theorem 4.1 (**Mumford's Theorem [11]**)**.** *If the geometric genus* $p_g(S) = \dim(H^0(S, \Omega_S^2)) \neq 0$*. Then*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$$

is not an isomorphism, i.e., $\mathsf{CH}_0(S)_{\deg=0}$ *is not representable.*

Bloch's conjecture is the converse of Mumford's theorem.

Conjecture 4.2 (**Bloch's Conjecture [3]**)**.** *If* $p_g(S) = 0$*, then*

$$alb_S : \mathsf{CH}_0(S)_{\deg=0} \to \mathrm{Alb}(S)$$

is an isomorphism, i.e., $\mathsf{CH}_0(S)_{\deg=0}$ *is representable.*

Bloch's conjecture has been proved for surfaces of special type, that is, for surfaces with Kodaira dimension $\kappa(S) < 2$ by Bloch, Kas, and Lieberman in [5]. For surfaces of general type (sgt), i.e., surfaces with $\kappa(S) = 2$, the conjecture has not yet been proved except for some cases.

If $\kappa(S) = 2$, i.e., S is of general type, vanishing of the geometric genus of S implies the vanishing of the irregularity of S (so then $Alb(S) = 0$) and Bloch's conjecture simply states

Conjecture 4.3 (Bloch's conjecture for sgt)**.** *Let S be a surface of general type. If* $p_g(S) = 0$*, then*

$$\mathsf{CH}_0(S)_{\deg=0} \cong 0,$$

i.e., $\mathsf{CH}_0(S)_{\deg=0}$ *is representable.*

Note that $\mathsf{CH}_0(S)_{\deg=0} \cong 0$ means that any two points on S are rationally equivalent to each other.

Bloch's conjecture for surfaces of general type is the hard case of Bloch's conjecture and only known for some particular cases.

4.1. Relation with the representability of Chow groups

Let S be a surface as in the hypothesis of Theorem 3.3.

The connection of our results, i.e., the study of the Gysin kernel (Gk) with the representability of Chow groups of 0-cycles on surfaces, is stated in the following corollary

Corollary 4.4 (Gysin kernel and representability)**.** *If G_t is uncountable for a very general $t \in U$, then $\mathsf{CH}_0(S)_{\deg=0}$ is representable.*

Proof. This follows immediately from our Theorem 3.5 and the definition of the representability of $\mathsf{CH}_0(S)_{\deg=0}$ (see Definition 2.18). □

This result gives a hypothesis that will ensure that $\mathsf{A}^2(S) = \mathsf{CH}_0(S)_{\deg=0}$ is representable through the study of the Gysin kernel, and it is an interesting approach to prove which surfaces are representable.

If in addition the surface S is such that $q(S) = 0$, we get the following hypothesis that will ensure that $\mathsf{A}^2(S) = \mathsf{CH}_0(S)_{\deg=0}$ is representable through the study of the Gysin kernel

Corollary 4.5. *Assume, in addition, that $q(S) = 0$. If $r_{t*} = 0$, for a very general $t \in U$, then $\mathsf{CH}_0(S)_{\deg=0}$ is representable.*

Proof. It follows from Corollary 3.9 and from the definition of representability for surfaces with $q(S) = 0$ (see Definition 2.18 and note that $q(S) = 0$ implies $\mathrm{Alb}(S) = 0$). □

Remark 4.6. In the above corollary we can replace the hypothesis "$r_{t*} = 0$, for a very general $t \in U$" by the hypothesis "$G_t = J_t$, for a very general $t \in U$", since they are equivalent.

For surfaces of general type, we also obtain the following criteria to study the representability of $\mathsf{CH}_0(S)_{\deg=0}$ through the study of the Gysin kernel

Corollary 4.7 (Gk and representability for sgt). *Let S be a surface of general type with $p_g(S) = 0$. Then $r_{t*} = 0$, for a very general $t \in U$, if and only if $\mathsf{CH}_0(S)_{\text{deg}=0}$ is representable.*

Proof. Since S is a surface of general type with $p_g(S) = 0$ we have $q(S) = 0$.

Now, if $r_{t*} = 0$, for a very general $t \in U$, by Corollary 4.5 $\mathsf{CH}_0(S)_{\text{deg}=0}$ is representable. Reciprocally, assume that $\mathsf{CH}_0(S)_{\text{deg}=0}$ is representable, i.e., $\mathsf{CH}_0(S)_{\text{deg}=0} \cong 0$, then clearly $r_{t*} = 0$, for a very general $t \in U$. □

Note that in fact this result gives us a new definition of the representability through the notion of the Gysin kernel for surfaces of general type with geometric genus zero.

5. The Gysin kernel and the Constant cycles curves

In this section, we will define the notions of Pointwise constant cycle curves (pccc) and Constant cycle curves (ccc), and we will explain the connection of the study of the Gysin kernel and the representability with the study of these notions.

5.1. Pointwise constant cycle curves (pccc)

The notion of pointwise constant cycle curves originated when Beauville and Voisin in [2] described a distinguished element $c_S \in \mathsf{CH}_0(S)$ when S is a K3 surface over $\mathbb{C}$. More precisely, they proved that there exists a class $c_S \in \mathsf{CH}_0(S)$ of degree 1 such that

$$\text{If } x \in C \subset S \text{ with } C \text{ rational, then } [x] = c_S.$$

The distinguishing class c_S is also called the *Beauville-Voisin class.*

This property inspired the definition of a pointwise constant cycle curve introduced by D. Huybrechts in [9].

Definition 5.1 (Pointwise constant cycle curve (pccc))**.** *Let S be a projective K3 surface over a field k. A curve C in S is a pointwise constant cycle curve (pccc) if all closed points $x \in C$ define the same class $[x]$ in $\mathsf{CH}_0(S)$.*

Over an algebraically closed field, we have the following equivalent definition.

Definition 5.2 (Pointwise constant cycle curve (pccc))**.** *Let S be a projective K3 surface over an algebraically closed field k. A curve C in S is a pointwise constant cycle curve (pccc) if one of the following equivalent conditions holds:*

1. *for all closed points $x \in C$ we have $[x] = c_S \in \mathsf{CH}_0(S)$.*
2. *the natural map*
$$r_{C*} : J(\tilde{C}) = \mathsf{CH}_0(\tilde{C})_{\deg=0} \to \mathsf{CH}_0(S)$$
is the zero map. Here $r_C : \tilde{C} \to S$ is the composition of the normalization $\tilde{C} \to C$ with the closed embedding $C \hookrightarrow S$.
3. $$G_C = \mathsf{Ker}(r_{C*}) = J(\tilde{C}),$$
that is, the kernel of the Gysin homomorphism r_{C*} coincides with $J(\tilde{C}) = \mathrm{Pic}^0(\tilde{C})$.

Note that this definition relates the study of the Gysin kernel with the study of pointwise constant cycle curves, Chow groups, and the distinguished cycle c_S on $K3$ surfaces when the ground field is algebraically closed.

Example 5.3. *Let S be a $K3$ surface over an algebraically closed field k. A rational curve $C \subset S$ in S is a pccc.*

Then D. Huybrechts in [9] introduced, over an algebraically closed field, a finer version of pccc, i.e., he introduced the notion of constant cycles curves (ccc).

5.2. Constant cycles curves (ccc)

Let S be a projective K3 surface over an algebraically closed field k.

In order to define the ccc we need the following definition.

Definition 5.4 (The class of a integral curve). *Let $C \subset S$ be an integral curve in S. Let $\Delta_C \subset S \times C$ be the graph of the inclusion $C \hookrightarrow S$ and $x_0 \in S$ be a point such that $[x_0] = c_S$. The class of C is the class*

$$\kappa_C \in \mathsf{CH}_0(S \times k(\eta_C))$$

of the restriction of the cycle $\Delta_C - \{x_0\} \times C$ in $S \times C$ to the generic fiber $S_{k(\eta_C)} = S \times_k k(\eta_C)$ of the second projection $S \times C \to C$.

Definition 5.5 (Constant cycle (integral) curve). *Let $C \subset S$ be an integral curve in S. We say that C is a constant cycle curve if its class $\kappa_C \in \mathsf{CH}_0(S \times k(\eta_C))$ is a torsion class, i.e., it is an element of $\mathsf{CH}_0(S \times k(\eta_C))$ of finite order (see [9, §3.2]). The order of the constant cycle curve C is the order of the torsion class κ_C.*

Definition 5.6 (Constant cycle curve (ccc)). *An arbitrary curve $C \subset S$ is a* constant cycle curve (ccc) *if every integral component of C is a constant cycle curve. The order of an arbitrary constant cycle curve C is the maximal order of its integral components.*

Finding a criterion that decides whether a given curve is a constant cycle curve seems as hard as finding a criterion that would ensure the opposite (see [9]).

5.3. Relation between pccc and ccc

When the ground field is algebraically closed and uncountable (e.g. $\mathbb{C}$), the notion of pccc coincides with the notion of ccc. More precisely, we have the following proposition.

Proposition 5.7. *Let S be a projective K3 surface over an algebraically closed field k. Then a ccc $C \subset S$ is also a pccc. If k is uncountable, the converse holds true as well ([9, Proposition 3.7]).*

5.4. Pointwise constant cycle curves and Constant cycle curves on arbitrary surfaces

The notion of ccc can be defined for any arbitrary surface (see [9, §3.2]). Thanks to Lemma 3.8 due to D. Huybrechts we have the following definition (see [9, Lemma 3.8]).

Definition 5.8 (ccc on arbitrary surface)**.** *Let S be a smooth projective surface over an algebraically closed field k. An integral curve $C \subset S$ is a constant cycle curve if one of the following equivalent conditions holds:*

1. *There exists a positive integer n such that*

$$n \cdot [\eta_C] \in \mathrm{im}(\mathsf{CH}_0(S) \to \mathsf{CH}_0(S \times_k k(\eta_C))),$$

   where the generic point $\eta_C \in C$ is viewed as a closed point in $S \times_k k(\eta_C)$.

2. *If $\eta_C \in C$ is viewed as a point in the geometric fiber $S \times_k k(\eta_C)$, then*

$$[\eta_C] \in \mathrm{im}(\mathsf{CH}_0(S) \to \mathsf{CH}_0(S \times_k k(\eta_C))).$$

An arbitrary curve $C \subset S$ is a constant cycle curve (ccc) *if every integral component of C is a constant cycle curve.*

Example 5.9. *Let S be a smooth projective surface over $\mathbb{C}$ with $p_g(S) = q(S) = 0$ satisfying Bloch's conjecture, then every curve on S is a ccc (see [9, Proposition 4.1]).*

Similarly, the notion of pointwise constant cycle curves (pccc) can be defined for any arbitrary surface as follows

Definition 5.10. *Let S be a smooth projective surface over a field k. A curve C on S is a pointwise constant cycle curve (pccc) if all closed points $x \in C$ define the same class $[x]$ in $\mathsf{CH}_0(S)$.*

Over an algebraically closed field, we have the following equivalent definition.

Definition 5.11 (pccc on arbitrary surface)**.** *Let S be a smooth projective surface over an algebraically closed field k. A curve C in S is a pointwise constant cycle curve (pccc) if the natural Gysin map*

$$r_{C*} : J(\tilde{C}) = \mathsf{CH}_0(\tilde{C})_{\deg=0} \to \mathsf{CH}_0(S)$$

is the zero map. Here $r_C : \tilde{C} \to S$ is the composition of the normalization $\tilde{C} \to C$ with the closed embedding $C \hookrightarrow S$.

5.5. Constant cycle curves and the Representability

Let S be a surface as in the hypothesis of Theorem 3.3 such that the notion of pccc coincides with the notion of ccc, for example, projective K3 surfaces over $\mathbb{C}$.

The connection of our results with the notion of ccc is stated in the following result

Corollary 5.12 (Gysin kernel and ccc)**.** *If in addition $q(S) = 0$. Then*

a) remains the same as in Theorem 3.3.

b) For very general $t \in U$, either G_t is countable or C_t is a ccc.

For the above, we mean: there exists a c-open subset $U_0 \subset U$ such that for all $t \in U_0$ we have G_t is countable or for all $t \in U_0$ we have that C_t is a ccc.

Proof. This follows immediately from Corollary 3.8, by the definition of a ccc (see Definition 5.11) and the fact that $A_t = 0$ implies that G_t is countable and reciprocally, G_t is countable implies that $A_t = 0$. $\square$

Note that this result relates the study of ccc with the study of the Gysin kernel, i.e., this result gives a criterion deciding when a curve in a linear system is a ccc for surfaces as in the hypothesis of Theorem 3.3 with $q(S) = 0$ and where the notion of ccc coincides with the notion of pccc.

On the other hand, we have the following.

Corollary 5.13. *Assume, in addition, that $q(S) = 0$. If C_t is a ccc, for a very general $t \in U$, then $\mathsf{CH}_0(S)_{\deg=0}$ is representable.*

Proof. This follows immediately from Corollary 3.9 by the definition of a ccc (see Definition 5.11) and by the definition of representability for surfaces with $q(S) = 0$. $\square$

This result gives us a new hypothesis to determine the representability of $\mathsf{CH}_0(S)_{\deg=0}$ through the study of the ccc for surfaces S as in this section and with $q(S) = 0$.

The following result gives us a criterion to determine the representability of $\mathsf{CH}_0(S)_{\deg=0}$ for surfaces of general type with $p_g(S) = 0$ through the study of the ccc.

Corollary 5.14 (ccc and representability for sgt)**.** *Let S be a surface of general type with $p_g(S) = 0$. Then C_t is a ccc, for a very general $t \in U$, if and only if $\mathsf{CH}_0(S)_{\deg=0}$ is representable.*

Proof. Since S is a surface of general type and $p_g(S) = 0$ we have $q(S) = 0$.

Now if C_t is a ccc, for a very general $t \in U$, then by Corollary 5.13, $\mathsf{CH}_0(S)_{\deg=0}$ is representable. Reciprocally, if $\mathsf{CH}_0(S)_{\deg=0}$ is representable, then $\mathsf{CH}_0(S)_{\deg=0} \cong 0$, so any two points on S are rationally equivalent to each other, i.e., every curve in S is a ccc (recall that pccc and ccc coincide in this subsection), then, in particular, C_t is a ccc, for a very general $t \in U$. □

Remark 5.15. The second part of this proof can also be done as follows: if $\mathsf{CH}_0(S)_{\deg=0}$ is representable, then every curve in S is a ccc by [9, Proposition 4.1], then in particular C_t is a ccc, for a very general $t \in U$.

So, if S is a surface as in this subsection of general type with $p_g(S) = 0$ and we want to prove the representability of $\mathsf{CH}_0(S)_{\deg=0}$ is enough to prove that C_t is a ccc, for a very general $t \in U$.

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Resumen

El estudio de los ciclos algebraicos es un tema muy desarrollado y fascinante, que no solo ocupa un lugar destacado en Geometría Algebraica, sino que también comparte puntos en común con, por ejemplo, la Geometría Compleja (e.g., vía la *Conjetura de Hodge*) la Geometría Aritmética (e.g., vía el estudio de los *puntos racionales*), Teoría de Números (e.g., via la *Conjetura de Tate*), K-Teoría Algebraica (e.g., vía la *cohomología motívica*) y Física Matemática. En este trabajo abordaré el problema de la representabilidad de los grupos de Chow de 0-ciclos algebraicos en superficies mediante el estudio del Gysin kernel que permite obtener resultados útiles para el estudio de la conjetura de Bloch y las curvas de ciclos constantes (ccc).

Palabras clave: Ciclos algebraicos; Grupos de Chow; Gysin kernel; Representabilidad; Curvas ciclos constantes.

Presentado: agosto del 2025.
Aceptado: junio del 2026.

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