

SPACE-SEMIDISCRETE APPROXIMATION AND QUENCHING-TIME CONVERGENCE FOR A NONLOCAL NEUMANN PROBLEM WITH POWER-LAW SINGULAR REACTION

Thibaut K. KOUAKOU¹

May, 2026

(Presented by J. Farfan)

Abstract

We study a space-semidiscrete approximation of a nonlocal Neumann-type quenching problem with a power-law singular reaction. Continuous extremal and Kaplan–Jensen estimates are first recorded as analytical benchmarks.

The main analysis concerns a quadrature-based cooperative system with nonnegative weights: positivity, comparison, finite-time quenching, remaining-time estimates, and a discrete average bound are established. A transformed-gap estimate for the semidiscrete flow makes the quadrature kernel mass enter quenching-time bounds explicitly under a reaction-dominance condition. Under a second-order trapezoidal consistency assumption, the semidiscrete solution converges uniformly on compact pre-quenching intervals, the semidiscrete quenching time converges to the continuous one, and subcritical hitting times converge under a strict-crossing assumption. Adaptive explicit and implicit–explicit (IMEX) Euler computations illustrate these results.

MSC(2020): 35B44, 35K57, 45K05. (primary) 65M12, 65M20 (secondary).

Keywords: *Nonlocal diffusion; Neumann condition; quenching; singular reaction; semidiscrete approximation; quenching time.*

1. *Université Nangui Abrogoua, (UNA) Abidjan, Côte d'Ivoire*

1. Introduction

Nonlocal diffusion equations arise when the rate of variation at a point depends on values of the unknown over a surrounding region rather than only on infinitesimal spatial derivatives. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. In this paper the diffusion is governed by

$$\mathcal{L}_J v(x) = \int_{\Omega} J(x-y)(v(y) - v(x)) \, dy, \quad x \in \bar{\Omega}.$$

Because the integration is restricted to Ω , no exterior contribution enters the equation. This is the standard nonlocal analogue of homogeneous Neumann confinement. The framework is related to the Neumann problem studied by Andreu, Mazón, Rossi and Toledo [2] and to the Neumann blow-up problem of Pérez-Llanos and Rossi [23]. Related convolution and boundary-flux models can be found in [3, 4, 12, 15].

We consider

$$\begin{cases} u_t(x, t) = \mathcal{L}_J u(x, t) + \lambda a(x)(1 - u(x, t))^{-p}, & x \in \bar{\Omega}, \, t > 0, \\ u(x, 0) = u_0(x), & x \in \bar{\Omega}, \end{cases} \quad (1.1)$$

where $\lambda > 0$, $p > 0$, a is positive, and $0 \leq u_0 < 1$. The kernel is nonnegative and symmetric. The factor $(1 - u)^{-p}$ becomes singular as u approaches the critical level 1.

Classical quenching for local parabolic equations goes back to Kawarada [20]; see Levine [21] for an early survey and Quittner-Souplet [24] for broader singular and superlinear parabolic theory.

The continuous quenching problem is closely related, after the change of variable $v = 1 - u$, to nonlocal diffusion equations with singular absorption. Ferreira [13] studied the corresponding Dirichlet-type setting, including quenching rates and quenching sets. Zhou et al. [26] treated a nonlocal Neumann problem with a general singular absorption term and obtained finite-time quenching together with rate and localization results under additional assumptions. Related rate questions

for coupled singular systems were considered in [18, 25]. These works show that the continuous extremal arguments used below belong to an established analytical framework.

The numerical literature is also directly relevant. Boni and Kouakou [5] studied an adaptive *time-semidiscrete* approximation of the unweighted nonlocal Neumann quenching problem and proved finite-time semidiscrete quenching together with convergence of the corresponding quenching time as the time mesh tends to zero. Boni and Nabongo [8] subsequently analyzed a space–time discrete approximation; under their regularity and consistency hypotheses they obtained a first-order-in- h pre-quenching error estimate for first-order-consistent grid data and proved convergence of the numerical quenching time as $h \rightarrow 0$. Earlier and related singular-time discretizations include [1, 11, 14, 16, 17, 22]. Nonlocal Neumann blow-up semidiscretizations were considered in [6, 7], while local quenching with a spatial potential was studied in [9, 10]. The present work addresses the complementary *space-semidiscrete* problem: time remains continuous, whereas the nonlocal operator is replaced by a nonnegative quadrature operator and the limiting process is $h \rightarrow 0$, rather than a time-step limit. In this setting we retain the spatially varying coefficient $a(x)$, prove an $O(h^2)$ pre-quenching estimate under the trapezoidal consistency assumption, establish $T_h \rightarrow T_q$, prove convergence of subcritical hitting times under a strict-crossing assumption, and derive a conditional diffusion-dependent bound involving the quadrature masses $j_{h,i}$. Adaptive explicit and IMEX Euler methods are used only afterward to integrate the semidiscrete system numerically.

Sections 3 and 4 provide the continuous analytical estimates that serve as a reference for the subsequent numerical analysis. The main study is then developed for the space-semidiscrete quadrature system introduced in Section 5, with its convergence properties established in Section 6. The continuous bounds depend only on the extremal values a_* and a^* , while the Neumann setting is used as part of the mathematical

framework rather than as an independent source of novelty.

The paper is organized as follows. Section 2 states the assumptions and the definition of quenching. Sections 3 and 4 record the continuous benchmark estimates. Section 5 introduces the space-semidiscrete problem and proves its comparison and singular-time properties. Section 6 establishes convergence before quenching, convergence of quenching times and convergence of subcritical hitting times. Section 7 gives numerical illustrations.

2. Assumptions and quenching

Let $X = C(\overline{\Omega})$ with the norm $\|v\|_\infty = \max_{x \in \overline{\Omega}} |v(x)|$. We assume:

(H1) Ω is a bounded domain of $\mathbb{R}^N$ with compact closure.

(H2) $J \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $J \geq 0$, $J(z) = J(-z)$, and $\int_{\mathbb{R}^N} J(z) \, dz = 1$.

(H3) $a \in C(\overline{\Omega})$ and there exist $a_*, a^* > 0$ such that

$$0 < a_* \leq a(x) \leq a^*, \quad x \in \overline{\Omega}.$$

(H4) $\lambda > 0$, $p > 0$, and $u_0 \in C(\overline{\Omega})$ satisfies

$$0 \leq u_0(x) < 1, \quad x \in \overline{\Omega}.$$

The use of $C(\overline{\Omega})$ ensures that spatial minima and maxima are attained. Since J is continuous and bounded, $\mathcal{L}_J : X \rightarrow X$ is bounded. On the open set

$$\mathcal{O} = \{v \in X : \max_{\overline{\Omega}} v < 1\},$$

the map $v \mapsto \lambda a(1-v)^{-p}$ is locally Lipschitz. Hence the right-hand side of (1.1) is locally Lipschitz on $\mathcal{O}$, and the Picard–Lindelöf theorem gives a unique maximal solution

$$u \in C^1([0, T_{\max}); X), \quad u(0) = u_0,$$

T. K. KOUAKOU

as long as the solution remains below the singular level.

Definition 2.1. *The maximal solution of (1.1) is said to quench in finite time if $T_{\max} < \infty$ and*

$$\lim_{t \rightarrow T_{\max}^-} \|u(\cdot, t)\|_{\infty} = 1.$$

In that case we write $T_q = T_{\max}$ and call T_q the quenching time.

Before quenching has been established, for $0 \leq t < T_{\max}$ set

$$m(t) = \min_{x \in \bar{\Omega}} u(x, t), \quad M(t) = \max_{x \in \bar{\Omega}} u(x, t).$$

We denote their initial values by $m_0 := m(0)$ and $M_0 := M(0)$.

3. Continuous benchmark: extremal estimates

The extremal functions can be treated with ordinary derivatives after establishing their local Lipschitz continuity.

Lemma 3.1. *Let $w \in C^1([0, T]; C(\bar{\Omega}))$. Then*

$$\mu(t) = \min_{x \in \bar{\Omega}} w(x, t), \quad \nu(t) = \max_{x \in \bar{\Omega}} w(x, t)$$

are Lipschitz on $[0, T]$. Consequently they are differentiable for almost every t . At every common differentiability time there exist active points $x_{\mu}(t)$ and $x_{\nu}(t)$ such that

$$\mu'(t) = w_t(x_{\mu}(t), t), \quad \nu'(t) = w_t(x_{\nu}(t), t).$$

Proof. For $s, t \in [0, T]$,

$$|\nu(t) - \nu(s)| \leq \|w(\cdot, t) - w(\cdot, s)\|_{\infty} \leq \sup_{r \in [0, T]} \|w_t(\cdot, r)\|_{\infty} |t - s|,$$

and the same estimate holds for μ . Rademacher's theorem gives differentiability almost everywhere. The identities at differentiability times follow from the standard envelope argument: choose maximizing or minimizing points at nearby times, use compactness of $\bar{\Omega}$, and pass to a convergent subsequence. $\square$

The first qualitative property is invariance of the nonnegative cone.

Lemma 3.2. *Assume (H1)–(H4). Then the maximal solution satisfies*

$$0 \leq u(x, t) < 1, \quad x \in \bar{\Omega}, \quad 0 \leq t < T_{\max}.$$

Proof. The upper inequality is part of the maximal interval before quenching. Suppose that nonnegativity fails and let t_0 be a first contact time with the level 0. At a point x_0 where $u(x_0, t_0) = 0$ and $u(y, t_0) \geq 0$,

$$\mathcal{L}_J u(x_0, t_0) = \int_{\Omega} J(x_0 - y) u(y, t_0) \, dy \geq 0,$$

so that $u_t(x_0, t_0) \geq \lambda a(x_0) > 0$. This contradicts a first downward crossing. Hence $u \geq 0$. $\square$

The next inequalities are the continuous benchmarks used later.

Proposition 3.3. *For almost every $0 < t < T_{\max}$,*

$$m'(t) \geq \frac{\lambda a_*}{(1 - m(t))^p}, \quad (3.1)$$

$$M'(t) \leq \frac{\lambda a^*}{(1 - M(t))^p}, \quad (3.2)$$

and

$$M'(t) \geq -M(t) + \frac{\lambda a_*}{(1 - M(t))^p}. \quad (3.3)$$

Proof. At a differentiability time, let x_m and x_M be active points supplied by Lemma 3.1. At x_m , all differences $u(y, t) - u(x_m, t)$ are

T. K. KOUAKOU

nonnegative, so $\mathcal{L}_J u(x_m, t) \geq 0$, which yields (3.1). At x_M , the differences are nonpositive, so $\mathcal{L}_J u(x_M, t) \leq 0$, which yields (3.2). Finally, Lemma 3.2 and the normalization of J imply

$$\mathcal{L}_J u(x_M, t) \geq -M(t) \int_{\Omega} J(x_M - y) \, dy \geq -M(t),$$

and (3.3) follows. $\square$

The inequalities immediately give two-sided time bounds.

Theorem 3.4. *The maximal time is finite, the solution quenches with $T_q = T_{\max}$, and*

$$\frac{(1 - M_0)^{p+1}}{\lambda(p+1)a^*} \leq T_q \leq \frac{(1 - m_0)^{p+1}}{\lambda(p+1)a_*}. \quad (3.4)$$

Proof. Since (3.2) holds almost everywhere, comparison with

$$Y' = \lambda a^*(1 - Y)^{-p}, \quad Y(0) = M_0,$$

gives $M(t) \leq Y(t)$ as long as both solutions exist below 1. The scalar solution reaches 1 at

$$T_Y = \frac{(1 - M_0)^{p+1}}{\lambda(p+1)a^*}.$$

If $T_{\max} < T_Y$, then $M(t)$ remains uniformly separated from 1 as $t \uparrow T_{\max}$, and local well-posedness would extend the solution past $T_{\max}$. Hence $T_{\max} \geq T_Y$.

Similarly, (3.1) and comparison with

$$Z' = \lambda a_*(1 - Z)^{-p}, \quad Z(0) = m_0,$$

show that $m(t) \geq Z(t)$. Since Z reaches 1 at

$$T_Z = \frac{(1 - m_0)^{p+1}}{\lambda(p+1)a_*},$$

the maximal solution cannot remain below 1 beyond T_Z . Thus $T_{\max} \leq T_Z < \infty$. Finally, if $M(t)$ failed to approach 1 as $t \uparrow T_{\max}$, the trajectory would remain in a subset of $\mathcal{O}$ separated from the singular level and could again be extended. Therefore $M(t) \rightarrow 1$, so the solution quenches and $T_q = T_{\max}$. The two bounds follow. $\square$

A remaining-time estimate will be used in the convergence proof.

Proposition 3.5. *There exists $\rho \in (0, 1)$ such that, whenever $M(t) \geq \rho$,*

$$T_q - t \leq \int_{M(t)}^1 \frac{ds}{\lambda a_*(1-s)^{-p} - s}. \quad (3.5)$$

Proof. Choose ρ so close to 1 that

$$q(s) = \lambda a_*(1-s)^{-p} - s > 0 \quad (\rho \leq s < 1).$$

Equation (3.3) gives $M'(t) \geq q(M(t))$ almost everywhere once $M(t) \geq \rho$. Separation of variables and integration up to the quenching time give (3.5). $\square$

4. Continuous benchmark: Kaplan–Jensen estimate

The average argument follows the classical Kaplan approach [19], combined here with Jensen’s inequality. Let

$$Y(t) = \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx, \quad Y_0 = Y(0).$$

The symmetry of the kernel removes the diffusion term after integration.

Lemma 4.1. *For every $v \in C(\overline{\Omega})$,*

$$\int_{\Omega} \mathcal{L}_J v(x) \, dx = 0.$$

T. K. KOUAKOU

Proof. By Fubini's theorem,

$$\int_{\Omega} \mathcal{L}_J v(x) \, dx = \int_{\Omega} \int_{\Omega} J(x-y)(v(y) - v(x)) \, dy \, dx.$$

Interchanging x and y in the first term and using $J(x-y) = J(y-x)$ gives cancellation. $\square$

The average therefore satisfies a scalar differential inequality.

Proposition 4.2. For $0 < t < T_q$,

$$Y'(t) \geq \frac{\lambda a_*}{(1 - Y(t))^p}. \quad (4.1)$$

Consequently,

$$T_q \leq \frac{(1 - Y_0)^{p+1}}{\lambda(p+1)a_*}. \quad (4.2)$$

Proof. Integrating (1.1) and using Lemma 4.1,

$$\begin{aligned} Y'(t) &= \frac{\lambda}{|\Omega|} \int_{\Omega} a(x)(1 - u(x, t))^{-p} \, dx, \\ Y'(t) &\geq \frac{\lambda a_*}{|\Omega|} \int_{\Omega} (1 - u(x, t))^{-p} \, dx. \end{aligned}$$

The function $s \mapsto (1 - s)^{-p}$ is convex on $[0, 1)$, so Jensen's inequality yields (4.1). Integration gives (4.2). $\square$

Remark 4.3. Since $m_0 \leq Y_0 \leq M_0$, (4.2) improves the minimum-based upper bound in (3.4) whenever $Y_0 > m_0$.

5. Space-semidiscrete approximation

We now take $\Omega = (\alpha, \beta)$ and write $\overline{\Omega} = [\alpha, \beta]$. Let $I \geq 2$ be the number of spatial subintervals and set

$$h = \frac{\beta - \alpha}{I}, \quad x_i = \alpha + ih, \quad 0 \leq i \leq I.$$

For nonnegative quadrature weights ω_i , define

$$(\mathcal{L}_J^h U)_i = \sum_{j=0}^I \omega_j J(x_i - x_j)(U_j - U_i).$$

For the trapezoidal rule,

$$\omega_0 = \omega_I = \frac{h}{2}, \quad \omega_i = h \quad (1 \leq i \leq I-1).$$

The space-semidiscrete problem is

$$U_i'(t) = (\mathcal{L}_J^h U(t))_i + \lambda a_i (1 - U_i(t))^{-p}, \quad U_i(0) = u_0(x_i), \quad (5.1)$$

where $a_i = a(x_i)$. The off-diagonal coefficients of $\mathcal{L}_J^h$ are nonnegative, hence the ODE system is cooperative.

Let

$$\mathcal{O}_h = \{V \in \mathbb{R}^{I+1} : \max_i V_i < 1\}.$$

The right-hand side of (5.1) is locally Lipschitz on $\mathcal{O}_h$. Hence there is a unique maximal solution

$$U_h = (U_0, \dots, U_I) \in C^1([0, T_{h,\max}); \mathbb{R}^{I+1}), \\ T_{h,\max} \in (0, \infty].$$

Definition 5.1. *The maximal solution is said to quench in finite time if $T_{h,\max} < \infty$ and*

$$\lim_{t \rightarrow T_{h,\max}^-} \max_i U_i(t) = 1.$$

In that case we write $T_h = T_{h,\max}$ and call T_h the space-semidiscrete quenching time.

Before quenching has been established, set

$$m_h(t) = \min_i U_i(t), \quad M_h(t) = \max_i U_i(t), \\ 0 \leq t < T_{h,\max}.$$

The invariant interval is preserved by the semidiscrete flow.

T. K. KOUAKOU

Lemma 5.2. *Every maximal solution of (5.1) satisfies*

$$0 \leq U_i(t) < 1, \quad 0 \leq i \leq I, \quad 0 \leq t < T_{h,\max}.$$

Proof. The strict upper bound follows from $U_h(t) \in \mathcal{O}_h$ on its maximal interval. At a first contact with 0, say $U_k = 0$ and $U_j \geq 0$ for all j , one has $(\mathcal{L}_J^h U)_k \geq 0$ and therefore $U'_k \geq \lambda a_k > 0$. A downward crossing is impossible. $\square$

The following comparison statement makes explicit the order-preserving property used in the semidiscrete analysis.

Proposition 5.3. *Let $V, W \in C^1([0, S]; \mathbb{R}^{I+1})$ remain below 1 and satisfy*

$$\begin{aligned} V'_i &\leq (\mathcal{L}_J^h V)_i + \lambda a_i(1 - V_i)^{-p}, \\ W'_i &\geq (\mathcal{L}_J^h W)_i + \lambda a_i(1 - W_i)^{-p}. \end{aligned}$$

If $V_i(0) \leq W_i(0)$ for every i , then $V_i(t) \leq W_i(t)$ for $0 \leq t \leq S$.

Proof. This is the standard first-contact argument for cooperative systems. If $V - W$ had a first nonnegative crossing at component k , then at the contact time $V_k = W_k$ and $V_j - W_j \leq 0$ for all j . Hence

$$(\mathcal{L}_J^h(V - W))_k = \sum_j \omega_j J(x_k - x_j)((V_j - W_j) - (V_k - W_k)) \leq 0,$$

while the reaction terms at component k coincide. Thus $(V_k - W_k)' \leq 0$ at first contact, contradicting an upward crossing. A standard ε -perturbation handles the case of equality. $\square$

As in the continuous case, m_h and M_h are Lipschitz on compact subintervals of $[0, T_{h,\max})$ and hence differentiable almost everywhere. At such times the following estimates hold.

Proposition 5.4. *For almost every $0 < t < T_{h,\max}$,*

$$m'_h(t) \geq \frac{\lambda a_*}{(1 - m_h(t))^p}, \quad (5.2)$$

$$M'_h(t) \leq \frac{\lambda a^*}{(1 - M_h(t))^p}, \quad (5.3)$$

and

$$M'_h(t) \geq -C_h M_h(t) + \frac{\lambda a_*}{(1 - M_h(t))^p}, \quad (5.4)$$

where

$$C_h = \max_i \sum_{j=0}^I \omega_j J(x_i - x_j).$$

For trapezoidal weights, $C_h \leq \|J\|_\infty (\beta - \alpha)$ uniformly in h .

Proof. At an active minimum, $(\mathcal{L}_j^h U)_k \geq 0$; at an active maximum, $(\mathcal{L}_j^h U)_\ell \leq 0$. These facts give (5.2) and (5.3). Since $U_j \geq 0$,

$$(\mathcal{L}_j^h U)_\ell = \sum_j \omega_j J(x_\ell - x_j) U_j - M_h \sum_j \omega_j J(x_\ell - x_j) \geq -C_h M_h,$$

which gives (5.4). The uniform bound on C_h follows by summing the trapezoidal weights. $\square$

The extremal inequalities now give finite-time semidiscrete quenching without assuming it in advance.

Theorem 5.5. *The maximal solution of (5.1) quenches in finite time. With $T_h = T_{h,\max}$, its quenching time satisfies*

$$\frac{(1 - M_h(0))^{p+1}}{\lambda(p+1)a^*} \leq T_h \leq \frac{(1 - m_h(0))^{p+1}}{\lambda(p+1)a_*}. \quad (5.5)$$

Proof. Let Y solve

$$Y' = \lambda a^* (1 - Y)^{-p}, \quad Y(0) = M_h(0).$$

T. K. KOUAKOU

The almost-everywhere inequality (5.3) and scalar comparison give $M_h(t) \leq Y(t)$ as long as both trajectories remain below 1. The scalar solution reaches 1 at

$$T_Y = \frac{(1 - M_h(0))^{p+1}}{\lambda(p+1)a^*}.$$

Thus the semidiscrete trajectory cannot reach the singular boundary before T_Y ; finite-dimensional continuation excludes any other finite endpoint while it remains in a compact subset of $\mathcal{O}_h$. Hence $T_{h,\max} \geq T_Y$.

Similarly, let

$$Z' = \lambda a_*(1 - Z)^{-p}, \quad Z(0) = m_h(0).$$

By (5.2), $m_h(t) \geq Z(t)$ while $Z(t) < 1$. Since Z reaches 1 at

$$T_Z = \frac{(1 - m_h(0))^{p+1}}{\lambda(p+1)a_*},$$

the maximal semidiscrete solution cannot remain in $\mathcal{O}_h$ beyond T_Z . Therefore $T_{h,\max} \leq T_Z < \infty$.

Finally, if $M_h(t)$ did not converge to 1 as $t \uparrow T_{h,\max}$, Lemma 5.2 would keep the entire finite-dimensional trajectory in a compact subset of $\mathcal{O}_h$ on which the vector field is locally Lipschitz and bounded. The solution could then be extended beyond $T_{h,\max}$, a contradiction. Hence $M_h(t) \rightarrow 1$, the solution quenches, and $T_h = T_{h,\max}$. The two bounds above give (5.5). $\square$

The preceding bounds use only the sign of the diffusion term at active extrema. The next semidiscrete estimate retains the quadrature kernel mass explicitly. Define

$$j_{h,i} = \sum_{j=0}^I \omega_j J(x_i - x_j), \quad 0 \leq i \leq I.$$

Proposition 5.6 (Diffusion-dependent semidiscrete estimate). *Assume that*

$$\varepsilon_{J,h} := \max_{0 \leq i \leq I} \frac{j_{h,i}}{\lambda a_i} < 1. \quad (5.6)$$

Set

$$\begin{aligned} \Theta_{J,h}^- &= \min_{0 \leq i \leq I} \frac{(1 - U_i(0))^{p+1}}{(p+1)(\lambda a_i + j_{h,i})}, \\ \Theta_{J,h}^+ &= \min_{0 \leq i \leq I} \frac{(1 - U_i(0))^{p+1}}{(p+1)(\lambda a_i - j_{h,i})}. \end{aligned} \quad (5.7)$$

Then

$$\Theta_{J,h}^- \leq T_h \leq \Theta_{J,h}^+.$$

Moreover, with

$$\Theta_{0,h} = \min_{0 \leq i \leq I} \frac{(1 - U_i(0))^{p+1}}{(p+1)\lambda a_i},$$

one has

$$\frac{\Theta_{0,h}}{1 + \varepsilon_{J,h}} \leq T_h \leq \frac{\Theta_{0,h}}{1 - \varepsilon_{J,h}}. \quad (5.8)$$

Proof. Set

$$z_i(t) = 1 - U_i(t), \quad w_i(t) = z_i(t)^{p+1}.$$

Lemma 5.2 gives $0 < z_i \leq 1$ for $t < T_h$. Since $\mathcal{L}_J^h \mathbf{1} = 0$, equation (5.1) yields

$$z_i' = (\mathcal{L}_J^h z)_i - \lambda a_i z_i^{-p},$$

and hence

$$w_i' = (p+1)z_i^p (\mathcal{L}_J^h z)_i - (p+1)\lambda a_i.$$

Furthermore,

$$|(\mathcal{L}_J^h z)_i| \leq \sum_{j=0}^I \omega_j J(x_i - x_j) |z_j - z_i| \leq j_{h,i},$$

T. K. KOUAKOU

because $0 < z_j, z_i \leq 1$. Therefore

$$-(p+1)(\lambda a_i + j_{h,i}) \leq w'_i \leq -(p+1)(\lambda a_i - j_{h,i}).$$

After integration,

$$w_i(t) \geq (1 - U_i(0))^{p+1} - (p+1)(\lambda a_i + j_{h,i})t,$$

and

$$w_i(t) \leq (1 - U_i(0))^{p+1} - (p+1)(\lambda a_i - j_{h,i})t.$$

If $T_h < \Theta_{J,h}^-$, then the first inequality leaves every w_i bounded away from zero as $t \uparrow T_h$, because the set of indices is finite. This contradicts semidiscrete quenching, and therefore

$$T_h \geq \Theta_{J,h}^-.$$

For the upper bound, let i_+ attain the minimum in (5.7). If $T_h > \Theta_{J,h}^+$, choose

$$\Theta_{J,h}^+ < t < T_h.$$

The upper estimate for $w_{i_+}(t)$ is then negative, which is impossible. Hence $T_h \leq \Theta_{J,h}^+$. Finally, (5.6) gives

$$\begin{aligned} \lambda a_i + j_{h,i} &\leq (1 + \varepsilon_{J,h})\lambda a_i, \\ \lambda a_i - j_{h,i} &\geq (1 - \varepsilon_{J,h})\lambda a_i, \end{aligned}$$

and (5.8) follows. $\square$

Remark 5.7. The quantity $j_{h,i}$ is the quadrature mass of the nonlocal kernel at the node x_i . Thus Proposition 5.6 makes both a_i and the diffusion weights enter the semidiscrete quenching-time estimate explicitly. The estimate is conditional on (5.6) and is complementary to the unconditional bounds (5.5); no universal improvement in sharpness is claimed.

A remaining-time estimate is needed uniformly in h .

Proposition 5.8. *Let $C_0 = \|J\|_\infty(\beta - \alpha)$. There exists $\rho \in (0, 1)$, independent of h , such that if $M_h(t) \geq \rho$, then*

$$T_h - t \leq \int_{M_h(t)}^1 \frac{ds}{\lambda a_*(1-s)^{-p} - C_0 s}. \quad (5.9)$$

Proof. Choose ρ so that $q_0(s) = \lambda a_*(1-s)^{-p} - C_0 s > 0$ on $[\rho, 1)$. Since $C_h \leq C_0$, (5.4) implies $M'_h \geq q_0(M_h)$ almost everywhere once $M_h \geq \rho$. Integration yields (5.9). $\square$

The discrete average reproduces the cancellation of the continuous operator. Put

$$W_h = \sum_{i=0}^I \omega_i, \quad Y_h(t) = \frac{1}{W_h} \sum_{i=0}^I \omega_i U_i(t).$$

Proposition 5.9. *For $0 < t < T_h$,*

$$Y'_h(t) \geq \frac{\lambda a_*}{(1 - Y_h(t))^p}, \quad (5.10)$$

and therefore

$$T_h \leq \frac{(1 - Y_h(0))^{p+1}}{\lambda(p+1)a_*}. \quad (5.11)$$

Proof. Multiplying (5.1) by ω_i and summing gives

$$\sum_i \omega_i (\mathcal{L}_J^h U)_i = \sum_{i,j} \omega_i \omega_j J(x_i - x_j) (U_j - U_i) = 0$$

by symmetry. Weighted Jensen inequality then gives (5.10), and integration gives (5.11). $\square$

6. Convergence of the semidiscrete solution and quenching time

Throughout this section the trapezoidal rule is used. We assume that for every $\tau \in (0, T_q)$ there exists $K_\tau > 0$ such that

$$\max_{0 \leq i \leq I} \left| \int_{\alpha}^{\beta} J(x_i - y)(u(y, t) - u(x_i, t)) dy - \sum_{j=0}^I \omega_j J(x_i - x_j)(u(x_j, t) - u(x_i, t)) \right| \leq K_\tau h^2$$

for $0 \leq t \leq T_q - \tau$.

The next theorem separates the spatial consistency error from any later time integration.

Theorem 6.1. *For every $\tau \in (0, T_q)$ there exist $h_0 > 0$ and $C_\tau > 0$ such that, for $0 < h < h_0$,*

$$\max_{0 \leq t \leq T_q - \tau} \max_{0 \leq i \leq I} |U_i(t) - u(x_i, t)| \leq C_\tau h^2. \quad (6.1)$$

Proof. Fix τ . Since the exact solution is separated from 1 on $[0, T_q - \tau]$, there is $\delta_\tau > 0$ such that $u(x, t) \leq 1 - \delta_\tau$. Set

$$e_i(t) = U_i(t) - u(x_i, t), \quad E(t) = \max_i |e_i(t)|.$$

Work first on the maximal interval on which $E(t) \leq \delta_\tau/2$. There the singular reaction is Lipschitz, so subtraction of the exact and semidiscrete equations gives

$$e'_i = (\mathcal{L}_j^h e)_i + \lambda a_i((1 - U_i)^{-p} - (1 - u(x_i, t))^{-p}) + R_i(t),$$

with $|R_i(t)| \leq K_\tau h^2$. At an index of a positive maximum of e , $(\mathcal{L}_j^h e)_i \leq 0$; applying the same argument to $-e$ gives

$$E'(t) \leq C_\tau E(t) + C_\tau h^2$$

for almost every t on the bootstrap interval. Since $E(0) = 0$, Gronwall's inequality yields $E(t) \leq C_\tau h^2$. For h small this closes the bootstrap and proves (6.1). $\square$

The upper limiting inequality for the quenching times requires a remaining-time estimate whose threshold is independent of the mesh. Proposition 5.8 provides exactly this uniform control.

Proposition 6.2. *The semidiscrete quenching times satisfy*

$$\lim_{h \rightarrow 0} T_h = T_q. \quad (6.2)$$

Proof. Let $0 < \sigma < T_q$. Since the exact solution is uniformly below 1 on $[0, \sigma]$, Theorem 6.1 implies that U_h is also below 1 there for all small h . Hence $T_h > \sigma$, and therefore

$$\liminf_{h \rightarrow 0} T_h \geq T_q.$$

For the converse inequality, fix $\varepsilon > 0$. Let ρ be the mesh-independent threshold in Proposition 5.8. Because

$$\int_r^1 \frac{ds}{\lambda a_*(1-s)^{-p} - C_0 s} \rightarrow 0 \quad \text{as } r \rightarrow 1^-,$$

choose $r \in (\rho, 1)$ so that this integral is smaller than $\varepsilon/2$. Since $M(t) \rightarrow 1$ as $t \rightarrow T_q^-$, choose $t_\varepsilon \in (T_q - \varepsilon/2, T_q)$ such that $M(t_\varepsilon) > r$. The grid is dense in $[\alpha, \beta]$, so

$$\max_i u(x_i, t_\varepsilon) \rightarrow M(t_\varepsilon).$$

Together with Theorem 6.1, this gives $M_h(t_\varepsilon) > r$ for all sufficiently small h . Proposition 5.8 then yields

$$T_h - t_\varepsilon \leq \int_{M_h(t_\varepsilon)}^1 \frac{ds}{\lambda a_*(1-s)^{-p} - C_0 s} < \frac{\varepsilon}{2}.$$

Thus $T_h < T_q + \varepsilon$ for small h , so $\limsup_{h \rightarrow 0} T_h \leq T_q$. Combining both inequalities proves (6.2). $\square$

T. K. KOUAKOU

The final convergence result concerns the subcritical levels used in computations. For $0 < \eta < 1 - M_0$, define

$$\begin{aligned} T_\eta &= \inf\{t > 0 : M(t) \geq 1 - \eta\}, \\ T_{h,\eta} &= \inf\{t > 0 : M_h(t) \geq 1 - \eta\}. \end{aligned}$$

Proposition 6.3. *Assume $T_\eta < T_q$ and that the crossing of $1 - \eta$ is strict: for every sufficiently small $\delta > 0$ there is $\gamma_\delta > 0$ such that*

$$M(t) \leq 1 - \eta - \gamma_\delta \quad (0 \leq t \leq T_\eta - \delta),$$

and

$$M(T_\eta + \delta) \geq 1 - \eta + \gamma_\delta.$$

Then

$$\lim_{h \rightarrow 0} T_{h,\eta} = T_\eta. \quad (6.3)$$

Proof. Fix $\delta > 0$ with $T_\eta + \delta < T_q$. Theorem 6.1 gives uniform grid convergence on $[0, T_\eta + \delta]$. The strict lower-side separation implies $T_{h,\eta} \geq T_\eta - \delta$ for all small h . For the upper bound, choose $x_\delta \in \bar{\Omega}$ with $u(x_\delta, T_\eta + \delta) > 1 - \eta$. A grid point approaches x_δ as $h \rightarrow 0$, and continuity together with the uniform convergence gives $M_h(T_\eta + \delta) \geq 1 - \eta$ for small h . Hence $T_{h,\eta} \leq T_\eta + \delta$. Letting $\delta \downarrow 0$ proves (6.3). $\square$

7. Numerical results

We take

$$\Omega = (-1, 1), \quad p = 1, \quad \lambda = 1,$$

$$J(z) = \frac{3}{4}(1 - z^2)_+, \quad a(x) = 1 + 0.25 \exp(-12(x - 0.25)^2),$$

where $(r)_+ := \max\{r, 0\}$, and $u_0(x) = 0.22 + 0.05 \cos(\pi(x + 0.2))$. For these data,

$$m_0 = 0.17, \quad M_0 = 0.27, \quad Y_0 = 0.22, \quad a_* = 1, \quad a^* = 1.25.$$

The continuous estimates give

$$0.21316 \leq T_q \leq 0.34445,$$

while the Kaplan–Jensen estimate gives $T_q \leq 0.30420$.

The semidiscrete system is integrated in time by adaptive explicit Euler,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t_n} = \sum_{j=0}^I \omega_j J(x_i - x_j)(U_j^n - U_i^n) + a_i(1 - U_i^n)^{-p},$$

and adaptive IMEX Euler,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t_n} = \sum_{j=0}^I \omega_j J(x_i - x_j)(U_j^{n+1} - U_i^{n+1}) + a_i(1 - U_i^n)^{-p}.$$

Here $\|U^n\|_\infty := \max_{0 \leq i \leq I} U_i^n$. For both methods,

$$\Delta t_n = \min \left\{ 0.5h^2, \frac{0.2(1 - \|U^n\|_\infty)^{p+1}}{\max_i a_i} \right\}.$$

The first branch imposes the diffusive $O(h^2)$ scale, whereas the second reduces the time step as the discrete gap to the singular level decreases. The computation stops when $\Delta t_n \leq 10^{-16}$.

To distinguish the continuous-time semidiscrete quenching time T_h from the output of a chosen time integrator, the reported stopping time is denoted by $\hat{T}_h$. The convergence theorem concerns T_h ; the numerical values below are computational approximations of it. The observed quantity

$$s = \frac{\log((\hat{T}_{4h} - \hat{T}_{2h})/(\hat{T}_{2h} - \hat{T}_h))}{\log 2}$$

is reported only as a mesh-consistency diagnostic and is not used as a theoretical convergence-rate claim for the quenching time.

Table 1: Computed quenching times for the adaptive explicit Euler integration.

I	n	$\widehat{T}_h$	CPU time (s)	s
16	104	0.24638220	0.001	–
32	191	0.24108408	0.001	–
64	554	0.23949258	0.004	1.735
128	2018	0.23887395	0.021	1.363
256	7880	0.23871038	0.136	1.919
512	31337	0.23866166	2.570	1.747

Table 1 shows stabilization of the computed stopping time under mesh refinement. The values remain within the analytical interval above. The fluctuations of s are simply recorded; Proposition 6.2 proves convergence of T_h but does not assign a second-order rate to the quenching time itself.

Table 2: Computed quenching times for the adaptive IMEX Euler integration.

I	n	$\widehat{T}_h$	CPU time (s)	s
16	104	0.24666686	0.002	–
32	191	0.24116943	0.003	–
64	554	0.23951470	0.012	1.732
128	2018	0.23887936	0.061	1.381
256	7880	0.23871175	0.547	1.922
512	31337	0.23866199	12.522	1.752

The explicit and IMEX values become nearly identical on the fine meshes. This agreement is consistent with both integrations approximating the same semidiscrete flow; it is not used as a proof of a space–time discrete convergence theorem.

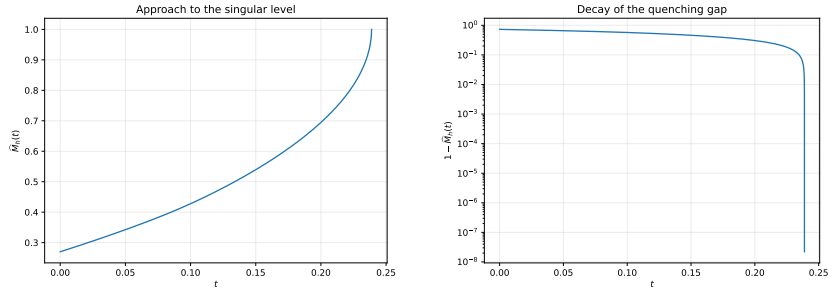


Figure 1: Explicit integration with $I = 128$. Left: growth of the computed maximum toward the singular level. Right: decay of the computed gap, which drives the adaptive reduction of Δt_n .

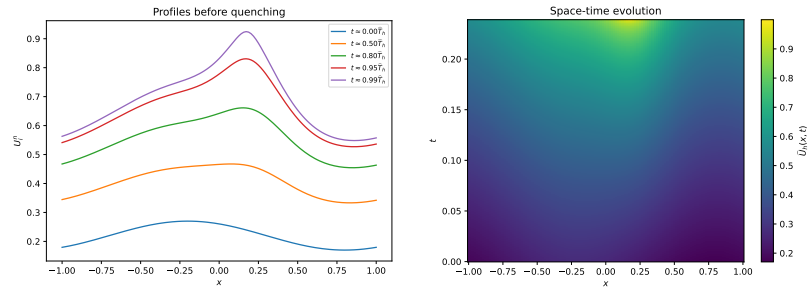


Figure 2: Explicit integration with $I = 128$. Left: profiles before the computed quenching time. Right: space-time evolution.

For this particular test, Figure 2 shows larger values near the region where $a(x)$ is larger. This is a numerical observation only. The analytical estimates in Sections 3–6 use a_* and a^* for time control and do not establish a localization theorem for the spatially varying coefficient.

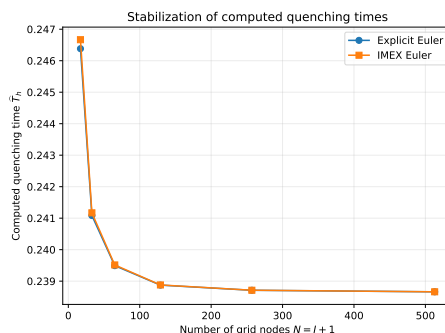


Figure 3: Computed stopping times for the explicit and IMEX integrations. Both sequences stabilize near the same value as the mesh is refined.

Conclusion and perspectives

The continuous extremal and Kaplan–Jensen estimates serve as benchmarks for the space-semidiscrete analysis. For the cooperative quadrature system, we prove positivity and comparison, finite-time quenching, an h -uniform remaining-time estimate, and a diffusion-dependent transformed-gap bound involving a_i and $j_{h,i}$ under a reaction-dominance condition. Under second-order trapezoidal consistency, the semidiscrete solution converges uniformly on compact pre-quenching intervals, $T_h \rightarrow T_q$, and subcritical hitting times converge under a strict-crossing hypothesis. Explicit and IMEX computations are consistent with these results, with the computed stopping time distinguished from T_h .

Further work may address rigorous space–time convergence of adaptive integrators, perturbation stability with respect to a , J and u_0 , and localization under hypotheses retaining more spatial information than a_* and a^* .

Acknowledgements. The author sincerely thanks the anonymous referees for their careful reading and for their comments and observations, which helped improve the presentation of the paper.

References

- [1] L. M. ABIA, J. C. LÓPEZ-MARCOS & J. MARTÍNEZ. On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, *Appl. Numer. Math.* **26** (1998) 399–414. DOI: 10.1016/S0168-9274(97)00105-0
- [2] F. ANDREU, J. M. MAZÓN, J. D. ROSSI & J. TOLEDO. The Neumann problem for nonlocal nonlinear diffusion equations, *J. Evol. Equ.* **8** (2008) 189–215. DOI: 10.1007/s00028-007-0377-9
- [3] P. BATES & A. CHMAJ. An integrodifferential model for phase transitions: stationary solutions in higher dimensions, *J. Stat. Phys.* **95** (1999) 1119–1139. DOI: 10.1023/A:1004514803625
- [4] P. BATES, P. FIFE, X. REN & X. WANG. Travelling waves in a convolution model for phase transitions. *Arch. Rational Mech. Anal.* **138** (1997) 105–136. DOI: 10.1007/s002050050037
- [5] T. K. BONI AND T. K. KOUAKOU. Quenching semidiscretizations in time of a nonlocal parabolic problem with Neumann boundary condition. *International Journal of Open Problems in Computer Science and Mathematics* **2** (2009) 2, 311–331.
- [6] T. K. BONI & T. K. KOUAKOU. Blow-up semidiscretizations in time of some nonlocal parabolic problems with Neumann boundary condition, *Southern Africa Journal of Pure and Applied Mathematics* **4** (2009) 44–60.

- [7] T. K. BONI & T. K. KOUAKOU. On the blow-up semidiscretizations in time of some non-local parabolic problems with Neumann boundary conditions. *Scientia Series A: Mathematical Sciences* **18** (2009) 1–16.
- [8] T. K. BONI & D. NABONGO. Quenching for discretizations of a nonlocal parabolic problem with Neumann boundary condition, *CUBO* **12** (2010), 23–40. DOI: 10.4067/S0719-06462010000100004
- [9] T. K. BONI & T. K. KOUAKOU. Numerical quenching for a semilinear parabolic equation with a potential and general nonlinearities. *Proyecciones* **27** (2008), 259–287. DOI: 10.4067/S0716-09172008000300004.
- [10] T. K. BONI & T. K. KOUAKOU. Numerical quenching for a semilinear parabolic equation with Dirichlet–Neumann boundary conditions and a potential, *Rostocker Mathematisches Kolloquium* **64** (2009), 17–38.
- [11] C. BRÄNDLE, P. GROISMAN & J. D. ROSSI. Fully discrete adaptive methods for a blow-up problem, *Mathematical Models and Methods in Applied Sciences* **14** (2004), 1425–1450.
- [12] C. CORTÁZAR, M. ELGUETA, J. D. ROSSI & N. WOLANSKI, Boundary fluxes for nonlocal diffusion, *J. Differ. Equ.* **234** (2007), 360–390. DOI: 10.1016/j.jde.2006.12.002
- [13] R. FERREIRA. Quenching phenomena for a non-local diffusion equation with a singular absorption, *Israel J. Math.* **184** (2011), 387–402. DOI: 10.1007/s11856-011-0072-y
- [14] R. FERREIRA, P. GROISMAN & J. D. ROSSI, Adaptive numerical schemes for a parabolic problem with blow-up. *IMA J. Numer. Anal.* **23** (2003), 439–463. DOI: 10.1093/imanum/23.3.439.

- [15] P. C. FIFE. Some nonclassical trends in parabolic and parabolic-like evolutions, in *Trends in Nonlinear Analysis*, Springer, Berlin, 2003, 153–191.
- [16] P. GROISMAN & J. D. ROSSI. Asymptotic behavior for a numerical approximation of a parabolic problem with blowing up solutions. *J. Comput. Appl. Math.* **135** (2001), 135–155. DOI: 10.1016/S0377-0427(00)00571-9
- [17] P. GROISMAN & J. D. ROSSI. Dependence of the blow-up time with respect to parameters and numerical approximations for a parabolic problem. *Asymptotic Analysis* **37** (2004), 79–91.
- [18] R. JI, S. ZHOU & S. ZHENG. Quenching behavior of solutions in coupled heat equations with singular multi-nonlinearities. *Appl. Math. Comput.* **223** (2013), 401–410. DOI: 10.1016/j.amc.2013.07.098
- [19] S. KAPLAN. On the growth of solutions of quasi-linear parabolic equations. *Commun. Pure Appl. Math.* **16** (1963), 305–330. DOI: 10.1002/cpa.3160160307
- [20] H. KAWARADA. On solutions of initial-boundary value problems for $u_t = u_{xx} + 1/(1 - u)$, *Publications of the Research Institute for Mathematical Sciences* **10** (1975), 729–736.
- [21] H. A. LEVINE. The phenomenon of quenching: a survey, in *Trends in the Theory and Practice of Nonlinear Analysis*, North-Holland, Amsterdam, 1985, 275–286.
- [22] T. NAKAGAWA. Blowing up on the finite difference solution to $u_t = u_{xx} + u^2$, *Appl. Math. Optim.* **2** (1976), 337–350. DOI: 10.1007/BF01448176

T. K. KOUAKOU

- [23] M. PÉREZ-LLANOS & J. D. ROSSI, Blow-up for a nonlocal diffusion problem with Neumann boundary conditions and a reaction term. *Nonlinear Anal. Theory Methods Appl.* **70** (2009), 1629–1640. DOI: 10.1016/j.na.2008.02.076
- [24] P. QUITTNER & P. SOUPLET. *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, 2nd ed., Birkhäuser/Springer, Cham, 2019.
- [25] S. ZHENG & W. WANG, Non-simultaneous versus simultaneous quenching in a coupled nonlinear parabolic system. *Nonlinear Anal. Theory Methods Appl.* **69** (2008), 2274–2285. DOI: 10.1016/j.na.2007.08.007
- [26] S. ZHOU, C. MU & R. ZENG, Quenching for a non-local diffusion equation with a singular absorption term and Neumann boundary condition. *Z. Angew. Math. Phys.* **62** (2011), 483–493. DOI: 10.1007/s00033-010-0103-2

Título. APROXIMACIÓN SEMIDISCRETA EN EL ESPACIO Y CONVERGENCIA DEL TIEMPO DE EXTINCIÓN PARA UN PROBLEMA DE NEUMANN NO LOCAL CON REACCIÓN SINGULAR DE LEY DE POTENCIA

Resumen. Estudiamos una aproximación semidiscreta en espacio de un problema no local de tipo Neumann con apagamiento y una reacción singular de ley de potencia. Las estimaciones continuas basadas en extremos y en el argumento de Kaplan–Jensen se presentan primero como referencias analíticas. El análisis principal se dedica a un sistema cooperativo obtenido por cuadratura con pesos no negativos: se establecen positividad, comparación, apagamiento en tiempo finito, estimaciones del tiempo restante y una cota promedio discreta. Una estimación basada en la transformación de la distancia al nivel singular hace intervenir explícitamente la masa de cuadratura del núcleo en cotas del tiempo de apagamiento semidiscreto bajo una condición de dominación de la reacción. Bajo una hipótesis de consistencia trapezoidal de segundo orden, se prueban la convergencia uniforme antes del apagamiento, la convergencia del tiempo de apagamiento semidiscreto y, bajo una hipótesis de cruce estricto, la convergencia de los tiempos de cruce subcríticos. Cálculos adaptativos de Euler explícito e implícito–explícito (IMEX) ilustran los resultados.

Palabras clave: Difusión no local; condición de Neumann; apagamiento; reacción singular; aproximación semidiscreta; tiempo de apagamiento.

Presentado: Mayo 2026.

Aceptado: Septiembre 2026.

Thibaut K. KOUAKOU 

<https://orcid.org/0009-0003-6792-251X>

Université Nangui Abrogoua.

UFR des Sciences Fondamentales et Appliquées.

Département de Mathématiques et Informatique.

02 BP 801 Abidjan 02, Côte d’Ivoire.

Email: kkthibaut@yahoo.fr