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# CONVERGENCE OF SINGULAR INTEGRALS IN WEIGHTED $L^1$ SPACES

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*It is shown that for a singular integral operator $K$ and a given function $f$ in $L^1_w$, $w$ belonging to the $A_1$ class of Muckenhoupt, if the image $Kf$ is also in $L^1_w$, then the truncated operator $K_\varepsilon$ applied to $f$ converges in $L^1_w$ to $Kf$. This is a generalization to the weighted case of a result due to A.P. Calderón and O.N. Capri* [2]. *As an application of the method developed, a new proof of a result on $H^1_w$ of R.L. Wheeden* [9] *is given.*

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## 1. Introduction

Let us denote by $R^n$ the $n$-dimensional euclidean space. By $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ we denote the scalar product of $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots y_n)$.

The norm of $x \in R^n$ is given by $| x |= (x \cdot x)^{1/2}$ and $B(x,r)$ designates the ball $\{y :| y - x |< r\}$, where $x \in R^n$ and $r > 0$.

All the functions considered here are Lebesgue measurable and $| E |$ stands for the Lebesgue measure of a measurable set $E$. Let $f$ be a locally integrable function. The Hardy-Little wood maximal function of $f$ is defined as

$$Mf(x) = \sup_{r>0} | B(x,r) |^{-1} \int_{B(x,r)} | f(y) | dy.$$

Let $w(x)$ be a non-negative and locally integrable function. The weight $w(x)$ is said to belong to the $A_p$ class of Muckenhoupt, $1 < p < \infty$, if there exists a constant $c$ such that the inequality

$$(| B |^{-1} \int_B wdx)(| B |^{-1} \int_B w^{-1/(p-1)}dv)^{p-1} \leq c$$

holds for every ball $B \subset R^n$. We say that $w$ belongs to $A_1$ if there exists a constant $c$ such that $Mw(x) \leq cw(x)$ holds for almost every point $x$ in $R^n$. Results concerning $A_p$ weights can be founded in B. Muckenhoupt [6], R. Coiman and C. Fefferman [3] and R. Hunt, B. Muckenhoupt and R. Wheeden [4]. We define the measure $w(E) = \int_E w(x)dx$, where $E$ is any Lebesgue measurable subset of $R^n$. $\Omega_n$ denotes the volume of the unit ball of $R^n$ and $\omega_{n-1}$ the surface area of the unit sphere. The letter $c$ designates a constant not necessarily the same at different ocurrences.

The type of singular integral kernels $k(x)$ that shall be considered in this paper will satisfy the following conditions:

(i) $| k(x) |\leq c | x |^{-n}$, if $x \neq 0$;

(ii) there exists an increasing function $\theta(t)$ such that $\theta(2t) \leq c\theta(t)$,

$$| k(x-y) - k(x) | \leq c\theta(| y | / | x |) / | x |^n$$

if $| x | \geq 2 | y |$, and $\int_0^1 \theta(t)d(t)/t < \infty$;

(iii) for every $0 < a < b$, $| \int_{a<x<b} k(x)dx | \leq c$;

(iv) the limit $\lim\limits_{n\to 0} \int_{n<x<1} k(x)dx$ exists.

A kernel $k(x)$ satisfying these properties allows us to define a temperate distribution, which is called the principal value of $k$, by means of

$$< p.v.k, \varphi > = \lim_{\varepsilon\to 0} \int_{x>\varepsilon} k(x)\varphi(x)dx =$$

$$\int_{|x|\leq 1} k(x)[\varphi(x) - \varphi(0)]dx +$$

$$+ \varphi(0). \lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} k(x)dx + \int_{|x|\geq 1} k(x)\varphi(x)dx$$

for every $\varphi \in S$. In order to calculate the Fourier transform of the distribution p.v.k, we observe that the conditions assumed on the kernel imply the hypothesis of Theorem in [1].
From the proof of that theorem it follows that the Fourier transforms of the truncated kernels, $k_\varepsilon(x) = k(x)$ if $| x | \geq \varepsilon$ and $k_\varepsilon(x) = 0$ if $| x | < \varepsilon$, are uniformly bounded, i.e. for every $\varepsilon > 0$.

$$| \widehat{k}_\varepsilon(x) | \leq M. \tag{1}$$

Moreover, we have

$$\widehat{k}_\varepsilon(x) = \int_{\varepsilon<|y|<1} [e^{-2\pi i x\cdot y}]k(y)dy + \widehat{k}_1(x)$$
$$= \int_{\varepsilon<|y|<1} [e^{-2\pi i x\cdot y} - 1]k(y)dy + \int_{\varepsilon<|y|<1} k(y)dy + \widehat{k}_1(x).$$

Letting $\varepsilon$ go to zero and taking into account (i) and (iv) we obtain

$$\lim_{\varepsilon\to 0} \widehat{k}_\varepsilon(x) = \int_{|y|<1} [e^{-2\pi i x\cdot y} - 1]k(y)dy$$
$$+ \lim \int_{\varepsilon<|y|<1} k(y)dy + \widehat{k}_1(x).$$

This proves the existence of the limit. Let us denote by $\widehat{k}(x)$ the function $\lim_{\varepsilon\to 0}\widehat{k}_\varepsilon(x)$. We shall show that

$$\text{(2)} \qquad < (p.v.k)^\wedge, \varphi > = < p.v.k, \widehat{\varphi} > = \int \widehat{k}(x)\varphi(x)dx.$$

Indeed, from the relation $< \widehat{k}_\varepsilon, \varphi > = < k_\varepsilon, \widehat{\varphi} >$ and (1), by the Lebesgue dominated convergence theorem, it follows (2) for every $\varphi \in S$. In addition, by (1), we have

$$\text{(3)} \qquad | \widehat{k}(x) | \leq M.$$

Let $k(x)$ be a kernel satisfying conditions (i) to (iv). For $\varepsilon > 0$, we define the truncated singular integral operator $K_\varepsilon$ as

$$K_\varepsilon f(x) = \int k_\varepsilon(x-y)f(y)dy,$$

where $f$ belongs to $L^p_w$, $1 \leq p < \infty$, and $w \in A_p$. The maximal singular integral operator $K^*$ associated to $k(x)$ is defined as

$$K^* f(x) = \sup_{\varepsilon>0} | K_\varepsilon f(x) | .$$

Next, we gather in theorems A and B some known results that will be needed in the sequel.

**Theorem A.** Let $K^*$ be the maximal singular integral operator associated to a kernel $k(x)$ which satisfies conditions (i), (ii), (iii) and (iv).

(i) if $f \in L^p_w$, $1 < p < \infty$, and $w \in A_p$, then $\|K^* f\|_{L^p_w} \leq c\|f\|_{L^p_w}$, where the constant $c$ does not depend on $f$.

(ii) If $f \in L^1_w$ and $w \in A_1$, then there exists a constant $c$ such that $w(\{x : K^* f(x) > \lambda\}) \leq c\lambda^{-1}\|f\|_{L^1_w}$ holds for every $\lambda > 0$.

Theorem A is a consequence of Theorem 5 of [5] since hypothesis (a) of that theorem holds by (3) and the other hypothesis (b) and (c) of the same theorem coincide with our assumptions (i) and (ii) on the kernel.

**Theorem B.** Let $k(x)$ be a kernel satisfying the same assumptions as in Theorem A.

(i) If $f \in L^P_w$, $1 \leq p < \infty$, $w \in A_p$, then $Kf(x) = \lim_{\varepsilon \to 0} K_\varepsilon f(x)$ exists almost everywhere.

(ii) If $f \in L^p_w$, $1 < p < \infty$ and $w \in A_p$, then $\|Kf\|_{L^p_w} \leq c\|f\|_{L^p_w}$, where $c$ does not depend on $f$.

(iii) If $f \in L^1_w$ and $w \in A_1$, then, there exists a constant $c$ such that $w(\{x :| Kf(x) |> \lambda\}) \leq c\lambda^{-1}\|f\|_{L^1_w}$ holds for every $\lambda > 0$.

(iv) If $f \in L^p_w$, $1 < p < \infty$ and $w \in A_p$, then $\lim_{\varepsilon \to 0}\|K_\varepsilon f - Kf\|_{L^p_w} = 0$.

It is easy to verify that (i) holds for $f \in C_0^\infty$. Arguing as in [7], p. 45, and taking into account Theorem A we obtain (i). Parts

(ii) and (iii) follow from part (i) and Theorem A. As for (iv), it follows from (i) and part (i) of Theorem A by applying the Lebesgue dominated convergence theorem.

## 2. Convergence in $L^1_w$.

Let $\varphi(x)$ be a function defined on $R^n$. The least decreasing radial majorant function $\psi(x)$ of $\varphi(x)$ is given by

$$\psi(x) = \sup_{|y| \geq |x|} | \varphi(y) | .$$

With an abuse of notation, we write $\psi(r) = \psi(x)$ if $| x |= r$. For $\varepsilon > 0$, we set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

**Lemma 1.** Let $\varphi(x)$ be a function whose least decreasing radial majorant $\psi(x)$ is integrable. Assume that $f(x)$ belongs to $L^p_w$, $1 \leq p < \infty$, $w \in A_p$. If for $\varepsilon > 0$ we define $f_\varepsilon(x) = \int f(x-y)\varphi_\varepsilon(y)dy$ then,

(i) for every $\varepsilon > 0$, $\quad | f_\varepsilon(x) | \leq Mf(x) \int \psi(y)dy$,

(ii) for every $\varepsilon > 0$, $\quad \|f_\varepsilon\|_{L^p_w} \leq c \int \psi(y)dy \|f\|_{L^p_w}$,

(iii) for every Lebesgue point $x$ of $f$ and therefore, for almost every point $x \in R^n$, $\quad \lim\limits_{\varepsilon\to 0} f_\varepsilon(x) = f(x) \int \psi(y)dy$ and

(iv) $\lim\limits_{\varepsilon\to 0} \int | f_\varepsilon(x) - f(x) \int \varphi(y)dy |^p \, w(x)dx = 0$

To prove Lemma 1 we shall need some results that we state in Lemmas 2 and 3.

**Lemma 2.** Let $h(x) \geq 0$ be a locally integrable function and let $\psi(x) \geq 0$ be a decreasing radial and integrable function. Then, the following properties hold

(i) $\int h(y)\psi(y)dy \leq Mh(0) \int \psi(y)dy$,

(ii) $\int_{|y|\leq p} h(y)\psi(y)dy$
$$\leq \sup_{0<r\leq p} (\Omega_n r^n)^{-1} \int_{|y|\leq r} h(y)dy \int_{|y|\leq p} \psi(y)dy,$$

(iii) $\int_{|y|>p} h(y)\psi(y)dy$
$$\leq 2^n(2^n-1)^{-1} \sup_{r>p}(\Omega_n r^n)^{-1} \int_{p<|y|\leq r} h(y)dy \int_{|y|>\frac{p}{2}} \psi(y)dy.$$

*Proof.* Part (i), which is the core of the lemma is well known (see [7], Theorem 2, p.62). Part (ii) follows from (i) considering $h(y)$ and $\psi(y)$ as equal to zero for $|y|>\rho$. To get part (iii) we observe that since $\psi$ is a radial and decreasing function, we have

$$[1-2^{-n}]\Omega_n\rho^n\psi(\rho) \leq \int_{\frac{\rho}{2}<|y|<\rho} \psi(y)dy.$$

If we apply part (i) to $h_\rho(y) = h(y)$ if $|y|>\rho$, $h_\rho(y) = 0$ if $|y|\leq\rho$ and $\psi_\rho(y) = \psi(y)$ if $|y|>\rho$ and $\psi_\rho(y) = \psi(\rho)$ if $|y|\leq\rho$, then

$$\begin{aligned}\int_{|y|>\rho} h(y)\psi(y)dy &= \int h_\rho(y)\psi_\rho(y)dy \\ &\leq Mh_\rho(0)[\Omega_n\rho^n\psi(\rho) + \int_{|y|>\rho}\psi(y)dy] \\ &\leq 2^n(2^n-1)^{-1}Mh_\rho(0)\int_{|y|>\frac{\rho}{2}}\psi(y)dy,\end{aligned}$$

which is (iii).

**Lemma 3.** Let $w \in A_1$ and $f \in L^1_w$. If, for $\rho > 0$, we set $\Delta_f(\rho) = (\Omega_n\rho^n)^{-1}\int_{|y|\leq\rho}\int |f(x-y)-f(x)|\,w(x)dxdy$, then,

(i) $\Delta_f(\rho) \leq c\|f\|_{L^1_w}$,

(ii) $\lim_{\rho\to 0}\Delta_f(\rho) = 0$.

*Proof.* We have,

$$\Delta_f(\rho) \leq (\Omega_n \rho^n)^{-1} \int_{|y|\leq\rho} \int | f(x) | w(x+y) dx dy + \int | f(x) | w(x) dx$$

$$= \int | f(x) | [(\Omega_n \rho^n)^{-1} \int_{|y|\leq\rho} w(x+y) dy] dx + \|f\|_{L^1_w}$$

$$\leq c \int | f(x) | w(x) dx + \|f\|_{L^1_w}$$

$$= c\|f\|_{L^1_w}$$

This proves (i). In order to prove part (ii), let $g$ be a continuous function with compact support such that $\|f-g\|_{L^1_w} < \varepsilon$. Then, there exists $\rho_0$ such that

$$\int | g(x-y) - g(x) | w(x) dx < \varepsilon,$$

for every $| y |< \rho_0$. Therefore, by part (i), we have

$$\begin{aligned} \Delta_f(\rho) &\leq (\Delta_{(f-g)}(\rho) + \Delta_g(\rho) \\ &\leq c\|f-g\|_{L^1_w} + \varepsilon \\ &< (c+1)\varepsilon, \quad \text{if } \rho < \rho_0. \end{aligned}$$

*Proof of Lemma 1.* Part (i) of Lemma 1 is an easy consequence of part (i) of Lemma 2. Part (i) and the assumption $w \in A_p$ imply (ii) for $p > 1$. If $p = 1$, by part (i) of Lemma 1, we have

$$\begin{aligned} \int | f_\varepsilon(x) | w(x) dx &\leq \int \{ \int | f(y) | \psi_\varepsilon(x-y) dy \} w(x) dx \\ &= \int | f(y) | \{ \int w(x+y) \psi_\varepsilon(x) dx \} dy \\ &\leq \int | f(y) | Mw(y) dy \\ &\leq c \int | f(y) | w(y) dy. \end{aligned}$$

since $w \in A_1$.
Let us prove part (iii). If $B$ is a ball, we have

$$(4) \qquad |B|^{-1} \int_B |f(y)|\,dy \le c\|f\|_{L^p_w} \Big(\int_B w(y)dy\Big)^{-\frac{1}{p}}$$

which shows that $f$ is a locally integrable function and therefore, that almost every point $x \in R^n$ is a Lebesgue point for $f$. Then, assuming that $x$ is a Lebesgue point for $f$, there exists $\rho_1 > 0$ such that

$$(\Omega_n r^n)^{-1} \int_{|x-y|\le r} |f(y) - f(x)|\,dy \le 1$$

holds for every $0 < r \le \rho_1$. Thus,

$$(5) \qquad (\Omega_n r^n)^{-1} \int_{|x-y|\le r} |f(y)|\,dy \le 1 + |f(x)|,$$

for $0 < r \le \rho_1$. Taking into account (4) with $B = \{y : |x-y| \le r\}$, $r > \rho_1$ and (5) it follows that $Mf(x) < \infty$. Now, we have

$$|f_\varepsilon(x) - f(x) \int \varphi(y)dy|$$
$$\le \{\textstyle\int_{|y|\le\rho} + \int_{|y|>\rho}\} |f(x-y) - f(x)|\,\psi_\varepsilon(y)dy.$$

By Lemma 2, we obtain

$$|f_\varepsilon(x) - f(x) \int \varphi(y)dy|$$
$$\le \sup_{0<r\le\rho} (\Omega_n r^n)^{-1} \int_{|y|\le\rho} |f(x-y) - f(x)|\,dx\,\cdot$$

$$\cdot \int_{|y|\le\rho} \psi_\varepsilon(y)dy + Mf(x) \int_{|y|>\frac{\rho}{2}} \psi_\varepsilon(y)dy + |f(x)| \int_{|y|>\rho} \psi_\varepsilon(y)dy$$
$$= I_1 + I_2 + I_3.$$

Choosing $\rho$ small enough, $I_1$ can be made less than any given $n > 0$ no matter the value of $\varepsilon$. Once $\rho$ is fixed, both $I_2$ and $I_3$ go to zero for $\varepsilon$ going to zero.
Therefore, (ii) is proved.
As for part (iv), we observe that the case $p > 1$ follows inmediately from (i) and (iii) by applying the Lebesgue dominated convergence theorem. If $p = 1$, we have

$$\int \mid f_\varepsilon(x) - f(x) \int \varphi(y)dy \mid w(x)dx$$

$$\begin{aligned}
&\leq \int \{ \int \mid f(x-y) - f(x) \mid \psi_\varepsilon(y)dy\} w(x)dx \\
&\leq \int \{ \int \mid f(x-y) - f(x) \mid w(x)dx\} \psi_\varepsilon(y)dy \\
&= \int h(y)\psi_\varepsilon(y)dy,
\end{aligned}$$

where

$$h(y) = \int \mid f(x-y) - f(x) \mid w(x)dx.$$

By Lemma 3, the function $h(y)$ satisfies $Mh(0) < \infty$ and $y = 0$ is a Lebesgue point for $h$. The proof of part (iii) of this lemma shows that

$$\lim_{\varepsilon \to 0} \int h(y)\psi_\varepsilon(y)dy = h(0) = 0,$$

which ends the proof of part (iv).

Let $m$ be a positive integer and $k = (k_1, \ldots, k_n) \in Z^n$. Let $y_k^m$ stand for the point $2^{-m}k \in R^n$ and $Q_k^m = \{y \in R^n : 2^{-m}k_i \leq y_i < 2^{-m}(k_i + 1), 1 \leq i \leq n\}$. We observe that $y_k^m \in Q_k^m$ and that the length of the sides of $Q_k^m$ are equal to $2^{-m}$. In addition, for any given $m$, the family $\{Q_k^m : k \in Z^n\}$ is a partition of $R^n$. Let us assume that $g$ is a bounded function with bounded support,

$w \in A_1$ and $f \in L^1_w$. Then, for any given positive integer $m$, we define

$$(6) \qquad c_m(f)(x) = \sum_{k \in Z^n} f(x - y_k^m) \int_{Q_k^m} g(y)dy$$

The set $I$ of indices $k$ with the property that $Q_k^m$ has a non-empty intersection with the support of $g$ is a finite set. If the support of $g$ is contained in the unit ball, the points $\{y_k^m\}_{k \in I}$ satisfy $| y_k^m | \leq 1 + 2^{-m}\sqrt{n}$. In the sequel we shall assume that $m$ is large enough so that $| y_k^m | \leq 2$.

**Lemma 4.** Let $w \in A_1, f \in L^1_w$ and let $g$ be bounded function with bounded support. Then, for every $R > 0$,

$$\lim_{m \to \infty} \int_{|x| \leq R} | C_m(f)(x) - (f * g)(x) | \, dx = 0.$$

*Proof.* Without loss of generality, we may assume that the support of $g$ is contained in the unit ball. If we take $x$, $| x | \leq R$ then since $| x - y_k^m | \leq R + 2$ and $| x - y | \leq R + 1$ for $| y | \leq 1$, it follows that

$$C_m(f)(x) - (f * g)(x) = C_m(f\chi_{R+2})(x) - ((f\chi_{R+2}) * g)(x),$$

where $\chi_{R+2}(y)$ stands for the characteristic function of $| y | \leq R+2$. Therefore, it is enough to prove the lemma for functions which are Lebesgue integrable. Then, assuming that $f \in L^1(R^n)$, we have

$$\int | C_m(f)(x) - (f * g)(x) | \, dx$$
$$\leq \sum_{k \in Z^n} \int_{Q_k^m} \int | f(x - y_k^m) - f(x - y) | \, dx \, | g(y) | \, dy$$

which tends to zero as $m$ tends to infinity.

*Lemma 5.* Let $w \in A_1$ and let $g$ be a bounded function with bounded support. If $f$ and $\tilde{f} = Kf$ belong to $L^1_w$, then

$$(f * g)^{\sim}(x) = (\tilde{f} * g)(x)$$

holds for almost every $x$ in $R^n$.

*Proof.* Without loss of generality we may assume that the support of $g$ is contained in the unit ball. From Theorem B, we know that the operator $k$ is defined on $L^1_w$ and since $f * g$ belongs to $L^1_w$, then it follows that $(f * g)^{\sim}(x)$ is defined almost everywhere. The assumption that $\tilde{f} \in L^1_w$ implies the almost everywhere existence of $(\tilde{f} * g)(x)$. Let $T > 0$ and $\varepsilon > 0$ Choose $R$ such that $R > T + 6$ and

$$\int_{|y|>R-2} | f(y) | \, w(y) dy < \varepsilon^2. \tag{7}$$

By Lemma 4 applied to $f$ and $\tilde{f}$ there exists $m$ such that

$$\int_{|y|\leq R} | C_m(f)(y) - (f * g)(y) | \, dy < \varepsilon^2 \text{and} \tag{8}$$

$$\int_{|y|\leq R} | C_m(\tilde{f})(y) - (\tilde{f} * g)(y) | \, dy < \varepsilon^2. \tag{9}$$

Then, denoting $B_T = \{x :| x |< T\}$ we have

$$\begin{aligned} (10) \quad & | \{x :| (f * g)^{\sim}(x) - (\tilde{f} * g)(x) |> \varepsilon\} \cap B_T | \\ & \leq | \{x :| (f * g)^{\sim}(x) - (\chi_R(f * g))^{\sim}(x) |> \tfrac{\varepsilon}{4}\} \cap B_T | \\ & \quad + | \{x :| (\chi_R(f * g))^{\sim}(x) - (\chi_R C_m(f))^{\sim}(x) |> \tfrac{\varepsilon}{4}\} \cap B_T | \\ & \quad + | \{x :| (\chi_R C_m(f))^{\sim}(x) - C_m(\tilde{f})(x) |> \tfrac{\varepsilon}{4}\} \cap B_T | \\ & \quad + | \{x :| C_m(\tilde{f})(x) - (\tilde{f} * g)(x) |> \tfrac{\varepsilon}{4}\} | \\ & = \alpha + \beta + \gamma + \delta. \end{aligned}$$

Let us estimate $\alpha$. By the $A_1$ condition on $w$ and the $(1,1)$-weak type of the operator $K$, we have

$$\alpha \leq c(| B_T | / w(B_T)) w(\{x : | (f * g)^{\sim}(x) - (\chi_R (f * g))^{\sim}(x) | > \frac{\varepsilon}{4}\})$$

$$\text{(11)} \qquad \leq c(| B_T | / w(B_T)) w(\{x : | [(1 - \chi_R)(f * g)]^{\sim}(x) | > \frac{\varepsilon}{4}\})$$
$$\leq c' \varepsilon^{-1} (| B_T | / w(B_T)) \| (1 - \chi_R)(f * g) \|_{L^1_w}.$$

For this norm, we have

$$\| (1 - \chi_R)(f * g) \|_{L^1_w} \leq \int_{|x| \geq R} \{ \int | f(y) || g(x - y) | dy \} w(x) dx.$$

Since the support of $g$ is contained in the unit ball, the relevant values of $x$ and $y$ in the integral satisfy $| x | \geq R$ and $| x - y | \leq 1$. Therefore, $R - | y | \leq | x | - | y | \leq 1$, which implies $R - 1 \leq | y |$. Then, by condition $A_1$ and (7), we obtain

$$\| (1 - \chi_R)(f * g) \|_{L^1_w}$$
$$\leq \int_{|y| \geq R-1} | f(y) | \{ \int_{|x-y| \leq 1} | g(x - y) | w(x) dx \} dy$$
$$\leq \Omega_n \| g \|_\infty \int_{|y| \geq R-1} | f(y) | \{ \Omega_n^{-1} \int_{|x-y| \leq 1} w(x) dx \} dy$$
$$\leq \Omega_n c \| g \|_\infty \int_{y \geq R-1} | f(y) | w(y) dy$$
$$\leq c \| g \|_{\infty} \varepsilon^2$$

which together with (11) gives

$$\text{(12)} \qquad \alpha \leq c(| B_T | / w(B_T)) \| g \|_\infty \varepsilon.$$

Estimation of $\beta$. By (8) and the (1,1)-weak type of the operator $k$, we get

$$(13)\quad \beta \leq \mid \{x : \mid (\chi_R C_m(f))^{\sim}(x) - (\chi_R(f * g))^{\sim}(x) \mid > \varepsilon/4\} \mid$$

$$\leq 4c\varepsilon^{-1} \int_{|x| \leq R} \mid C_m(f)(x) - (f * g)(x) \mid dx$$

$$\leq 4c\varepsilon.$$

Estimation of $\gamma$. We have

$$(\chi_R C_m(f))^{\sim}(x) - C_m(\tilde{f})(x)$$

$$= \sum_k pv \int k(x-y)\chi_R(y) f(y - y_k^m) dy \int_{Q_k^m} g(z) dz$$

$$- \sum_k p \cdot v \cdot \int k(x - y_k^m - y) f(y) dy \int_{Q_k^m} g(z) dz$$

$$= \sum_k pv \int k(x - y - y_k^m)[\chi_R(y + y_k^m) - 1] f(y) dy \int_{Q_k^m} g(z) dz.$$

Thus

$$(14)\quad \mid (\chi_R C_m(f))^{\sim}(x) - C_m(\tilde{f})(x) \mid \leq$$

$$\sum_k \int \mid k(x - y - y_k^m) - k(x-y) \mid \mid \chi_R(y + y_k^m) - 1 \mid \mid f(y) \mid dy \cdot$$

$$\cdot \int_{Q_k^m} \mid g(z) \mid dz$$

$$+ \mid \int k(x-y) f(y) \sum_k [\chi_R(y + y_k^m) - 1] dy \int_{Q_k^m} g(z) dz \mid$$

$$= I(x) + J(x).$$

If $|y| \leq R-2$, since $|y_k^m| \leq 2$ for $m$ large enough, then $|y+y_k^m| \leq R$ and thus $[\chi_R(y+y_k^m)-1]=0$. If $x \in B_T, T < R-6$ and $|y| > R-2$, we have

$$|x-y| \geq |y| - |x| \geq (R-2)-(R-6) = 4 \geq 2 |y_k^m| .$$

Therefore, by condition (ii) on the kernel $k(x)$ we get

$$|k(x-y-y_k^m) - k(x-y)| \leq c |x-y|^{-n} \Theta(|y_k^m| / |x-y|).$$

Then, taking into account that $|x-y| \geq 4$, we can write

$$(15) \quad \int_{B_T} I(x)w(x)dx \leq$$

$$\sum_k \{ \int_{|y|\geq R-1} [\int_{|x-y|\geq 4} c |x-y|^{-n} \Theta(2/|x-y|)w(x)dx] |f(y)| dy \cdot \\ \cdot \int_{Q_k^m} |g(z)| dz \}$$

To estimate the integral with respect to $dx$, we apply part (iii) of Lemma 2, obtaining

$$\int_{|x-y|\geq 4} |x-y|^{-n} \Theta(2/|x-y|)w(x)dx$$

$$\leq Mw(y) \int_{|x|\geq 2} |x|^{-n} \Theta(2/|x|)dx$$

$$\leq cw(y) \int_0^1 \Theta(t)\frac{dt}{t} \text{ ,since } w \in A_1 .$$

Returning to (15), the estimate obtained together with (7) gives

$$\int_{B_T} I(x)w(x)dx \leq c\|g\|_{L^1} \int_{|y|>R-2} |f(y)| w(y)dy$$

$$\leq c\|g\|_{L^1} \epsilon^2 .$$

Then, since $w \in A_1$ and by Tchebichef's inequality it follows that

$$
\begin{aligned}
(16) \qquad & \mid \{x : I(x) > \frac{\varepsilon}{8}\} \cap B_T \mid \\
& \leq c(\mid B_T \mid / w(B_T)) w(\{x : I(x) > \frac{\varepsilon}{8}\} \cap B_T) \\
& \leq c(\mid B_T \mid / w(B_T))(\frac{8}{\varepsilon} \int_{B_T} I(x) w dx \\
& \leq c(\mid B_T \mid / w(B_T))\varepsilon.
\end{aligned}
$$

By the $(1,1)$-weak type of the operator $K$ and (7), we obtain

$$
\begin{aligned}
(17) \qquad & \mid \{x : J(x) > \frac{\varepsilon}{8}\} \cap B_T \mid \\
& \leq c(\mid B_T \mid / w(B_T)) w(\{x : J(x) > \frac{\varepsilon}{8}\}) \\
& \leq c\varepsilon^{-1}(\mid B_T \mid / w(B_T))(\int_{|y|>R-2} \mid f(y) \mid w(y) dy) \|g\|_{L^1} \\
& \leq c(\mid B_T \mid / w(B_T))\varepsilon.
\end{aligned}
$$

Then, from (14), (16) and (17), it follows that

$$
\begin{aligned}
(18) \qquad \gamma & \leq \mid \{x : I(x) > \frac{\varepsilon}{8}\} \cap B_T \mid + \mid \{x : J(x) > \frac{\varepsilon}{8}\} \cap B_T \mid \\
& \leq c(\mid B_T \mid / w(B_T))\varepsilon.
\end{aligned}
$$

Let us estimate $\delta$. By Tchebichef's inequality and (9), we have

$$
\begin{aligned}
(19) \qquad \delta & = \mid \{x : \mid C_m(\tilde{f})(x) - (\tilde{f} * g)(x) \mid > \frac{\varepsilon}{4}\} \cap B_T \mid \\
& \leq (\frac{4}{\varepsilon}) \int_{|x| \leq R} \mid C_m(\tilde{f})(x) - (\tilde{f} * g)(x) \mid dx \\
& \leq 4\varepsilon.
\end{aligned}
$$

The estimates (12), (13), (18) and (19) for $\alpha$, $\beta$, $\gamma$ and $\delta$ respectively show that (10) is smaller than a constant $c_T$ times $\varepsilon$. This ends the proof of the lemma.

**Lemma 6.** Let $w \in A_1$ and $f \in L^1_w$. If $g$ is a function with a decreasing radial majorant $\psi$ such that $\psi \in L^1 \cap L^{p_0}$, $1 < p_0 \leq \infty$ then

$$(f * g)^{\sim}(x) = (f * \tilde{g})(x)$$

holds almost everywhere.

*Proof.* If $w \in A_1$, it follows from the reverse Hôlder inequality that there exists $\delta > 1$ such that for every $p$, $1 < p \leq \delta$, $w^p \in A_1$. We choose $p$ satisfying $1 < p \leq min(p_0, \delta)$. By Lemma 2, part (i), and the assumption that $w \in A_1$, we have

$$\begin{aligned} (20) \qquad \int |\, g(x-y) \,|^p \, w(x)^p dx &\leq M w^p(y) \int \psi(y)^p dy \\ &\leq c w(y)^p \|\psi\|^p_{L^p}. \end{aligned}$$

Then, by Minkowski's integral inequality and (20), we obtain

$$\begin{aligned} (21) \qquad \|f * g\|_{L^p_{w^p}} &\leq \int |\, f(y) \,| \Big( \int |\, g(x-y) \,|^p \, w(x)^p dx \Big)^{1/p} dy \\ &\leq c \|\psi\|_{L^p} \int |\, f(y) \,| \, w(y) dy \\ &= c \|f\|_{L^1_w} \|\psi\|_{L^p}. \end{aligned}$$

On the other hand, since $w^p \in A_1 \subset A_p$ and recalling that the operator $K$ is of strong type $(p,p)$ for weights belonging to $A_p$, we have

$$\int |\, \tilde{g}(x-y) \,|^p \, w(x)^p dx \leq c \int |\, g(x-y) \,|^p \, w(x)^p dx.$$

Then, applying Minkowski's integral inequality and (20), we obtain

$$\|f * \tilde{g}\|_{L^p_{w^p}} \leq \int | f(y) | (\int | \tilde{g}(x-y) |^p w(x)^p dx)^{1/p} dy$$
$$\leq c\|f\|_{L^1_w} \|\psi\|_{L^p}.$$

Applying Fubini's theorem, it is easy to show that for every $\varepsilon > 0$

$$(f * g) * k_\varepsilon = f * (g * k_\varepsilon).$$

Since, as we have shown in (21), $f * g \in L^p_{w^p}$, then by Theorem B, part (iv), we have that $f * g) * k_\varepsilon$ converges to $(f * g)^\sim$ in $L^p_w$ as $\varepsilon$ goes to zero. The proof of the lemma will be completed once we show that

$$\lim_{\varepsilon \to 0} \|f * (g * k_\varepsilon) - f * \tilde{g}\|_{L^p_{w^p}} = 0. \tag{22}$$

We observe that (20) says that $g(x-y)$ belongs to $L^p_{w^p}$ for almost every $y$. Therefore, in virtue of Theorem B, part (iv), if $\tilde{g}_\varepsilon = K_\varepsilon g$, we have that

$$\int | \tilde{g}_\varepsilon(x-y) - \tilde{g}(x-y) |^p w^p(x) dx \tag{23}$$

tends to zero as $\varepsilon \to 0$ for almost every $y \in R^n$. Moreover, since $K^* g(x) = \sup_{\varepsilon > 0} | \tilde{g}_\varepsilon(x) |$, then, by Theorem A, part (i), we have

$$\int | \tilde{g}_\varepsilon(x-y) - \tilde{g}(x-y) |^p w(x)^p dx$$
$$\leq 2^p \int (K^* g(x-y))^p w(x)^p dx$$
$$\leq c \int | g(x-y) |^p w(x)^p dx$$
$$\leq c w(y)^p \|\psi\|^p_{L^p}.$$

Then, by Minkowski's integral inequality, we get

$$(24) \qquad \|f*(g*k_\varepsilon) - f*\tilde{g}\|_{L^p_{w^p}}$$
$$\leq \int |f(y)| \left(|\tilde{g}_\varepsilon(x-y) - \tilde{g}(x-y)|^p w^p(x)dx\right)^{\frac{1}{p}} dy,$$

and, by the considerations just made in (23) and the Lebesgue dominated convergence theorem, we conclude that the right hand side of (24) tends to zero with $\varepsilon$. This ends the proof of the lemma.

**Lemma 7.** Let $k(x)$ be a singular integral kernel and $w \in A_1$. Then,

$$\int_{|x|\geq 2|y|} |k(x) - k(x-y)| \, w(x)dx \leq cw(y)$$

holds for almost every $y \in R^n$.

*Proof.* By condition (ii) on the kernel and Lemma 2, part (iii), we have

$$\int_{|x|\geq 2|y|} |k(x-y) - k(x)| \, w(x)dx$$
$$\leq c \int_{|x|\geq 2|y|} |x|^{-n} \, \Theta(|y|/|x|)w(x)dx$$
$$\leq c \sup_{r>2|y|} \left(\frac{1}{\Omega_n r^n} \int_{2|y|<|z|<r} w(x)dx\right) \cdot$$
$$\cdot \left(\int_{|x|>|y|} |x|^{-n} \, \Theta(|y|/|x|)dx\right).$$

Taking polar coordinates, we get

$$\int_{|x|\geq|y|} |x|^{-n} \, \Theta(|y|/|x|)dx = w_{n-1} \int_{|y|}^{\infty} r^{-1}\Theta(|y|/r)dr$$
$$= w_{n-1} \int_0^1 \Theta(t)dt/t < \infty.$$

Now, since $\{z : 2 \mid y \mid < \mid z \mid < r\} \subset B(y, 3r/2)$, we have

$$\frac{1}{\Omega_n r^n} \int_{|y|<|x|} w(z)dz \leq c \mid B(y,3r/2) \mid^{-1} \int_{B(y,3r/2)} w(z)dz \\ \leq cw(y),$$

for almost every $y$, ending the proof of the lemma.

**Theorem 1.** Let $Kf = \tilde{f}$ be a singular integral operator with a kernel $k(x)$ and let $w \in A_1$. If $f$ and $\tilde{f}$ belong to $L^1_w$, then

1) for every $\varepsilon > 0$, $k_\varepsilon * f$ belongs to $L^1_w$ and

2) $\|k_\varepsilon * f - \tilde{f}\|_{L^1_w} \to 0$ as $\varepsilon \to 0$.

*Proof.* Let $\varphi \geq 0, \varphi \in C_0^\infty$, supp $\varphi \subset \{x : \mid x \mid \leq 1\}$ and $\int \varphi(x)dx = 1$. Let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ and set $\delta_\varepsilon(x) = \tilde{\varphi}_\varepsilon(x) - K_\varepsilon(x)$. If $g$ is a continuous function with bounded support we shall show that $\|g * \delta_\varepsilon\|_{L^2_w}$ tends to zero as $\varepsilon$ tends to zero. In fact, we have $g * \tilde{\varphi}_\varepsilon = (g * \varphi_\varepsilon)^\sim = \tilde{g} * \varphi_\varepsilon$ (see [2] , Lemma 3). Therefore, $(g * \delta_\varepsilon)(x) = (\tilde{g} * \varphi_\varepsilon)(x) - (k_\varepsilon * g)(x)$, a.e. Adding and substracting $\tilde{g}$, and taking into account Lemma 1 and Theorem B, part (iv) we have

$$(25) \qquad \lim_{\varepsilon \to 0} \|g * \delta_\varepsilon\|_{L^2_w} \\ \leq \lim_{\varepsilon \to 0} \|\tilde{g} * \varphi_\varepsilon - \tilde{g}\|_{L^2_w} + \lim_{\varepsilon \to 0} \|\tilde{g} - k_\varepsilon * g\|_{L^2_w} = 0.$$

Now, arguing as in the proof of Lemma 5 of [2], we shall prove that $g * \delta_\varepsilon$ also tends to zero in $L^1_w$ when $\varepsilon$ goes to zero. In fact, if the support of $g$ is contained in the ball $\mid x \mid \leq N$, then support of $g * \varphi_\varepsilon$ is contained in $\mid x \mid \leq 2N$ provided $0 < \varepsilon < N$. By Lemma 3 of [2], if $\mid x \mid \geq 4N$ and $0 < \varepsilon < N$, we have

$$(\tilde{\varphi}_\varepsilon * g)(x) = (\varphi_\varepsilon * g)^\sim(x) \\ = \int_{|y| \leq 2N} k(x-y)(\varphi_\varepsilon * g)(y)dy.$$

Moreover, since

$$(k_\varepsilon * g)(x) = \int_{|y|\leq 2N} k(x-y)g(y)dy,$$

it follows that

$$(g * \delta_\varepsilon)(x) = \int_{|y|\leq 2N} k(x-y)[(\varphi_\varepsilon * g)(y) - g(y)]dy,$$

for $| x |\geq 4N$ and $0 < \varepsilon < N$.
Observing that

$$\int_{|y|\leq 2N} [(\varphi_\varepsilon * g)(y) - g(y)]dy = 0,$$

We write,

$$(g * \delta_\varepsilon)(x) = \int_{|y|\leq 2N} [k(x-y) - k(x)][(\varphi_\varepsilon * g)(y) - g(y)]dy,$$

for $| x |\geq 4N$ and $0 < \varepsilon < N$. Then, multiplying by $w(x)$ and integrating on $| x |\geq 4N$, we get

$$\int_{|x|\geq 4N} | (g * \delta_\varepsilon)(x) | w(x)dx \leq$$
$$\int_{|y|\leq 2N} \{ \int_{|x|\geq 2y} | k(x-y) - k(x) | w(x)dx \} | (\varphi_\varepsilon * g)(y) - g(y) | dy.$$

By Lemma 7, the right hand side of the inequality above is bounded by a constant times $\|\varphi_\varepsilon * g - g\|_{L^1_w}$ which in turn tends to zero when $\varepsilon \to 0$, by Lemma 1. If $| x |\leq 4N$, we have

$$\int_{|x|\leq 4N} | (g * \delta_\varepsilon)(x) | w(x)dx$$
$$\leq \left( \int_{|x|\leq 4N} w(x)dx \right)^{\frac{1}{2}} \|g * \delta_\varepsilon\|_{L^2_w},$$

Which, by (25), tends to zero with $\varepsilon$. Next, we shall find a decreasing, radial and integrable function $\Delta(x)$ such that $|\delta_\varepsilon(x)| \leq \Delta_\varepsilon(x) = \varepsilon^{-n}\Delta(x/\varepsilon)$. In fact, if $|x| \leq 2\varepsilon$, we have

$$\begin{aligned}\delta_\varepsilon(x) &= \tilde{\varphi}_\varepsilon(x) - k_\varepsilon(x)\\ &= \lim_{n\to 0}\int_{n<|y|<4\varepsilon} k(y)[\varphi_\varepsilon(x-y) - \varphi_\varepsilon(x)]dy\\ &\qquad + \varphi_\varepsilon(x)\lim_{n\to 0}\int_{n<|y|<4\varepsilon} k(y)dy - k_\varepsilon(x).\end{aligned}$$

Hence, by conditions (i), (iii) and (iv) on $k(x)$, we obtain

$$(26) \qquad |\delta_\varepsilon(x)| \leq c\varepsilon^{-n}, \quad \text{for } |x| \leq 2\varepsilon.$$

Let $|x| \geq 2\varepsilon$. Then,

$$\delta_\varepsilon(x) = \int_{|y|\leq\varepsilon}[k(x-y) - k(x)]\varphi_\varepsilon(y)dy.$$

Thus, by condition (ii) on $k(x)$, we have

$$(27) \qquad |\delta_\varepsilon(x)| \leq c\int_{|y|\leq\varepsilon} |x|^{-n}\,\Theta(|y|/|x|)\varphi_\varepsilon(y)d(y).$$

Defining $\Delta(x) = c$ for $|x| \leq 2$ and

$$\Delta(x) = c\int_{|y|\leq 1} |x|^{-n}\,\Theta(|y|/|x|)\varphi_\varepsilon(y)d(y) \quad \text{for} \quad |x| \geq 2.$$

We have that by (26) and (27), $|\delta_\varepsilon(x)| \leq \Delta_\varepsilon(x) = \varepsilon^{-n}\Delta(x/\varepsilon)$. As for the integrability of $\Delta(x)$.

$$\begin{aligned}\int_{|y|\geq 2}\Delta(x)dx &= \int_{|y|\leq 1}\varphi(y)\Big(\int_{|x|\geq 2} |x|^{-n}\,\Theta(|y|/|x|)dx\Big)dy\\ &= {}_{n-1}\int\varphi(y)dy\int_0^{1/2}\Theta(t)dt/t \quad < \infty,\end{aligned}$$

shows that $\Delta(x)$ is integrable.
By Lemma 5 and 6, we know that $(\tilde{f} * \varphi_\varepsilon)(x) = (f * \tilde{\varphi}_\varepsilon)(x)$ almost everywhere. Hence, $(f * k_\varepsilon)(x) = (\tilde{f} * \varphi_\varepsilon)(x) - (f * \delta_\varepsilon)(x)$, a.e. Therefore,

$$\|f * k_\varepsilon\|_{L^1_w} \leq \|\tilde{f} * \varphi_\varepsilon\|_{L^1_w} + \|f * \delta_\varepsilon\|_{L^1_w}.$$

Since $| \delta_\varepsilon(x) | \leq \Delta_\varepsilon(x)$, by Lemma 1, part (ii), we obtain

$$\|f * k_\varepsilon\|_{L^1_w} \leq \|f\|_{L^1_w} + \|\tilde{f}\|_{L^1_w}.$$

For $n > 0$, let $g$ be a continuous function with bounded support such that $\|f - g\|_{L^1_w} < n$. Then,

$$\|f * \delta_\varepsilon - \tilde{f}\|_{L^1_w} \leq \|\tilde{f} * \varphi_\varepsilon - \tilde{f}\|_{L^1_w} + \|(f - g) * \delta_\varepsilon\|_{L^1_w} + \|g * \delta_\varepsilon\|_{L^1_w}.$$

Since $\|(f - g) * \delta_\varepsilon\|_{L^1_w} \leq \| \mid f - g \mid * \Delta_\varepsilon\|_{L^1_w}$, by Lemma 1, part (ii), we get $\|(f - g) * \delta_\varepsilon\|_{L^1_w} \leq cn$. On the other hand, since we have already shown that $\|g * \delta_\varepsilon\|_{L^1_w}$ thends to zero and by Lemma 1 applied to $\|\tilde{f} * \varphi_\varepsilon - \tilde{f}\|_{L^1_w}$ we get

$$\limsup_{\varepsilon \to 0} \|f * k_\varepsilon - \tilde{f}\|_{L^1_w} \leq cn.$$

Then arbitrariness of $n > 0$ proves that the limit exists and that it is equal to zero.

Lemmas 5 and 6 show that for a function $f$ such that $f$ and $\tilde{f}$ belong to $L^1_w$, $w \in A_1$,

$$(\tilde{f} * g)(x) = (f * \tilde{g})(x)$$

holds almost everywhere provided that $g$ is a bounded function with bounded support. This result can be generalized as follows.

**Theorem 2.** Let $w \in A_1$ and let $f$ be a function such that $f$ and $\tilde{f}$ belong to $L^1_w$. Let us assume that $g$ is a function with a decreasing and radial majorant $\psi$ such that $\psi \in L^1 \cap L^{p_0}$, $1 < p_0 \leq \infty$. Then

$$(\tilde{f} * g)(x) = (f * \tilde{g})(x)$$

holds almost everywhere on $x$.

*Proof.* By Fubini's theorem, $(f * K_\varepsilon) * g = f * (k_\varepsilon * g)$.
Since $w \in A_1$, we have

$$\|(f * k_\varepsilon) * g - \tilde{f} * g\|_{L^1_w} \le c\|f * k_\varepsilon - \tilde{f}\|_{L^1_w}\|\psi\|_{L^1}$$

By Theorem 1, $f * k_\varepsilon - \tilde{f}$ converges to zero in $L^1_w$ as $\varepsilon \to 0$. Therefore, $(f * k_\varepsilon) * g$ converges to $\tilde{f} * g$ in $L^1_w$. On the other hand, observing that the hypothesis of the theorem imply those of Lemma 6, by (22) we have that $f * (k_\varepsilon * g)$ converges in $L^p_{w^p}$, $1 < p \le p_0$ to $f * \tilde{g}$ when $\varepsilon \to 0$. This completes the proof of the theorem.

## 3. Application $\mathbf{H^1_w}$.

Let $F(x,t) = (u(x,t), v_1(x,t), \dots, v_n(x,t))$, $x \in R^n$, $t > 0$ be a vector function satisfying the Cauchy-Riemann equations in the sense of Stein and Weiss [8]. The vector $F(x,t)$ is said to belong to $H^1_w$ if

$$|||F|||_{H^1_w} = \sup_{t>0} \int_{R^n} |\, F(x,t)\, |\, w(x)dx \quad < \infty.$$

The Poisson integral of a function $f$ is defined as

$$Pf(x,t) = \int f(x-y)P(y,t)dy,$$

where $P(y,t) = c_n t(t^2 + |\, y\,|^2)^{-(n+1)/2}$.
The $j$-conjugate Poisson integral, $1 \le j \le n$, is given by

$$Q_j f(x,t) = \int f(x-y)Q_j(y,t)dy,$$

where $Q_j(y,t) = c_n y_j(t^2 + |\, y\,|^2)^{-(n+1)/2}$.

Let $R_j$ denote the $j$-Riesz transform, i.e.

$$R_j f(x) = p.v. \int c_n y_j \mid y \mid^{-n-1} f(x-y)dy.$$

It is well known that $Q_j(x,t) = R_j(P(.,t))(x)$.

We shall apply Theorem 2 to give a proof of a result due to R. L. Wheeden in [9].

**Theorem 3.** (see [9], Theorem 1, part (ii)). Let $w \in A_1$ and $f \in L^1_w$. If each $R_j f \in L^1_w$, $1 \leq j \leq n$, then, the vector

$$F = (Pf, Q_1 f, \ldots, Q_n f)$$

belongs to $H^1_w$. Moreover, for $1 \leq j \leq n$,

$$Q_j f = P(R_j f) \ , \quad \text{and} \tag{28}$$

$$c_1 |||F|||_{H^1_w} \leq \|f\|_{L^1_w} + \sum_{j=1}^{n} \|R_j f\|_{L^1_w} \leq c_2 |||F|||_{H^1_w},$$

where the constants $c_1$ and $c_2$ do not dependent on $f$.

*Proof.* Since $R_j(P) = Q_j$ and observing that $P(x,t)$ is a radial decreasing function in $L^1 \cap L^\infty$ , from Theorem 2 we have that if $f$ and $R_j f$, $1 \leq j \leq n$, belong to $L^1_w$, then

$$\int f(y) Q_j(x-y,t)dy = \int R_j f(y) P(x-y,t)dy \tag{29}$$

holds for almost every point $x \in R^n$ for each give $t > 0$. It is easy to show that both sides of (29) are continuous function of $x$ and $t > 0$. Therefore (29) holds for every $x \in R^n$ and $t > 0$. This proves (28). Now, since

$$F = (Pf, Q_1 f, \ldots, Q_n f) = (Pf, P(R_1 f), \ldots, P(R_n f)),$$

we have

$$\int | f(x,t) | w(x) dx$$
$$\leq \int | Pf(x,t) | w(x) dx + \sum_{j=1}^{n} \int | P(R_j f)(x,t) | w(x) dx.$$

By Lemma 1, the right hand side of this inequality is bounded by

$$c \int | f(x) | w(x) dx + \sum_{j=1}^{n} \int | R_j f(x) | w(x) dx.$$

Conversely, since by Lemma 1, $\|f\|_{L^1_w} = \lim_{t\to 0} \|Pf(x,t)\|_{L^1_w}$ and $\|R_j f\|_{L^1_w} = \lim_{t\to 0} \|P(R_j f)(x,t)\|_{L^1_w}$, we obtain

$$\|f\|_{L^1_w} + \sum_{j=1}^{n} \|R_j f\|_{L^1_w} \leq \sqrt{n} \||F\||_{H^1_w},$$

which ends the proof of the theorem.

## References


[1] *A. Benedek, A. P. Calderón and R. Panzone,* Convolution operators on Banach space valued functions, Proc. Nat. Acad. Sci. U.S.A. 48 (1962), 356 - 365.

[2] *A. P. Calderón and O. N. Capri,* On the convergence in $L^1$ of singular integrals, Studia Math. 78. (1984), 321 - 327.

[3] *R. Coifman and C. Fefferman,* Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974), 241 - 250.

[4] *R. Hunt, B. Muckenhoupt and R. Wheeden,* Weighted norm inequalities for the conjugate function and the Hilbert transform, Trans. Amer. Math. Soc. 176 (1973), 227 - 251

[5] *M. Kaneko and S. Yano,* Weighted norm inequalities for singular integrals, J. Math. Soc. Japan 27 (1975), 570 - 588.
[6] *B. Muckenhoupt,* Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972), 207 - 226.
[7] *E. M. Stein,* Singular Integrals and differentiability properties of functions, Princeton Univ. Press; Princeton, N. J. 1970.
[8] *E. M. Stein and G. Weiss,* On the theory of harmonic functions of several variables, I. The Theory of $H^P$ spaces, Acta Math. 103 (1960), 25 - 62.
[9] *R. L. Wheeden,* A boundary value characterization of weighted $H^1$, L'Ens. Math. 22 (1976), 121 - 134.