Pro Mathematica : *Vol. III*, *Nos.* 5 – 6, 1989

ON SOME RECENT VARIATIONAL PRINCIPLES

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In this paper we survey some recent variational principles, which have proved to be very useful in the applications to the theory of differential equations, both ordinary and partial. We start with a basic principle due to Ekeland [4], which provides new proofs to the well known minimax theorems of Ambrosetti - Rabinowitz [2] and Rabinowitz [7], [8]. For proofs of these results we refer to [8]. We also mention some applications to semilinear elliptic equations.

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1. The Ekeland Variational Principle.

Let (χ, d) be a complete metric space, and $\Phi : \chi \to \Re \cup \{+\infty\}$ a lower semicontinuous functional which is bounded from below. Then given $\varepsilon > 0$ there exists $x_\varepsilon \in \chi$ such that

$$\Phi(x_\varepsilon) \leq \inf_\chi \Phi + \varepsilon \tag{1}$$

$$\Phi(x_\varepsilon) < \Phi(x) + \varepsilon d(x, x_\varepsilon), \quad \forall x \neq x_\varepsilon \tag{2}$$

2. Remark.

For comparision we recall a minimization, theorem from Topology. "Let χ be a compact topological space, and $\Phi : \chi \to \Re \cup \{+\infty\}$ a lower semicontinuous functional. Then Φ is bounded below and the inf Φ is actually achieved". In the principle stated in paragraph 1 above, although the functional is bounded from below, it is not true that the infimun is achieved in all cases. The reason for this is a lack of compactness. However that principle states the possibility of obtaining a minimizing sequence with a very special property. Such a property will be better understood if we assume more structure on the space χ, namely that $\chi$is a Banach space. In this case if Φ is assumed to be Gâteaux differentiable, condition (2) simply states that $\|D\Phi(x_\varepsilon)\|_{\chi^*} \leq \varepsilon$ when $D\Phi(x_\varepsilon)$ denotes the Gâteaux derivative of Φ at x_ε.

3. The Palais-Smale condition.

In the next paragraphs we will state results on the existence of critical points for functionals which are in general unbounded below. We will use the strong (i.e. the norm) topology of the Banach space. Since the spaces considered are in general of infinite dimension we will not have compactness of bounded sets. So even restricting the functional to a ball we cannot use the minimization theorem stated in paragraph 2. In the theorems stated below we require instead a compactness condition on the functional itself.

Let us define it. Let $\Phi : \chi \to \Re$ be a C^1 functional defined in a Banach space χ.

We say that Φ satisfies the *Palais-Smale condition* (for short, the (PS) condition) if given any sequence (u_n) in χ such that $| \Phi(u_n) | \leq C$ and $\Phi'(u_n) \to 0$, for some constant C, then we can find a subsequence u_{n_j} which converges in the norm - topology. In other words, we first define a Palais-Smale sequence as being a sequence (u_n) such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$. Then Φ satisfies the (PS) condition if every Palais-Smale sequence is relatively compact.

4. A sufficient condition for (PS).

"Let $\Phi : \chi \to \Re$ be a C^1 functional on a Hilbert space χ. Suppose that $\Phi' = I + K$, where I is the identity operator and K is a compact operator. Then Φ satisfies the (PS) condition provided all Palais-Smale sequences are bounded". We recall that $\Phi' : \chi \to \chi^*$, where χ^* is the dual space of χ; here we identify χ^* with χvia the Riesz representation theorem. We say that $K : \chi \to \chi$is *compact* if it is continuous and takes bounded sets into relatively compact ones. Let us prove the statement above. Take a Palais-Smale sequence (u_n). By hypothesis (u_n) is bounded. So (Ku_n) contains a convergent subsequence (Ku_{n_j}), in view of the compactness of K. On the other hand, since $\Phi'(u_n) = u_n + Ku_n \to 0$ we conclude that (u_{n_j}) converges. Q.E.D.

5. An specific example of a functional in the conditions of the previous paragraph.

Let us consider the functional $\Phi : H_0^1(\Omega) \to \Re$ defined by

$$\Phi(u) = \frac{1}{2}\int | \nabla u |^2 - \int F(x,u) \tag{3}$$

where the integrals are taken over the whole of Ω. Here we assume Ω to be a bounded open connected subset of $\Re^{\mathcal{N}}$. We recall that $H_0^1(\Omega)$ denotes the Sobolev space of L^2 functions, whose first derivatives in the distribution sense are in L^2, and moreover they satisfy a zero boundary condition. See, for instance, Adams [1] or Brézis [3]. To have the functional Φ well defined in H_0^1 some growth condition on F is necessary. Namely, there are positive constants c and σ and an L^1 function b such that

$$(4) \qquad | F(x,s) | \leq c | s |^\sigma + b(x), \quad \forall x \in \Omega, \forall s \in \Re,$$

where

$$(5) \qquad 1 \leq \sigma < \infty \qquad \text{if } N = 2 \text{ , or}$$

$$(6) \qquad 1 \leq \sigma \leq \frac{2N}{N-2} \qquad \text{if } N \geq 3.$$

Indeed, by the Sobolev immersion theorem we know that H_0^1 is continuously embedded in $L^\sigma(\Omega)$ with σ restricted as in (5) and (6) above. This together with the growth condition (4) gives that the second integral in (3) is well defined. As for the first integral, we recall that the norm in H_0^1 is taken as

$$(7) \qquad \|u\|_{H^1} = \left(\int | \nabla u |^2 \right)^{1/2}$$

Let us now suppose that $F(x,s)$ is differentiable with respect to s, and let us denote its derivative by $f(x,s)$. We *assume* that f is continuous and it satisfies also a growth condition: there exists positive constants c and α and a L^p-function $\beta(x)$ such that

$$(8) \qquad | f(x,s) | \leq c | s |^\alpha + \beta(x) \quad \forall x \in \Omega, \forall s \in \Re$$

where

$$(9) \qquad 1 \leq \alpha < \infty, \qquad 1 \leq p < \infty \qquad \text{if } N = 2$$

$$(10) \qquad 1 \leq \alpha < \frac{N+2}{N-2}, \quad p \geq \frac{2N}{N+2} \qquad \text{if } N \geq 3.$$

This growth restriction on f is actually necessary in order to have Φ as a C^1 functional in H_0^1. Under these restrictions we conclude that

$$< \Phi'(u), v >= \int \nabla u \nabla v - \int f(x,u)v \qquad \forall v \in H_0^1(\Omega),$$

where $<,>$ denotes the inner product in $H_0^1(\Omega)$.

Let us now define a mapping $K : H_0^1 \to H_0^1$ by

$$< Ku, v >= \int f(x,u)v \qquad \forall u, v \in H_0^1.$$

It is easy to see that K is compact if we assume (8) with (9) and in (10) α is restricted to be *strictly less* than $(N+2)/(N-2)$. In this argument one uses the fact that H_0^1 is compactly embedded in L^p with $1 \leq p < \infty$ if $N = 2$ and $1 \leq p < 2N/(N-2)$ if $N \geq 3$. See [1] or [3]. So under these conditions, in order to check that the functional Φ defined in (3) satisfies the (PS) condition, we have only to check that Palais-Smale sequences are bounded. See several examples in [5]. Observe that (8) - (9) - (10) imply (4) - (5) - (6).

6. The Mountain Pass Theorem.

Let $\Phi : \chi \to \Re$ be a C^1 functional defined in a Banach space χ. Suppose that Φ satisfies the (PS) condition. Let S be a subset of χ which disconnects χ[for instance, S could be a hyperplane or the boundary of an open connected set]. Let x_0 and x_1 be in distinct connected components of $\chi \backslash S$. *Suppose that*

(11) $\inf\{\Phi(x) : x \in S\} > \max\{\Phi(x_0), \Phi(x_1)\}$.

Let

(12) $\Gamma = \{\gamma : [0,1] \to \chi;$ continuous, $\gamma(0) = x_0, \gamma(1) = x_1\}$

and

(13) $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t))$.

Then c is a critical value. That is, there exists a $u_0 \in \chi$ *such that $\Phi(u_0) = c$ and $\Phi'(x_0) = 0$.*

The proof of this result via the so-called Clark deformation can be seen in [8]. This result was first proved in [2]. See also a proof using the Ekeland variational principle in [5].

7. An example.

Let us consider the Dirichlet problem

$$(14) \qquad -\Delta u = f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

when $f : \Re \to \Re$ is a continuous function satisfying condition (8) - (9) - (10) with $\alpha < (N+2)/(N-2)$. Then the functional

$$(15) \qquad \Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u)$$

where $F(s) = \int_0^s f(t)dt$, is C^1 and

$$< \Phi'(u), v >= \int \nabla u \nabla v - \int f(u)v,$$

which implies that the critical points of Φ are precisely the weak (or generalized) solutions of (14). Now *assume* that $f(s) = o(s)$ as $s \to 0$. So $u \equiv 0$ is a solution of (14). If we *assume further* that these exists a constant $\Theta > 2$ and an $s_0 > 0$ such that

$$(16) \qquad 0 < \Theta F(s) \leq sf(s) \quad \forall \mid s \mid > s_0$$

then one can prove without difficulty that Φ satisfies the (PS) condition. Also (16) implies that

$$(17) \qquad \lim_{s\to+\infty} \frac{f(s)}{s} = +\infty.$$

Now we assert that (14) has a solution $u \neq 0$. For that matter we apply the Mountain Pass Theorem. Here we take $x_0 = 0$ and

$x_1 = R\bar{u}$ where $\bar{u}$ is any positive function in $H_0^1(\Omega)$ and $R > 0$ is conveniently chosen. Indeed, from (17) given any $M > 0$ there is a constant C_M such that

$$f(s) \geq M_s - C_M, \quad \forall s > 0.$$

Hence

$$F(s) \geq \frac{M}{2}s^2 - C_M s, \quad \forall s > 0$$

and then

$$\Phi(R\bar{u}) \leq \frac{R^2}{2}\int |\nabla\bar{u}|^2 - \frac{MR^2}{2}\int \bar{u}^2 + C_M R\int \bar{u}. \tag{18}$$

So choose $M > 0$ such that

$$\int |\nabla\bar{u}|^2 - M\int |\bar{u}|^2 < 0$$

and then $R > 0$ so large such that the right side of (18) is negative. Next for S we select a sphere of a convenient radius r about O, as follows. Given $\varepsilon > 0$ let $\delta > 0$ be such that

$$|f(s)| \leq \varepsilon |s|, \quad \text{for } |s| \leq \delta \tag{19}$$

On the other hand using (8) we can find a constant $c > 0$ such that

$$|f(s)| \leq c|s|^\alpha, \quad \text{for } |s| \geq \delta$$

and without loss of generality we may assume $\alpha > 1$.
So

$$|f(s)| \leq \varepsilon|s| + c|s|^\alpha, \quad \text{for all } s$$

and this implies

$$(20) \qquad | f(s) | \leq \frac{\varepsilon}{2} s^2 + \frac{c}{\alpha+1} | s |^{\alpha+1}.$$

Now we use (20) to estimate Φ :

$$(21) \qquad \Phi(u) \geq \frac{1}{2} \int | \nabla u |^2 - \frac{\varepsilon}{2} \int u^2 - \frac{c}{\alpha+1} \int | u |^{\alpha+1}$$

To continue the estimate in (21) we use Poincaré inequality

$$\lambda_1 \int u^2 \leq \int | \nabla u |^2 \qquad \forall u \in H_0^1(\Omega)$$

where λ_1, is the first eigenvalue of the Laplacian subject to Dirichlet boundary condition, and also the continuity of the embedding of H_0^1 into $L^{\alpha+1}$. Thus

$$\Phi(u) \geq \frac{1}{2} \int | \nabla u |^2 - \frac{\varepsilon}{2\lambda_1} \int | \nabla u |^2 - \frac{cc_1}{\alpha+1} \left(\int | \nabla u |^2 \right)^{\frac{\alpha+1}{2}}$$

where c_1 is the constant that comes from the embedding.
Choose $\varepsilon < \lambda_1$. So $r > 0$ has to be chosen in such a way that

$$(22) \qquad \frac{1}{2}(1 - \frac{\varepsilon}{\lambda_1}) r^2 - \frac{cc_1}{\alpha+1} r^{\alpha+1} > 0.$$

This is possible becouse $\alpha > 1$. In this way we have satisfied all the hypothesis of the Mountain Pass Theorem, and the existence of a nontrivial solution u_0 of (14) follows. Observe that c is larger than the expression in (22).

8. The Saddle Point Theorem.

Let $\Phi : \chi \rightarrow \Re$ be a C^1 functional defined in a Banach space χ. Assume that Φ satisfies the (PS) condition. Suppose that $\chi =$

$V \oplus W$, where V is a finite dimensional subspace, and Φ satisfies the two conditions below

$$\Phi\big|_W \geq b \qquad\qquad \Phi\big|_{V\cap\partial B_r(0)} \leq a < b$$

where a and b are constants and $B_r(0)$ is a ball of radius r about the origin, with boundary $\partial B_r(0)$.

$$\Gamma = \{\gamma : V \cap \overline{B_r(0)} \to \chi \text{ continuous} : \ \gamma \mid V \cap \partial B_r(0) = id\}.$$

Then

$$C = \inf_{\gamma\in\Gamma} \max_{t\in V\cap\overline{B_r(0)}} \Phi\big(\gamma(t)\big)$$

is a critical value.

A proof of the above theorem can be seen in [8], and in [5], where the Ekeland principle is used.

9. An example.

Consider the Dirichlet problem

$$(23) \qquad -\Delta u = f(u) + h(x) \text{ in } \Omega \qquad u = 0 \text{ on } \partial\Omega,$$

where $h \in L^2(\Omega)$ and $f : \Re \to \Re$ is a continuous function such that the limits below exist

$$(24) \qquad \alpha = \lim_{s\to+\infty} \frac{f(s)}{s} \qquad\qquad \beta = \lim_{s\to-\infty} \frac{f(s)}{s}$$

and

$$(25) \qquad \lambda_n < \alpha, \beta < \lambda_{n+1}$$

Here λ_n, λ_{n+1} are two consecutive eigenvalues of the Laplacian subject to Dirichlet boundary condition. We claim that (23) *has a*

H_0^1 *solution for all* h. It follows from (24) that there are constants a and b such that

$$(26) \qquad |f(s)\leq a\,|\,s\,|+b \qquad \forall s.$$

So the functional

$$\Phi(u)=\frac{1}{2}\int|\nabla u|^2-\int F(u)-\int hu$$

is well defined in H_0^1 and its critical points are precisely the H_0^1 solutions (i.e. the weak solutions) of (23). As before $F(s)=\int_0^s f(t)dt$. Let us denote by V the space generated by $\varphi_1,\ldots,\varphi_n$, the eigenfunctions of $-\Delta$ corresponding to the eigenvalues $\lambda_1<\lambda_2\leq\cdots\leq\lambda_n$. And let $W=V^{\perp}$. It follows from (24) and (25) that there exist numbers $\mu_n<\mu_{n+1}$, with $\lambda_n<\mu_n<\mu_{n+1}<\lambda_{n+1}$, such that

$$(27) \qquad \frac{1}{2}\mu_n s^2-C\leq F(s)\leq\frac{1}{2}\mu_{n+1}s^2+C \qquad \forall s$$

for some constant C. Using (27) and the inequalities

$$\int|\nabla v|^2\leq\lambda_n\int v^2 \qquad \forall v\in V$$

$$\int|\nabla w|^2\geq\lambda_{n+1}\int w^2 \qquad \forall w\in W$$

we prove that Φ is bounded below in W and $\Phi(v)$ goes to $-\infty$ when $v\in V$ and $\|v\|\to\infty$. To apply the Saddle Point Theorem and then establish the claim it remains to prove the (PS) condition in the present hypothesis. Suppose by contradiction that we have Palais-Smale sequence (u_n) which is unbounded. Let us denote by $v_n=(u_n/\|u_n\|)$.

Passing to a subsequence we may assume that there is $v_0\in H_0^1$ such that v_n converges weakly in H_0^1 to v_0, strongly in L^2 and a.e; moreover we may assume that there is an L^2 function h such that $|\,v_n(x)\,|\leq h(x)$. Now we make use of the inequality

$$(28)\quad \left|\int \nabla u_n \nabla z - \int f(u_n)z - \int hz\right| \le \varepsilon_n \|z\|, \quad \forall z \in H_0^1$$

where $\varepsilon_n \to 0$. In view of (27) we can prove (see [5]) that

$$\frac{f(u_n)}{\|u_n\|} \overset{L^2}{\to} \gamma(x)v_0$$

where $\alpha \le \gamma(x) \le \beta$. So dividing (28) by $\|u_n\|$ and passing to the limit we obtain

$$\int \nabla v_0 \nabla z - \int \gamma(x) v_0 z = 0 \qquad\qquad \forall z \in H_0^1,$$

which implies $v_0 = 0$. On the other hand, using (28) with $z = v_n/\|u_n\|$ and passing to the limit, we see that the first term is constantly 1 and the other three go to zero. This is the contradiction. QED.

10. Final remarks.

There are other variational principles which are used to stablish existence of critical points for functionals coming from differential equations. We refer to the lecture notes by Rabinowitz [8], Mawhin [6], Struwe [9] and the author [5]. The present paper aims to call attention of the reader to certain interesting problems and inform him on recent trends in Differential Equations and the Calculus of Variations. The list of references supplies additional reading, which should be used if one wants to get a complete picture of this area. We take the opportunity to thank the hospitality of the Peruvian mathematicians during my visit to Lima in July 1989, under an agreement between the Sociedad de Matemática Peruana and the Sociedade Brasileira de Matemática, sponsored by International Centre for Theoretical Physics.

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