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ALGEBRAIC QUANTUM FIELD THEORY AND NONCOMMUTATIVE MOMENT PROBLEMS I*

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Let $\underline{S}$ denote Borchers' test function algebra and $\mathcal{I}_c$ the locality ideal. It is shown that the quotient algebra $\underline{S}/\mathcal{I}_c$ admits a continuous C^*-norm and thus has a faithful representation by bounded operators on Hilbert space. This representation can be chosen to be Poincaré-covariant. Some further properties of the topology defined by the continuous C^*-norms on this algebra are also established.

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1. Introduction

The mathematical theory of relativistic quantum fields has mainly been developed within two general frameworks. On the one hand there is the approach initiated by Wightman and Gårding [1], where the fields are considered as distributions with values in the unbounded, closable operators on a Hilbert space. On the other hand there is the theory of Haag, Kastler and Araki [2] [3] using nets of C*-resp. v. Neumann algebras associated with bounded domains of space-time. It is a long standing problem to establish conditions under which it is possible to pass from one scheme to the other. We shall not attempt to review all previous work on this subject here, but mention only the papers [4] [5] [6] and the recent publication [7], where further references can be found.

In this paper we propose to consider this question from a point of view that was developed in two papers by Dubois-Violette about 10 years ago [8,9]. In this papers it is shown how one can in a natural way associate a C*-algebra $\mathcal{B}$ with any *-algebra $\mathcal{U}$ that is equipped with a family of C*-seminorms. Moreover, every state ω on $\mathcal{U}$ that satisfies a certain positivity condition determines a state $\hat{\omega}$ on the C*-algebra $\mathcal{B}$ in such a way that ω can be reconstructed from $\hat{\omega}$. In the special case where $\mathcal{U}$ is an algebra of polynomials, the C*-algebra $\mathcal{B}$ is an algebra of continuous functions and the construction of $\hat{\omega}$ amounts to solving a classical moment problem. For this reason the term $\ll$ noncommutative moment problems $\gg$ has been used in [8] [9] for the general case.

This formalism can be applied to a Wightman-Gårding quantum field theory in the algebraic version due to Borchers [10] and Uhlmann [11]. A quantum field is here regarded as a representation of a tensor algebra over a space of test function; in the simplest case with Schwartz test functions this algebra is denoted by $\underline{S}$. The locality postulate of quantum field theory means that the representation should annihilate a two sided ideal, $\mathcal{I}_c$, that is generated by commutators of test functions with space-like separated supports. The relevant algebra for a local quantum field theory is therefore the quotient algebra $\underline{S}/\mathcal{I}_c$ rather than $\underline{S}$ itself.

In [8] [9] Dubois-Violette considered the algebra $\underline{S}$ and showed that every positive functional on it gives rise to a state on an associated C*-algebra, called the ≪quasi-localizable C*-algebra≫. While this C*-algebra is generated by nets of subalgebras corresponding to bounded domains of Minkowski space, it does not satisfy the locality postulate, i.e. commutativity for space-like separated domains. There thus remains the problem to decide which states on $\underline{S}$ give rise to representations of the quasi-localizable C*-algebra fulfilling the locality postulate. A sufficient condition was stated in [8] in terms of quasi-analyticity of the vacuum, that implies essential self-adjointness of the field operators on the natural domain.

In an endeavour to obtain more general criteria, we would like to apply the method of [8] [9] to the algebra $\underline{S}/\mathcal{I}_c$ instead of $\underline{S}$. The associated C*-algebra has the local commutativity built in and is thus a quasi-local algebra in the sense of [2]. It is not a *priori* clear, however, that this C*-algebra is a useful object. The main ingredient requiered for its construction is a family of continuous C*-seminorms on the algebra $\underline{S}/\mathcal{I}_c$. These C*-seminorms determine the positivity condition which a functional on $\underline{S}/\mathcal{I}_c$ has to satisfy in order to define a state on the C*-algebra: The functional has to be positive on the closure of the positive cone in $\underline{S}/\mathcal{I}_c$ w.r.t. the topology defined by the C*-seminorms. If this closure is too big, there may be no nontrivial functionals that satisfy this requirement. This happens for instance if one considers instead of $\mathcal{I}_c$ the ideal corresponding to the canonical commutation relations, that have no bounded representation at all.

In this paper we investigate the topology defined by the continuous C*-seminorms on $\underline{S}/\mathcal{I}_c$. We show that the algebra admits a continuous $C^* - \textit{norm}$, so the topology separates points. This means that the associated C*-algebra is nontrivial, and the functionals on $\underline{S}/\mathcal{I}_c$ which satisfy the positivity requirement span a dense set in the dual space $(\underline{S}/\mathcal{I}_c)'$. We also prove that the topology defined by the continuous C*-norms induces the original Fréchet-topology on each ≪ finite segment ≫ of $\underline{S}/\mathcal{I}_c$, i.e. the subspaces generated by tensor powers of $\mathcal{S}$ up to a finite order. This means in particular that every n-point Wightman-distribution can be written as a linear combination of n-point distributions that correspond to a representation of the test function algebra by bounded operators satisfying the condition of local commutativity.

Finally, we show that the algebra $\underline{S}/\mathcal{I}_c$ has a faithful, Poincaré covariant representation by bounded operators.

2. The General Formalism

In this section we review the general formalism developed by Dubois-Violette [8,9].

Let $\mathcal{U}$ be a *-algebra over $\mathbf{C}$ with unit element $\mathbf{1}$. A seminorm p on $\mathcal{U}$ is called a C*-*seminorm* if

$$p(a^*a) = p(a)^2$$

for all $a \in \mathcal{U}$ (1) Let Γ be a family of C*-seminorms on $\mathcal{U}$ and assume that the topology $\mathcal{T}_\Gamma$ generated by Γ on $\mathcal{U}$ is *Hausdorff*, i.e. for every $a \in \mathcal{U}$, $a \neq 0$, there is a $p \in \Gamma$ with $p(a) \neq 0$. Denote by $\mathcal{A}(\mathcal{U}, \Gamma)$ (or simply $\mathcal{A}$, if $\mathcal{U}$ and Γ are fixed) the completion of $\mathcal{U}$ w.r.t. $\mathcal{T}_\Gamma$. $\mathcal{A}$ is in a natural way a topological *-algebra. If $a \in \mathcal{A}$, then the spectrum $Sp(a)$ is the set of all $\lambda \in \mathbf{C}$ such that $a - \lambda\mathbf{1}$ is not invertible in $\mathcal{A}$. Using standard results of C*-theory one shows that

$$\mathcal{A}^+ := \{a \in \mathcal{A} \mid sp(a) \subset \Re^+\}$$

is a closed, convex cone in $\mathcal{A}$ and

$$\mathcal{A}^+ = \{a^*a \mid a \in \mathcal{A}\} = \{h^2 \mid h \in \mathcal{A}, h = h^*\} = \mathcal{T}_\Gamma\text{-closure}$$

(1) This implies that p satisfies $p(ab) \leq p(a)p(b)$ and $p(a^*) = p(a)$, for all $a, b = \mathcal{U}$, cf. [13].

of $\{\sum a_i^* a_i \mid a_i \in \mathcal{U}\}$.

We denote by $\mathcal{U}_\Gamma^+$ the intersection of $\mathcal{A}^+$ with $\mathcal{U}$; this is also the $\mathcal{T}_\Gamma$-closure in $\mathcal{U}$ of the convex cone $\mathcal{U}^+ = \{\sum a_i^* a_i \mid a_i \in \mathcal{U}\}$. A linear functional ω on $\mathcal{U}$ is called Γ-*strongly positive* (or,for fixed Γ, simply *strongly positive*), if ω is positive on $\mathcal{U}_\Gamma^+$.

The positive cone $\mathcal{A}^+$ defines an order relation on $\mathcal{A}$ denoted by $\leq$. Let $\mathcal{M} \equiv \mathcal{M}(\mathcal{U}, \Gamma)$ be the linear subspace of $\mathcal{A}$ generated by order intervals with endpoints in $\mathcal{U}$, i.e.

$$\mathcal{M} = \mathcal{M}_h + i\mathcal{M}_h$$
$$\mathcal{M}_h = \{x \in \mathcal{A} \mid x = x^*, \exists a, b \in \mathcal{U} \text{ with } a \leq x \leq b\}$$

$\mathcal{M}$ is a *-invariant, linear space, but in general not an algebra unless $\mathcal{U}$ is commutative.

We have the following theorem ([8], theorem 1 and prop. 4, cf. also [9], prop.26.12):

Theorem 2.1. – A linear functional ω on $\mathcal{U}$ has an extension $\hat{\omega}$ to a linear functional on $\mathcal{M}$ that is positive on $\mathcal{M}^+ = \mathcal{M} \cap \mathcal{A}^+$ if and only if ω is strongly positive.

The extension $\hat{\omega}$ satisfies

$$\omega_\#(h) \leq \hat{\omega}(h) \leq \omega^\#(h) \quad \text{for all} \quad h \in \mathcal{M}, \quad h = h^*$$

where

$$\omega^\#(h) := sup\{\omega(a) \mid a \in \mathcal{U}, a \leq h\}$$
$$\omega_\#(h) := inf\{\omega(b) \mid b \in \mathcal{U}, h \leq b\}$$

The extension is unique if and only if $\omega_\# = \omega^\#$.

Consider next the algebra

$$\mathcal{B}_\infty = \{a \in \mathcal{A} \mid \sup_{p \in \Gamma} p(a) < \infty\}.$$

It is straightforward to verify that $\mathcal{B}_\infty$ is a C*-*algebra*, when equipped with the norm

$$\| a \| := \sup_{p \in \Gamma} p(a).$$

Moreover $\mathcal{B}_\infty$ is contained in $\mathcal{M}$, for if $h \in \mathcal{L}_\infty, h = h^*$, then $-\| h \| \mathbf{1} \leq h \leq \| h \| = \mathbf{1}$. In general, $\mathcal{U}$ and $\mathcal{B}_\infty$ will only have multiples of the unit element in common. On the other hand, a strongly positive functional ω on $\mathcal{U}$ can by theorem 2.1 be extended to a positive functional $\hat{\omega}$ on $\mathcal{M}$ and $\hat{\omega}$ can be restricted to $\mathcal{B}_\infty$. This restriction in fact determines ω uniquely (cf. [8], theorem 5). In this way one has established a connection between strongly positive functionals on $\mathcal{U}$ and states on the C*-algebra $\mathcal{B}_\infty$. The situation is illustrated by the following diagram.

$$\begin{tikzcd}
& \mathcal{A} & \\
& \mathcal{M} \arrow[u, phantom, "\cup"] \arrow[dd, "\hat{\omega}"] & \\
\mathcal{U} \arrow[ur, hook] \arrow[dr, "\omega"'] & & \mathcal{B}_\infty \arrow[ul, hook'] \arrow[dl, "\hat{\omega}|\mathcal{B}_\infty"] \\
& \mathbf{C} &
\end{tikzcd}$$

The spaces $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{B}_\infty$ can be easily described in concrete terms if $\mathcal{U}$ is the commutative algebra $C[X_1, \ldots, X_n]$ of polynomials in n indeterminates and Γ is the family of all C*-seminorms on $\mathcal{U}$. $\mathcal{U}$ can be identified with the algebra of polynomial functions on $\Re^n$, and the C*-seminorm are of the form

$$p(P) = \sup_{x \in K} | P(x) |$$

with $K \subset \Re^n$ compact. $\mathcal{A}$ is the algebra of all continuous functions on $\Re^n$, whereas $\mathcal{M}$ consists of the polynomially bounded continuous functions. The algebra $\mathcal{B}_\infty$ is the C*-algebra of all bounded, continuous finctions on $\Re^n$ with the sup-norm. Strongly positive functionals on $\mathcal{U}$ are those that are positive on all positive polynomials. The positive functionals on $\mathcal{M}$ correspond to positive measures on $\Re^n$ of rapid decrease at infinity. The measures are uniquely determined by the states they define on $\mathcal{B}_\infty$. It is also clear that the subalgebra $\mathcal{C}_0(\Re)$ of functions vanishing at infinity is sufficiently large to determine the measure.

In the general situation it may also happend that a suitable subalgebra of $\mathcal{B}_\infty$ is a more natural object to deal with than $\mathcal{B}_\infty$ itself. To construct such subalgebras one can make use of the *functional calculus* established in [8]: Namely, for every hermitean $h \in \mathcal{U}$, there is a unique homomorphism φ of the algebra of continuous functions on $S = sp(h)$ into $\mathcal{A}$ such that $\varphi(Id_S) = h$ and $\varphi(1) = \mathbf{1}$. If f is a polynomially bounded, continuous function, then $\varphi(f) =: f(h) \in \mathcal{M}$, and if f is bounded, then $f(h) \in \mathcal{B}_\infty$. Let $\mathcal{G}$ be a subset of hermitean elements of $\mathcal{U}$. Define $\mathcal{B}(\mathcal{G},\Gamma)$ as the subalgebra of $\mathcal{B}_\infty$ generated by elements $f(h)$ with $h \in \mathcal{G}$ and $f \in \mathcal{C}_0(\Re)$. If $\mathcal{G}$ is a set of generators for $\mathcal{U}$, then the restriction of $\hat{\omega}$ to $\mathcal{B}(\mathcal{G},\Gamma)$ is sufficient to determine ω uniquely. On the other hand one has the following strengtening of theorem 2.1 (cf. [8], corollary 3).

Theorem 2.2. – The restriction $\hat{\omega} \mid \mathcal{B}(\mathcal{G},\Gamma)$ is uniquely determined by ω if and only if

$$\omega_{\#}(a) = \omega^{\#}(a) \quad \text{for all} \quad a = a^* \in \mathcal{B}(\mathcal{G},\Gamma).$$

In [9] it is shown that this uniqueness holds e.g. in the case that the elements in $\mathcal{G}$ are represented by essentially self-adjoint operators in the GNS-representation of $\mathcal{U}$ defined by ω.

Finally we remark that the choice of the algebras $\mathcal{B}(\mathcal{G},\Gamma)$ is somewhat arbitrary. Instead of using functions vanishing at infinity to generate an algebra, one could just as well consider the C*-subalgebra of $\mathcal{B}_\infty$ generated by the unitary elements $e^{i\lambda h}$, $h \in \mathcal{G}$, $\lambda \in \Re$.

3. C*-seminorms On Partially Symmetric Tensor Algebras

Let E be a linear space (over $\mathbf{C}$) and ρ a relation on E, i.e. a subset of $E \times E$. The partially symmetric tensor algebra $S_\rho(E)$ is defined as the quotient algebra

$$S_\rho(E) = T(E)/I_\rho$$

where $T(E)$ is the tensor algebra over E and I_ρ is the two-sided ideal in $T(E)$ generated by all commutators $a \otimes b - b \otimes a$ with $(a,b) \in \rho$. If E is a topological vector space, it is understood that $T(E)$ is the completion of the algebraic tensor produced w.r.t. some suitable tensor product topology and that I_ρ is a closed ideal.

We consider first the case that E is a finite dimensional space with basis $\{e_1, \ldots, e_N\}$ and ρ is a relation on the basis elements [14]. One can then also regard ρ as a relation on the set $\{1, \ldots, N\}$. $S_\rho(E)$ is in a natural way a *-algebra, if the basis elements are considered to be *-invariant. We shall construct representations of S_ρ by embedding this algebra into a group algebra.

Let G_ρ denote the ≪partially abelian free group≫ corresponding to the relation ρ. This is defined as the group with N generators $u_1, \ldots, u_N$ satisfying the relations

$$u_i u_j = u_j u_i \quad \text{if} \quad (i,j) \in \rho.$$

The group algebra $\mathcal{J}_1(G_\rho)$ consists of all formal sums

$$\sum_{g \in G_\rho} \alpha_g g$$

with $\alpha_g \in \mathbf{C}$ and

$$\|\Sigma \alpha_g g\|_1 := \Sigma \mid \alpha_g \mid < \infty.$$

The product in $\mathcal{J}_1(G_\rho)$ is the convolution

$$(\Sigma\alpha_g G) * (\Sigma\beta_h h) = \sum_g \Big(\sum_h \alpha_{gh^{-1}}\beta_h\Big) g$$

and the *-operation is

$$(\Sigma\alpha_g g)^* = \Sigma\bar{\alpha}_{g^{-1}} g.$$

We embed $S_\rho(E)$ into $\mathcal{J}_1(G_\rho)$ as follows: Define

$$\varphi(e_i) := \frac{1}{2}(u_i + u_i^{-1})$$

and extend φ to a homomorphism $S_\rho(E) \to \mathcal{J}_1(G_\rho)$. This is possible, because $[\varphi(e_i), \varphi(e_j)] = 0$ in $\mathcal{J}_1(G_\rho)$, if $[e_i, e_j] = 0$ in $S_\rho(E)$.

Lemma 3.1. – The homomorphism φ is injective.

Proof. – Let $\mathcal{J}_0(G_\rho)$ denote the subalgebra of $\mathcal{J}_1(G_\rho)$ consisting of all *finite* sums $\Sigma\alpha_g g$. We define a grading on $\mathcal{J}_0(G_\rho)$ with values in $\mathbf{Z}$ by putting

$$deg\, u_i = +1$$

$$deg\, u_i^{-1} = -1$$

and extending this to all monomials, i.e. elements of $G_\rho \subset \mathcal{J}_0(G_\rho)$, by the formula

$$deg\ gh = deg\ g + deg\ h \tag{3.1}$$

If $a = \Sigma\alpha_g g \in \mathcal{J}_0(G_\rho), a \neq 0$ we define

$$deg\ a = max\ \{deg\ g \mid \alpha_g \neq 0\}.$$

Now $S_\rho(E)$ is also a graded algebra with $deg\ e_i = 1$ for all i. Moreover, we have for all $i_1, \ldots, i_n$

$$\varphi(e_{i_1} \otimes \ldots \otimes e_{i_n}) = \frac{1}{2^n} u_{i_1} \otimes \ldots \otimes u_{i_n} + \text{ terms of lower degree.}$$

It follows that $\varphi : S_\rho(E) \to \mathcal{J}_0(G_\rho)$ preserves the degree. In particular we have $\ker \varphi = \{0\}$.

The (nondegenerate) representations of $\mathcal{J}_1(G_\rho)$ correspond uniquely to the unitary representations of G_ρ. If V is a representation of $\mathcal{J}_1(G_\rho)$, then $\pi = V \circ \varphi$ is a representation of $S_\rho(E)$, and

$$\pi(e_i) = \frac{1}{2}(V(u_i) + V(u_i)^*). \tag{3.2}$$

Since $V(u_i)$ is unitary, we have $\|\pi(e_i)\| \leq 1$. Conversely, if π is a representation of $S_\rho(E)$ with $\|\pi(e_i)\| \leq 1$ for all i, we can define a representation V of $\mathcal{J}_1(G_\rho)$ satisfying (3.2) by

$$V(u_i) = \pi(e_i) + i\sqrt{1 - \pi(e_i)}. \tag{3.3}$$

We may picture the situation by the diagram

$$\begin{tikzcd} S_\rho(E) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathcal{J}_1(G_\rho) \arrow[dl, "V"] \\ & \mathcal{B}(\mathcal{H}) & \end{tikzcd}$$

Since φ is injective, we obtain a faithful representation π of $S\rho(E)$ with $\|\pi(e_i)\| \leq 1$ by picking any faithful representation V of $\mathcal{J}_1(G_\rho)$ (it suffices that V is faithful on $\mathcal{J}_0(G_\rho)$) and combining it with φ. We may for instance consider the left regular representation $V_\mathcal{J}$, defined on

$$\mathcal{H} = \mathcal{J}_2(G\rho) = \{\Sigma\beta_h h \mid \|\Sigma\beta_h h\|_2 := (\Sigma \mid \beta_h \mid^2)^{1/2} < \infty\}$$

as follows:

$$V_\mathcal{J}(\Sigma\alpha_g g)(\Sigma\beta_h h) = \sum_g \Big(\sum_h \alpha_{gh^{-1}}\beta_h\Big) g.$$

This representation is faithful because the unit element of the group defines a separating vector in $\mathcal{H}$. We have thus proven

Theorem 3.2. – Suppose E is finite dimensional and ρ is a relation on a set of basis elements for E. Then the algebra $S_\rho(E)$ has a faithful representation by bounded operators on Hilbert space.

The construction above can be described quite explicity if $S\rho(E)$ is the totally symmetric tensor algebra $S(E)$. Here $G_\rho = \mathbf{Z}^n$. The left regular representation $V_{\mathcal{J}}$ is defined on $\mathcal{H} = \mathcal{J}_2^{\otimes N}$, and u_i resp u_i^{-1} is represented by a right resp. left shift in the i-th factor. The Fourier transform

$$\{\alpha\} \to \Sigma \alpha_n e^{inx}$$

establishes an isomorphism between $\mathcal{H}$ and $L_2([-\pi,\pi]^N, dx)$. Here u_i resp. u_i^{-1} is represented as multiplication with the function $\exp(ix_i)$ resp. $\exp(-ix_i)$. Hence $\pi(e_i)$ is multiplication by $\cos x_i$. The C*-algebra generated by $\pi(S_\rho(E))$ is the algebra of all continuous functions on $[-\pi,\pi]^N$, that are even w.r.t. inversion of each coordinate. After a variable transformation, $y_i = \cos x_i$, this algebra is manifestly isomorphic to the algebra of all continuous functions on $[-1,1]^N$ equipped with the sup norm. Under the same isomorphism the algebra $\pi(S(E))$ is mapped onto the algebra of polynomials restricted to $[-1,1]^N$.

Any bounded representation π of $S\rho(E)$ gives rise to a C*-seminorm

$$p(a) = \|\pi(a)\|.$$

Further bounded representation and C*-seminorms can be constructed by combining π with automorphism of the algebra. Thus one can for each $\lambda > 0$ define an automorphism α_λ of $S_\rho(E)$ by $\alpha_\lambda e_i := \lambda e_i$ and obtain a representation $\pi_\lambda = \pi \circ \alpha_\lambda$ and a C*-seminorm $p_\lambda = p \circ \alpha_\lambda$. In the case of the totally symmetric tensor algebra discussed above, p_λ is the sup norm on $[-\lambda,\lambda]^N$. One thus obtains a *basis* of C*-norms for the symmetric tensor algebra by combining the regular representation of the group algebra and the automorphisms α_λ. It is an open question whether this is generally true for the algebras $S_\rho(E)$.

Consider now the partially symmetric tensor algebra $\underline{S}/\mathcal{I}_c$ that is of interest in quantum field theory [13]. Here $\underline{S} = T(\mathcal{S})$ is the tensor algebra over Schwartz space of the test functions, $\mathcal{S} = \mathcal{S}(\Re^d)$, and $\mathcal{I}_c$ is the two sided ideal generated by commutators $f \otimes g - g \otimes f$, with $f, g \in \mathcal{S}$ having space-like separated supports.

Theorem 3.4. – The algebra $\underline{S}/\mathcal{I}_c$ admits a continuous C*-norms and thus has a faithful representation by bounded operators om Hilbert space.

Proof. – We use the method of [14] to map the algebra onto partially symmetric tensor algebras over finite dimensional spaces. Let $\mathcal{X} = \{\bar{x}_1, \ldots, \bar{x}_N\}$ be a set of points in $\Re^d$. Define a relation $\rho_\mathcal{X}$ on $\{1, \ldots, N\}$ by

$$(i,j) \in \rho_\mathcal{X} \quad \text{iff} \quad (\bar{x}_i - \bar{x}_j)^2 < 0.$$

Denote by $S_\mathcal{X}(E)$ the partially symmetric algebra corresponding to the relation $\rho_\mathcal{X}$. Let $e_1, \ldots, e_N$ be the generators for $S_\mathcal{X}(E)$ and define a homomorphism $\Phi_\mathcal{X} : \underline{S}/\mathcal{I}_c \to S_\mathcal{X}(E)$ by

$$\Phi_\mathcal{X}(f) = \sum_{i=1}^{N} f(\bar{x}_i) e_i \tag{3.4}$$

for $f \in \mathcal{S}$ and canonical extension to other elements of the algebra. It was shown in [14], lemma 4.3, that the homomorphism $\Phi_\mathcal{X}$ separate points in $\underline{S}/\mathcal{I}_c$ when $\mathcal{X}$ runs through all finite subsets of $\Re^d$. Suppose now that p is a C*-seminorm on $S_\mathcal{X}$. Then $p_\mathcal{X} := p \circ \Phi_\mathcal{X}$ is a C*-seminorm on $\underline{S}/\mathcal{I}_c$. Moreover,

$$p_\mathcal{X}(f) \leq \sup_x \mid f(x) \mid \cdot \sum_{i=1}^{N} p(e_i)$$

so $p_\mathcal{X}$ is continuous w.r.t. the LF-topology of $\underline{S}/\mathcal{I}_c$. If we combine this seminorm with an automorphism α_λ with $\lambda = \left(\sum_{i=1}^{N} p(e_i)\right)^{-1}$, we obtain a new C*-seminorm $q_\mathcal{X}$ with the same null space, and

$$q_\mathcal{X}(f) \leq \sup_x \mid f(x) \mid .$$

The family $\{q_{\mathcal{X}}\}$ is therefore equicontinuous, and

$$\|\cdot\|_\infty := \sup_{\mathcal{X}} q_{\mathcal{X}} \tag{3.5}$$

is a continuous C*-seminorm $\underline{\mathcal{S}}/\mathcal{I}_c$. The statement of the theorem now follows from lemma 4.3 in [14] and theorem 3.3.

To construct further C*-norms on $\underline{\mathcal{S}}/\mathcal{I}_c$ one can make use of automorphisms of the algebra. In particular, we can consider graded automorphisms that are generated by mappings $\mathcal{S} \to \mathcal{S}$ of the form

$$f \to Mf = P(D,x) f_{\{a,\Lambda\}} \tag{3.6}$$

where $P(D,x)$ is a linear differential operator with polynomially bounded C^∞-coefficients and $\{a,\Lambda\}$ is a Poincaré-transformation. This mapping leaves the locality ideal invariant and thus generates an automorphism $\alpha_M : \underline{\mathcal{S}}/\mathcal{I}_c \to \underline{\mathcal{S}}/\mathcal{I}_c$. If $\|\cdot\|$ is a continuous C*-norm on the algebra, then $\|\alpha_M(\cdot)\|$ is another continuous C*-norm. By using such automorphisms we can make more detailed statements than theorem 3.4 about the topology defined by the continuous C*-norms on $\underline{\mathcal{S}}/\mathcal{I}_c$.

We recall from [14] that $\underline{\mathcal{S}}/\mathcal{I}_c$ is a locally convex direct sum of nuclear Fréchet-spaces:

$$\underline{\mathcal{S}}/\mathcal{I}_c = \bigoplus_{n\geq 0} (\underline{\mathcal{S}}/\mathcal{I}_c)_n$$

with $(\underline{\mathcal{S}}/\mathcal{I}_c)_0 = \mathbf{C}$ and $(\underline{\mathcal{S}}/\mathcal{I}_c)_n = \mathcal{S}(\Re^{n.d})/(\mathcal{I}_c \cap \mathcal{S}(\Re^{n.d}))$ for $n \geq 1$. The topology on $(\underline{\mathcal{S}}/\mathcal{I}_c)_n$ can e.g. be defined by the Schwartz-norms on $\mathcal{S}(\Re^{n.d})$

$$\|f\|_k^n = \max_{|\alpha_i|\leq k} \sup_{x_i \in \Re^d} \left| \prod_i D^{\alpha_i} f(x_1, \ldots, x_n)(1+ \mid x_i \mid^k) \right|.$$

An explicit formula for the corresponding quotient norm is given in [14] (formula (4.2) and prop.(4.2)).

Theorem 3.5. – The topology defined by the continuous C*-norms on $\underline{S}/\mathcal{I}_c$ induces the original Frechét topology on the subspaces $\bigoplus_{n=0}^{N} (\underline{S}/\mathcal{I}_c)_n$ for all $N < \infty$.

Proof. – We have to show that every Schwartz-norm on $\bigoplus_{n=0}^{N} (\underline{S}/\mathcal{I}_c)_n$ can be dominated by a C*-norm on the algebra. Since we can combine C*-norms with mappings of the type (3.6), it suffices to consider the simplest Schwartz norms on $\mathcal{S}(\Re^{d.n})$:

$$\|f\|_0^n = \sup \mid f(x_1, \ldots, x_n) \mid \tag{3.7}$$

and the corresponding norm $\|\cdot\|_0 = \sum_{n=0}^{N} \|\cdot\|_0^n$ on $\bigoplus_{n=0}^{N} (\underline{S}/\mathcal{I}_c)_n$. Now for all $f \in \mathcal{S}(\Re^{d.n})$ the supremum in (3.7), is reached at a point $(\bar{x}_i, \ldots, \bar{x}_n) \in \Re^{d.n}$. Hence, for every $a \in \bigoplus_{n=0}^{N} (\underline{S}/\mathcal{I}_c)_n$, there is a set $\mathcal{X}$ with at most $N(N-1)/2$ points such that

$$\|a\|_0 \leq p(\Phi_{\mathcal{X}}(a)) \leq const_{\mathcal{X}} \cdot \|a\|_0 \tag{3.8}$$

where $\Phi_{\mathcal{X}}$ is the homomorphism $\underline{S}/\mathcal{I}_c \rightarrow S(E)$ defined by (3.4) and p is a norm on $S_{\mathcal{X}}(E)$. The norm p depends only on the algebra $S_{\mathcal{X}}(E)$, but is otherwise independent of a. Also the constant, $const_{\mathcal{X}}$, depends only on the relation $\rho_{\mathcal{X}}$. Moreover, $\Phi_{\mathcal{X}}$ maps $\bigoplus_{n=0}^{N} (\underline{S}/\mathcal{I}_c)_n$, onto a finite dimensional subspace of $S_{\mathcal{X}}(E)$, where all norms are equivalent to each other. By theorem 3.2 we may therefore assume that p is a C*-norm. As $\mathcal{X}$ runs through allset with at most $N(N-1)/2$ points the algebra $S_{\mathcal{X}}(E)$ will change. However, there are only finitely many different relations $\rho_{\mathcal{X}}$ and hence only finitely many different algebras to be taken into account. Because of this and (3.8), the family of C*-seminorms $p \circ \Phi_{\mathcal{X}}$ is equicontinuous. The supremum over $\mathcal{X}$ defines then a C*-seminorm that dominates $\|\cdot\|_0$ on $\bigoplus_{n=0}^{N} (\underline{S}/\mathcal{I}_c)_n$.

4. Poincaré-Covariance

The Poincaré group $\mathcal{P}$ operates in a natural way as a group of continuous automorphisms of $\underline{S}/\mathcal{I}_c$. If Γ is a family of C*-seminorms on $\underline{S}/\mathcal{I}_c$ and Γ is invariant under this action, then the group is also represented by automorphisms of the C*-algebra $\mathcal{B}(\underline{S}/\mathcal{I}_c,\Gamma)$, defined in section 2, cf. [9], prop. 3. However, his action is in general not continuous in the group elements. In fact, suppose Γ contains the C*-seminorms

$$\|a\|_K = \sup_{\chi\in K} \|\chi(a)\|$$

where K runs through the compact sets of characters on the algebra. For the C*-seminorm $\|\cdot\|$ on $\mathcal{B}$ we then have

$$\|f(a) - f(b)\| \geq \sup_{\chi} |f(\chi(a)) - f(\chi(b))|$$

for all $a,b \in \underline{S}/\mathcal{I}_c, f \in C_0(\Re)$. If $a,b \in (\underline{S}/\mathcal{I}_c)_1$ are linearly independent, there is a χ with $\chi(a) = 1$ and $\chi(b) = 0$. Thus we have

$$\|f(a) - f(b)\| \geq |f(1) - f(0)|.$$

Since a Poincaré-transformation L takes an element $a \in (\underline{S}/\mathcal{I}_c)$, $a \neq \mathbf{1}$ into a linearily independent element, it is clear that $L \to f(\alpha_L a)$ is not a continuous function on the group if $a \neq \mathbf{1}$, $f(0) \neq f(1)$.

This situation, however, is not unexpected; what matters is that the algebra should have many nontrivial covariant representations, where the group action is unitarily implemented in a continuous way and satisfies the phisical spectrum condition. (See [15] for a general characterization of representations that are quasi-equivalent to covariant representations.) The question whether this is the case for the C*-algebra $\mathcal{B}$ associated with the family of all continuous C*-seminorms on $\underline{S}/\mathcal{I}_c$ is a very difficult one, and we shall not deal with it here. Instead we present an example of a Poincaré-covariant representation (without spectrum condition) where the algebra $\underline{S}/\mathcal{I}_c$ is faithfully represented by *bounded* operators.

Consider the C*-norm $\|\cdot\|_\infty$ on $\underline{S}/\mathcal{I}_c$ defined by (3.5). This norm is obviously invariant under Poincaré-transformations. Moreover, the action is continuous in the group elements for every fixed element of the algebra. In fact, since the norm is invariant, it suffices to consider the action on the generators, i.e. on $\underline{S}/\mathcal{I}_c \cong \mathcal{S}(\Re^d)$. But there $\|\cdot\|_\infty$ is the usual sup-norm and the assertion is obviously true. We can now apply a standard method for constructing covariant representations by forming cross products (cf. [16], ch.7.6):

Let π be a representation of $\underline{S}/\mathcal{I}_c$ on a Hilbert space $\mathcal{H}$ with $\|\pi(a)\| = \|a\|_\infty$. Define a representation $\tilde{\pi}$ of the algebra and a unitary representation U of the Poincaré group $\mathcal{P}$ on $\mathcal{H} = L_2(\mathcal{H}, \mathcal{P})$ by

$$(\tilde{\pi}(a)\psi)(L) = \pi(\alpha_L^{-1}a)\psi(L)$$

and

$$(U(L_1)\psi)(L) = \psi(L_1^{-1} \circ L).$$

then

$$U(L)\tilde{\pi}(a)U(L)^{-1} = \tilde{\pi}(\alpha_L a)$$

for all $L \in \mathcal{P}$, $a \in \underline{S}/\mathcal{I}_c$. Moreover, $\tilde{\pi}$ is faithful since π is faithful.

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References

[1] *A. S. Wightman and L. Gårding,* Fields as operator valued distributions in relativistic quantum field theory. Ark. F. Fys., t. 28, 1965, p. 129.

[2] *R. Haag and D. Kastler,* An algebraic approach to quantum field theory. J. Math. Phys., t. 5, 1964, p. 848.

[3] *H. Araki,* Einführung in die axiomatische Quantaenfeldtheorie. ETH Zürich 1961-1962.

[4] *H. J. Borchers and W. Zimmermann,* On the self-adjointness of field operators. Nuovo Cim., t. 31, 1963, p. 1047.

[5] *W. Driessler and J. Fröhlich,* The reconstruction of local observable algebras from the Euclidean Green's functions of relativistic quantum field theory. Ann. Inst. H. Poincaré, t. 27, 1977, p. 221

[6] *Fredenhagen and J. Hertel,* Local algebras of observables and pointlike localizaed fields. Commun. Math. Phys., t. 80, 1981, p. 555.

[7] *W. Driessler, S. J. Summers and E.H. Wichmann,* On the connection between quantum fields and von Neumann algebras of local operators. Commun. Math. Phys., t. 105, 1986, p. 49.

[8] *M. Dubois-Violette,* A generalization of the classical moment problem on *-algebras with applications to relativistic quantum theory. I. Commun. Math. Phys., t. 43, 1975, p. 225.

[9] *M. Dubois-Violette,* A generalization of the classical moment problem on *-algebras with applications to relativistic quantum theory. II. Commun. Math. Phys., t. 54, 1977, p. 151.

[10] *H. J. Borchers,* On the structure of the algebra of field operators. Nuovo Cim., t. 24, 1962, p. 214.

[11] *A. Uhlmann,* Über die Definition der Quantenfelder nach Wightman und Haag. Wiss. Zeitschr., Karl Marx Univ., t. 11, 1962, p. 213.

[12] *G. Choquet,* Lectures on Analysis, vol 2, New York, Bemjamin, 1984.

[13] *H. Araki and G.A. Elliott,* On the definition of C*-algebras. Publ. RIMS Kyoto, t. 9, 1973, p. 93.

[14] *J. Yngvason,* On the locality ideal in the algebra of test functions for quantum fields. Publ. RIMS Kyoto., t. 20, 1984, p. 1063.

[15] *H.J. Borchers,* C*-algebras and auomorphism groups. Commun. Math. Phys., t. 88,1983, p. 95.

[16] *G. K. Pedersen,* C*-algebras and their Automorphism Groups. London, New York, San Francisco, Academic Press, 1979.