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REORDERINGS OF SOME DIVERGENT SERIES

Julio ***Alcantara-Bode***

The purpose of this note is to show that if $f: \mathbb{N} \to \mathbb{C}$ *is a function such that*

$$\sum_{n=1}^{\infty} |f(n) - \frac{1}{n}| < \infty \text{ and } \alpha \in [0,1] \text{ is irrational then}$$

$$\sum_{n=1}^{\infty} \{f([\frac{n}{\alpha}]) + f([\frac{n}{1-\alpha}]) - f(n)\} = -\alpha \ ln\alpha - (1-\alpha)\ ln(1-\alpha)$$

where [x] *denotes the integer part of x.*

It has been proven by Skölem and Bang [2] that if $\alpha \in]0,1[$ and $Q_\alpha = \{ [\frac{n}{\alpha}] \mid n \in \mathbb{N}\}$ then

i) $Q_\alpha \cap Q_{1-\alpha} = \phi$ iff $\alpha \notin \mathbb{Q}$ and

ii) $Q_\alpha \cup Q_{1-\alpha} = \mathbb{N}$ iff $\alpha \notin \mathbb{Q}$.

⇨ *Profesor del Departamento de Ciencias, Secc. Matemáticas, PUCP.*

Now if $\alpha = \frac{p}{q}$, $1-\alpha = \frac{q-p}{q}$, $q,p \in \mathbb{N}$, $(q,p) = 1$, $p < q$, then

i') $Q_{\frac{p}{q}} \cap Q_{\frac{q-p}{q}} = \{jq : j \in \mathbb{N}\}$ and

ii') $Q_{\frac{p}{q}} \cup Q_{\frac{q-p}{q}} = \mathbb{N} \setminus \{jq - 1 : j \in \mathbb{N}\}$

Using these results it is not difficult to prove that if $\alpha \in]0,1[$, $\alpha \notin \mathbb{Q}$ and $\rho(x) = x-[x]$ then

$$\rho\left(\frac{\alpha}{x}\right)+\rho\left(\frac{1-\alpha}{x}\right)-\rho\left(\frac{1}{x}\right)=\sum_{m=1}^{\infty}\chi_{]\frac{\alpha}{m},[\frac{m}{\alpha}]^{-1}]}(x)+\sum_{m=1}^{\infty}\chi_{]\frac{1-\alpha}{m},[\frac{m}{1-\alpha}]^{-1}]}(x), \quad x>0 \tag{1}$$

where χ_A denotes the characteristic function of the set A, and if $\alpha \in]0,1[\cap \mathbb{Q}$, $\alpha = \frac{p}{q}$, $(p,q) = 1$, $p,q \in \mathbb{N}$ then

$$\rho\left(\frac{\alpha}{x}\right)+\rho\left(\frac{1-\alpha}{x}\right)-\rho\left(\frac{1}{x}\right)=\sum_{m=1}^{\infty}\chi_{]\frac{\alpha}{m},[\frac{m}{\alpha}]^{-1}]}(x)+\sum_{m=1}^{\infty}\chi_{]\frac{1-\alpha}{m},[\frac{m}{1-\alpha}]^{-1}]}(x)+ \tag{2}$$

$$+\sum_{r=1}^{\infty}\chi_{]\frac{1}{qr},\frac{1}{qr-1}]}(x), \quad x>0$$

It can be shown that the intervals whose characteristic functions appear in the second members of equations (1) and (2) are pairwise disjoint.

If $\theta \in [0,1]$ and $\operatorname{Re} r > -1$ it is known that [1]

$$\int_0^1 \rho\left(\frac{\theta}{x}\right)x^r\,dx = \frac{\theta}{r} - \frac{\zeta(r+1)}{r+1}\theta^{r+1} \tag{3}$$

where ζ represents the Riemann's zeta function. Then taking the limit $r \to 0$ in (3) we get

$$\int_0^1 \rho\left(\frac{\theta}{x}\right)dx = -\theta \ln\theta + (1-\gamma)\theta \tag{4}$$

where γ represents Euler's constant.

Therefore integrating equations (1) and (2) with respect to x between 0 and 1 we get

$$-\alpha \ln\alpha - (1-\alpha)\ln(1-\alpha) = \sum_{m=1}^{\infty}\left\{\frac{1}{[\frac{m}{\alpha}]} + \frac{1}{[\frac{m}{1-\alpha}]} - \frac{1}{m}\right\} \tag{5}$$

and

$$-\alpha \ln\alpha - (1-\alpha)\ln(1-\alpha) = \sum_{m=1}^{\infty}\left\{\frac{1}{[\frac{m}{\alpha}]} + \frac{1}{[\frac{m}{1-\alpha}]} - \frac{1}{m}\right\} + \sum_{r=1}^{\infty}\frac{1}{qr(qr-1)} \tag{6}$$

respectively.

Now if $g\colon \mathbb{N} \to \mathbb{C}$ is such that $\sum_{n=1}^{\infty}|g(n)| < \infty$ we get from the results of Skölem and Bang that if $\alpha \in]0,1[$ is irrational then

$$\sum_{n=1}^{\infty}\left\{g\left(\left[\frac{n}{\alpha}\right]\right) + g\left(\left[\frac{n}{1-\alpha}\right]\right) - g(n)\right\} = 0 \tag{7}$$

and that if $\alpha \in]0,1[$ is rational, $\alpha = \frac{p}{q}$, then

$$\sum_{n=1}^{\infty}\left\{g\left(\left[\frac{n}{\alpha}\right]\right) + g\left(\left[\frac{n}{1-\alpha}\right]\right) - g(n)\right\} + \sum_{r=1}^{\infty}\left\{g(rq-1) - g(rq)\right\} = 0 \tag{8}$$

If we apply (7) and (8) to the function $g(n) = f(n) - \frac{1}{n}$, we get from (5) and (6) that

$$\sum_{n=1}^{\infty}\left\{f([\frac{n}{\alpha}])+f([\frac{n}{1-\alpha}])-f(n)\right\}=-\alpha\ln\alpha-(1-\alpha)\ln(1-\alpha) \qquad (9)$$

for $\alpha\in[0,1]$ and irrational and

$$\sum_{n=1}^{\infty}\left\{f([\frac{n}{\alpha}])+f([\frac{n}{1-\alpha}])-f(n)\right\}=-\alpha\ln\alpha-(1-\alpha)\ln(1-\alpha)$$

$$-\sum_{r=1}^{\infty}\left\{f(qr-1)-f(qr)\right\}$$

if $\alpha\in\,]0,1[$ is rational, $\alpha=\frac{p}{q}$, $(p,q)=1$, $p,q\in\mathbb{N}$.

References:

[1] A. Beurling, *A closure problem related to the Riemann zeta-function*, Proc. Nat. Acad. Sci. 41(1955), 312-314.

[2] I. Niven, *Diophantine Approximations*, Interscience, John Wiley, New York, 1963.

jalcant@pucp.edu.pe