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RELATIONS BETWEEN WAVELETS AND OPERATOR THEORY

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Introduction

In this writing we are interested in to place some known relations between the operator theory and the recent wavelet theory. It is our intention to remark some of the deep Calderón's ideas in his work on singular integral operators (pseudo-differential operators) and how the wavelets, a new tool in pure and applied mathematics, can be useful to brighten some difficulties in to solve problems with these operators. By the way, one road to arrive to the "wavelet world" was given by Calderón in his research on interpolation of function spaces (Calderón's reproducing identity, 1960).

It is curious that in "pure" ideas, like Littlewood-Paley theory and atomic decompositions, between other ideas, there were the seed of something common to explore later. The celebrated Calderón-Zygmund school, a branch of operator theory, is also a beautiful history related to this "wavelet adventure".

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In this way, it was also amazing the identification of the algorithms proposed, by "wavelet people", in the signal processing with some recourses developed by the cited school. As Meyer says: "time-frequency localization operators should definitively be understood before deriving any conclusion from a wavelet analysis. In the wavelet case, these operators happen to be Calderón-Zygmund operators, which in the windowed Fourier transform case, the corresponding operator theory is much wilder (it corresponds to the symbols $S^{o}{}_{o,o}$ in Hörmander's terminology". See [Me 4].

The general idea is to address the possibility of using wavelets for studying operators, like singular integral operator or other. Thus, we must able to find a wavelet based criterion to have the continuity of the operator T acting on some Hilbert space, like $L^2(\mathbb{R})$. An ideal case is to have orthonomal basis for H in which T is diagonalized. The generality is to have unconditional basis or frames in which we can to represent the operator T. The task is to construct a basis adapted to T and to have access for efficient algorithms. Before wavelet theory appear, "harmonic-people"uses the famous Cotlar's lemma for analysing operators, as we shall see later.

We finish this introduction with a message of Ch. Chui-R. Coifman-I. Daubechies, a trio of authorities in the wavelet-field.

"The past few years have seen a significant increase of interest in applications on the part of mathematicians with a "pure harmonic analysis" background, as well as a surge of curiosity, from applied scientists and engineers, in mathematical techniques that we shall label under the common denominator of harmonic analysis. Of course, all these different groups of people have used Fourier transforms for a long time and will continue to do so, but the emphasis here is on new tools that have recently made their appearance or the innovative use of any of the older tools.

A particularly well-known and currently very popular example is provided by wavelets. With hindsight there are many precursors for what we now call wavelet expansions, ranging from the use and refinement of Calderón's decomposition ideas in the study of singular integral operators at one end of the pure-versus-applied spectrum to the algorithms from subband filtering in electrical engineering at the other end, with many intermediate stages as well, such as the link with coherent states in quantum physics or connections with spline functions and subdivision schemes in approximation theory. However, none of these forerunners showed the full force of the wavelet tools, which stems from a combination of these different aspects. Before they were linked to efficient and fast algorithms, the deep theorems of pure harmonic analysis had a minimal impact on the study of applied problems. At the same time, the algorithms alone, no matter how efficient, would not have led to meaningful results for those same applied problems without the deeper underlying mathematical unders tanding. The development of the wavelet tools themselves gives many beatiful examples of a constant feedback between theoretical understanding and practical points of view, in both directions".

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A. Singular Integral Operators and Differential Operators

Motived by the several important applications of the singular integral operators in several branches of the analysis mainly in problems in partial differential equations, the theory of Calderón-Zygmund was extended to operators which are not of convolution type, like

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x,y)\, f(y)dy$$

where the kernel $K(x)$ satisfies certain adequate conditions. Between the different contributions to the theory, the work of R. Coifman-Y.Meyer (1978) is fundamental since they began a systematic study of such operators in a general context. These new operators were called "the new Calderón-Zygmund operators" (CZO). The reader can see Meyer [M.2] for more references.

Using the distribution language, we say that an **integral operator** is defined on the space $\mathcal{D}(\mathbb{R}^n)$ (test-functions) by

$$T\,\phi(x) = \int K(x,y)\ \phi(y)dy, \quad \phi \in \mathcal{D}(\mathbb{R}^n), \tag{1}$$

where the kernel k is a distribution on $\mathbb{R}^n \times \mathbb{R}^n$ and the operator T: $\mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is defined more precisely by

$$< T\psi, \phi > = < k, \phi \otimes \psi >, \quad \phi, \psi \in \mathcal{D}(\mathbb{R}^n),$$

where

$$(\phi \otimes \psi)\,(x,y) = \phi(x)\ \psi(y) \tag{2}$$

(in general $< g,h > = \int g\,h$).

A.P. Calderón studied kernels satisfying the conditions $|k(x,y)| \leq \frac{C_o}{|x-y|}$,

$|\frac{\partial}{\partial x} k(x,y)| \leq C_1 |x-y|^2$ and $k(x,y) = - k(y,x)$.

A famous result of L. Schwartz says that "every linear continous operator T: $\mathcal{D} \to \mathcal{D}'$ admit a kernel k such that we have (1) in the sense (2)". In this way, when a linear operator T is bounded on the Lebesgue space L^p, $1 \leq p < \infty$?... For example, if we add to Calderón's conditions $k(x,y) = k_1(x-y)$ (i.e T is a convolution operator), then T is bounded on L^2. Also we know that the classical CZO (convolution type) are bounded on L^p, $1<p<\infty$. For a general answer one consider the C-Z conditions:

CZ.1 Outside of the diagonale $x = y$, $k(x,y)$ is a locally bounded function ($k \in L^{\infty}_{loc}(\mathbb{R}^n)$), and there exits a constant $C>0$ such that

$$| k(x,y) | \leq \frac{C}{|x-y|^n} .$$

Unfortunately an operator whose kernel satisfies only this condition is not necessary bounded on $L^2(\mathbb{R}^n)$. Thus this condition is insufficient for the L^p-theory. For this reason one is carried to impose to k a minimal continuity property outside of the diagonal.

CZ.2 There exists a constant $C>0$ such that

$$\int_{|x-y| \geq 2|x-x'|} |k(x,y) - k(x',y)| dy \leq C \ \ldots \ \forall x,x' \in \mathbb{R}^n.$$

In this road one is also carried to consider the so called "standard kernels". Thus, k: $\Delta_c \to C$ (Δ_c is the complement of the diagonal $x=y$) is a **standard kernel** if it is a continous function for which $\exists\ C > 0$ such that $\forall (x,y) \in \Delta_c$ we have (like Calderón's conditions)

$$| k(x,y) | \leq C |x-y|^{-n} \tag{3}$$

$$|\nabla_x k(x,y)| + |\nabla_y k(x,y)| \leq C|x-y|^{-(n+1)} \tag{4}$$

(the gradient is in the distribution sense).

Definition. *The operator* T: $\mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ *is called a* ***Calderón-Zygmund operator*** *(CZO) if:*

(i) *T is extended to $L^2(\mathbb{R}^n)$ as a linear bounded operator; and*

(ii) *there exists a standard kernel k such that $\forall f \in L^\infty_0 \in (\mathbb{R}^n)$ (compact support), we have the representation $T f(x) = \int k(x,y) f(y)dy$* a.e. *on the complement of supp f.*

Now we have the fundamental (C-Z) result: << if T is a CZO, then

T: $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a linear bounded operator, $1 < p < \infty$ >>. A deep problem at the beginning 80's years was to find necessary and sufficient conditions to assure that T is bounded on $L^2(\mathbb{R}^n)$. In 1982, G. David - J.L. Journé find a solution to the problem.

Clasical examples of singular integral operators are the Hilbert and M. Riesz transforms. Thus,

$$\tilde{f}(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t} dt , \qquad x \in \mathbb{R}^1,$$

is the **Hilbert transform** of f; and the **Riesz transform** is its natural extension to $\mathbb{R}^n$,

$$(R_j f)(x) = \lim_{\varepsilon \to 0} c_n \int_{|x-t|>\varepsilon} \frac{x_j - t_j}{|x-t|^{n+1}} f(t)dt.$$

In this way we have an interesting relation between singular integral operators and differential operators. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$; if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we write

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \quad \text{and} \quad \left(\frac{\partial}{\partial x}\right)^\alpha f \equiv D^\alpha f = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f.$$

Let the polynomial $P(x) = \sum_{|\alpha| \le m} a_\alpha(x) x^\alpha$ and its associated partial differential operator $P(D) = \sum_{|\alpha| \le m} a_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha$.

If $f \in \mathcal{S}$ (rapidly decreasing distributions), we know that (using Fourier transform) $\left(\frac{\partial}{\partial x_j} f\right)^{\wedge}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$, and more generally

$[P(D)f]^{\wedge}(\xi) = P(2\pi i \xi)\, \hat{f}(\xi)$. As an example, let $P(x) = x_1^2 + \ldots + x_n^2 = |x|^2$, then $P(D)f = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2} \equiv \Delta f$, and therefore $[\Delta f]^{\wedge}(\xi) = -4\pi^2 |\xi|^2 \hat{f}(\xi)$.

If we define the operator Λ as

$$[\Lambda f]^{\wedge}(\xi) = 2\pi |\xi| \hat{f}(\xi) \tag{5}$$

then we obtain the representation $\Lambda f(x) = i \sum R_j \frac{\partial}{\partial x_j} f$.

In this direction we also have the relation $R_j = -i \frac{\partial}{\partial x_j} (-\Delta)^{-1/2}$, which was used by Calderón in his study about the uniqueness of Cauchy problem. If we reiterates (5) we obtain $[\Lambda^m f]^{\wedge}(\xi) = (2\pi |\xi|)^m \hat{f}(\xi)$; thus, in particular $[\Lambda^2 f]^{\wedge}(\xi) = (2\pi |\xi|)^2 \hat{f}(\xi) = -[\Delta f]^{\wedge}(\xi)$; i.e. we have obtained $\Lambda^2 = -\Delta$, a nice formula.

In a similar way,

$$\begin{aligned}[(\tfrac{\partial}{\partial x})^{\alpha} f]^{\wedge}(\xi) &= (2\pi i \xi)^{\alpha} \hat{f}(\xi) = (\tfrac{\xi}{|\xi|})^{\alpha} i^{|\alpha|} (2\pi |\xi|)^{|\alpha|} \hat{f}(\xi) = \\ &= i^{|\alpha|} (\xi')^{\alpha} [\Lambda^{|\alpha|} f]^{\wedge}(\xi)\end{aligned} \tag{6}$$

where $(\xi')^{\alpha} \equiv (\frac{\xi}{|\xi|})^{\alpha} \in C^{\infty}(\mathbb{R}^n - \{0\})$ is an homogeneous function of zero-degree; therefore (it is known that) there exists a (generalized) singular integral operator T_α such that $(x')^{\alpha}$ is the symbol σ_{T_α} of T_α (by definition, $\sigma_{T_\alpha}(x) = c(x) + \hat{k}(x)$). Therefore, $[(\frac{\partial}{\partial x})^{\alpha} f]^{\wedge}(\xi) = i^{|\alpha|} \sigma_{T_\alpha}(\xi) [\Lambda^{|\alpha|} f]^{\wedge}(\xi)$, from which $(\frac{\partial}{\partial x})^{\alpha} f = i^{|\alpha|} T_\alpha \Lambda^{|\alpha|} f$, where more precisely

$$T_\alpha f(x) = a(x) f(x) + \int k_\alpha(x, x-y) f(y)\, dy, \quad x \in \mathbb{R}^n.$$

Thus we have the interesting relation

$$P(D)f = \sum_{|\alpha| \le m} a_\alpha (\tfrac{\partial}{\partial x})^{\alpha} f = \sum_{|\alpha| \le m} i^{|\alpha|} a_\alpha T_\alpha \Lambda^{|\alpha|} f.$$

In particular, if $|\alpha| = m$ we have

$$P_m(D)f = \sum_{|\alpha|=m} a_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha f = i^m\left(\sum_{|\alpha|=m} a_\alpha(x)\, T_\alpha\right)\Lambda^m f \equiv i^m\, T\,\Lambda^m f,$$

a link between partial differential operators and singular integral operators. See [0] for other related commentaries.

B. Pseudo-Differential Operators.

We observe that

$$P(D)f(x) = \sum_{|\alpha|\le m} a_\alpha(x)\left(\frac{\partial}{\partial x}\right)^\alpha f(x) = \sum_{|\alpha|\le m} a_\alpha(x)\int e^{-2\pi i x\xi}\left[\left(\frac{\partial}{\partial \xi}\right)^\alpha f\right]^\wedge(\xi)\,d\xi =$$

$$= \sum_{|\alpha|\le m} a_\alpha(x)\int e^{-2\pi i x\xi}(2\pi i\xi)^\alpha \hat{f}(\xi)d\xi = \int e^{-2\pi i x\xi}\left[\sum_{|\alpha|\le m} a_\alpha(x)(2\pi i\xi)^\alpha\right]\hat{f}(\xi)d\xi.$$

The idea of the pseudo-differential operators is to replace the polynomials $\sum_{|\alpha|\le m} a_\alpha(x)(2\pi i\xi)^\alpha$ for more general functions $p(x,\xi)$, called **symbol** of the operator $P(D)$, $|\alpha| \le m$, such that the new class of operators (which contain the differential operators) must contain products (algebra), inverses, compositions, ... In the case of differential operators we have these properties if the coefficients are C^∞; therefore this way does not work for operators with coefficients which are not infinite differentiable. Thus, we must look another method as the following: we write $\sum_{|\alpha|\le m} a_\alpha(x)(2\pi i\xi)^\alpha = [Lq(x,\xi) + \tau(x,\xi)]\,\phi^m(\xi)$, where ϕ is a positive function, C^∞ and such that $\phi(\xi) = |\xi|$ if $|\xi| \ge 1$, and $q(x,\xi) = |\xi|^{-m} \sum_{|\alpha|\le m} a_\alpha(x)(2\pi i\xi)^\alpha$. If we put

$Tf(x) = \int q(x,\xi)\, e^{-2\pi i x.\xi}\, \hat{f}(\xi)d\xi + Sf(x)$ and $Sf(x) = \int \tau(x,\xi)\, e^{-2\pi i x.\xi}\, \hat{f}(\xi)d\xi$, we obtain, as before,

$$P(D)f = i^m\, T\,\Lambda^m f.$$

Now one observes that $q(x,\xi)$ is homogeneous of zero degree in ξ, and it can be shown that the operators S and $S\frac{\partial}{\partial x}$ are bounded on L^2 or on L^p, $1 < p < \infty$, provided that the $a_\alpha(x)$'s are bounded.

How to generalize the operator T? ... we replace $q(x,\xi)$ by any homogeneus function of zero-degree in ξ and also S is replace by an arbitrary operator with the boundedness condition mentioned above. Respect the smoothness of $q(x,\xi)$ in x, it is smaller than the smoothness of the coefficients in the differential operators which we are interested. In this way Calderón [C1] has considered a class of pseudo differential operators as follows.

Let the Lebesgue space (or Sobolev space)

$$L_k^2(\mathbb{R}^n) \equiv L_k^2 = \{f \in \mathcal{S}'(\mathbb{R}^n) \,/\, (1+|\xi|)^k \hat{f}(\xi) \in L^2(\mathbb{R}^n)\},\ k \in \mathbb{R},$$

where we consider the norm $\|f\|_{L_k^2} = \|(1+|\xi|)^k \hat{f}\|_{L^2}$. Let m be a positive integer and $f \in \mathcal{S}$. We say that the operator T belongs to the **class $\mathcal{S}_m$** if

$$Tf(x) = \sum_{j=0}^{n} \int p_j(x,\xi)\, e^{-2\pi i x.\xi} \hat{f}(\xi) d\xi + Sf(x) \tag{7}$$

where the $p_j(x,\xi)$ are bounded and for $|z| > c$ the $p_j(x,\xi)$ are homogeneous functions in z for each x and of degree $-d_j$ with $0 \le d_j < d_{j+1} < m$; moreover $\partial_x^\alpha \partial_\xi^\beta p_j(x,\xi)$ are continous bounded functions for $|\alpha| \le 2m - [d_j]$, $\forall\beta$. Moreover, the operator $S: \mathcal{S} \to L_m^2$ is such that S, $S(\frac{\partial}{\partial\xi})^\alpha$ and $(\frac{\partial}{\partial\xi})^\alpha S$ are bounded on L^2, $\forall\alpha$ with $|\alpha| = m$; i.e. $\|Sf\|_{L^2} \le C\|f\|_{L^2}$, $\|S(\frac{\partial}{\partial\xi})^\alpha f\|_{L^2} \le C \|f\|_{L^2}$ and $\|(\frac{\partial}{\partial\xi})^\alpha Sf\|_{L^2} \le C\|f\|_{L^2}$. For this reason (and in general for L_m^p, $1 < p < \infty$) we say that S belongs to the **class of operators $\mathcal{I}_m$**.

The operator Tf, according (7), is well defined since $p_j(x,\xi)\, e^{-2\pi i x.\xi} \hat{f}(\xi) \in L^1$ and S is assumed well defined. $\mathcal{S}_m$ is a self-adjoint algebra.

Now, following before motivation, we define pseudo-differential operators using singular integral operators and the operator Λ. In effect, let $\varphi \in C^\infty$ be a real function, $\varphi(x) \ge 1$ and $\varphi(x) = |x|$ for $|x| \ge 2$. If s is a real (or complex) number we define (as before) Λ^s via $[\Lambda^s f]^\wedge(\xi) = [\varphi(\xi)]^s \hat{f}(\xi)$. $\{\Lambda^s\}_{s\in\mathbb{R}}$ is a group, and $\Lambda^s : L_r^2 \to L_{r-s}^2$ is continuous and onto.

Definition. *T is called* ***a seudo-differential operator of class m and order*** $s \in \mathbb{R}$ if $T = \Lambda^s P$ *(or* $= P\Lambda^s$*), where* $P \in \mathcal{S}_m$.

We say that T is a pseudo-differential operator of mixed type if $T = \Lambda^{s_1} P \Lambda^{s_2}$ (See also [0] for other comentaries).

A wide class of pseudo differential operators was considered by L. Hörmander [H], who introduced the class of operators of the form $Tf(x) = \int e^{-2\pi i x.\xi}\, p(x,\xi)\, \hat{f}(\xi) d\xi$, where $f \in \mathcal{D}(\Omega)$, $x \in \Omega$ (open subset of $\mathbb{R}^n$) and where (the Kernel) $p(x,\xi) \in S^m_{\rho,\delta}(\Omega)$, being $S^m_{\rho,\delta}(\Omega)$ the set of $\rho(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ such that for every compact $K \subset \Omega$ and every multi indices α,β, there exists a constant $C_{\alpha,\beta,K} \equiv C$ such that $|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq C(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$ with m,ρ,δ real numbers, $\rho > 0$, $\delta \geq 0$, $x \in K$, $\xi \in \mathbb{R}^n$.

Also, between other, Calderón-Vaillancourt [C-V] had considered pseudo-differential operatos; they proved that a bounded symbol $p(x,\xi)$, with bounded derivatives, define a pseudo-differential operator $Tf(x) = \int e^{-2\pi i x \xi}\, p(x,\xi)\, \hat{f}(\xi) d\xi$, $f \in \mathcal{S}$, such that Tf is extended to a bounded operator from L^2 to L^2. If $p(x,\xi) \in S^0_{\rho,0}$, $\rho > 0$, then $p(x,\xi)$ is a symbol in the sense C-V. The cited work of C-V is related to Cotlars's lemma, to see later.

C. Wavelets.

Wavelets is an amazing discovery which is a synthesis over the last eighteen years of several ideas related to many different roads, like harmonic analysis, quantum physics, electrical engineering. The wavelet transforms are a tool for de composition functions or distributions and they have various and interesting applications in a growing number of fields, in pure and applied mathematics. Wavelets is an alternative to the classical Fourier analysis. We follow the standard way to construct orthonormal bases of wavelets for $L^2(\mathbb{R})$, which is due to S. Mallat [Ma]; see also Meyer [M.1] where the reader finds a complete exposition about this theory. The Daubechies's book [D1] is an excellent work about wavelets; she mades a profound presentation about several topics of the theory and its detour. A very useful concept is the multiresolution (Mallat-Meyer), as follows.

Definition. A *multiresolution analysis (MRA) for* $L^2(\mathbb{R})$ *on* $\mathbb{R}$ is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ such that

[1] $V_j \subset V_{j+1} \ldots \forall j\in\mathbb{Z}$

[2] $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \ldots \forall j\in\mathbb{Z}$, ie. $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$

[3] *There exists* $g\in V_0$ *such that the family* $(g(x-k))_{k\in\mathbb{Z}}$ *is a Riesz basis of* V_0. *This means that the finite linear combinations* $\sum c_k\, g(x-k)$ *are dense in* $L^2(\mathbb{R})$ *and there exist two constants* C_1 *and* C_2 *such that for every finite sequence* (c_k) *we have*

$$C_1 \left(\sum |c_k|^2\right)^{1/2} \leq \left\| \sum_k c_k\, g(x-k)\right\|_{L^2} \leq C_2 \left(\sum |c_k|^2\right)^{1/2}.$$

We note that if $C_1 = C_2 = 1$ then we obtain an orthonormal basis for $L^2(\mathbb{R})$ $(\| g \| = 1)$.

[4] $\bigcap_{j\in\mathbb{R}} V_j = \{0\}$

[5] $\overline{\bigcup_{j\in\mathbb{Z}} V_j}^{L^2} = L^2(\mathbb{R})$.

In particular, we are interested in **r-regular MRA's** , wich means that g and $D^\alpha g$ are of decreasing rapid for $|\alpha| \leq r$, ie. $\exists$ a constant $C_{M,\alpha}$ such that $|D^\alpha g(x) \leq \frac{C_{M,\alpha}}{(1+|x|)^M}$, $\forall \alpha, |\alpha| \leq r$ and $M \geq 0$ entire.

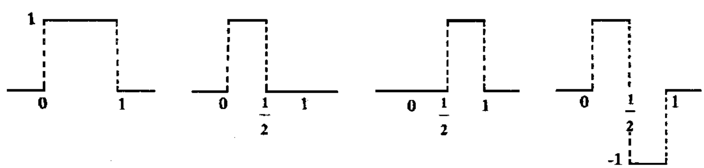
We note that $f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0$, and in general $f(x) \in V_j \Leftrightarrow f(x-\frac{k}{2^j})$ $\in V_j$, $\forall j\in\mathbb{Z}$. Also, $\{g(x-k)\}_{k\in\mathbb{Z}}$ is called an **unconditional basis** since $\sum_{k\in\mathbb{Z}} c_k\, g(x-k)$ is absolutely convergent (ie. it is independent of the form of summation). We must remark that under the above considerations, one can prove that there exists two positive constants C_1 and C_2 such that $C_1 \leq \left(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2\pi k)|^2\right)^{1/2} \leq C_2$. Thus, if we define the **scaling function** φ by $\hat{\varphi}(\xi) = \left(\sum |\hat{g}(\xi + 2\pi k)|^2\right)^{-\frac{1}{2}} \hat{g}(\xi)$ then $(\varphi(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0 ; and if g satisfies the r-regular condition, then φ thus do. By dilation property, if $\varphi_j(x-k) = 2^{1/2}\varphi(2^j x-k)$ then the sequence $(\varphi_j(x-k))_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j. We can think that the family

$\{V_j\}$ allows us to approximate any function $f \in L^2(\mathbb{R})$, with more and more accurately. It is clear that $\left(\sum_{k\in\mathbb{Z}} |\hat{g}(\xi + 2\pi k)|^2 = 1\right.$ (orthonormality condition). If we put

$c_{j,k} = 2^j \int f(y)\overline{\varphi(2^j y - k)}\,dy$, the *projection* P_j: $L^2(\mathbb{R}) \to V_j$, $j\in\mathbb{Z}$, is defined as $P_j f(x) = \sum_k c_{j,k}\, \varphi(2^j x\text{-}k)$. If φ is a regular function satisfying $|D^\alpha \varphi(x)| \le \frac{C}{(1+|x|)^M}$, then $\int \sum_{k\in\mathbb{Z}} \varphi(x\text{-}k)\overline{\varphi(y-k)}\,dy = 1 \Leftrightarrow \|P_j f - f\|_{L^2} < \varepsilon$.

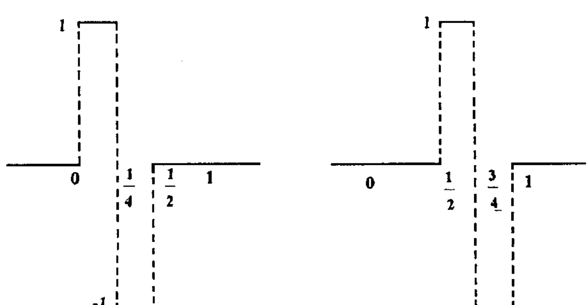
Examples:

1. **Haar System.** The oldest construction of a "wavelet basis" is due to A. Haar (1909). We begin with the characteristic function of the unit interval, $\varphi(x) = \begin{cases} 1 \ldots 0 \le x \le 1 \\ 0 \ldots \text{othervise} \end{cases}$. We can see that $\int_{-\infty}^{\infty} \varphi(x)\, \varphi(x\text{-}k)dx = 0$, $k \neq 0$, which is also true for $\varphi(2^j x)$ and $\varphi(2^j x\text{-}k)$, $\forall j\in Z$. We use φ to construct "the wavelet" ψ and its dilation-translation forms. One constructs ψ by the relation $\psi(x) = \varphi(2x) - \varphi(2x\text{-}1)$. Graphically we have



$\varphi(x)$ $\varphi(2x)$ $\varphi(2x\text{-}1)$ $\psi(x)$

And more generally,



$\psi(2x)$ $\psi(2x\text{-}1)$

We observe that all is okey with these $\psi(x)$'s, ie $\psi(x)$ and $\psi(x\text{-}k)$ are orthogonal for $k\neq0$; also $\varphi(x\text{-}k_1)$ and $\psi(x\text{-}k_2)$ are orthogonal $\forall k_1,k_2 \in\mathbb{Z}$. By dilation all this is also true for $\varphi(2^jx\text{-}k_1)$ and $\varphi(2^jx\text{-}k_2)$ $\forall j\in\mathbb{Z}$.

Conclusion: $\{\psi_{j,k}\} \equiv \{2^{j/2}\,\psi(2^j\,x\text{-}k)\}_{j,k\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$.

2. **Splines.** We shall talk on B-splines of order m, which can be seen as a "regularization"of the Haar system. We start with g_o, the characteristic function of $[0,1)$, and by definition the *B-spline*, of order m, is

$g_m = g_{m-1} * g_o$ ... $\forall m\geq 1$ entire.

Then we construct V_0^m as the closure (in L^2 ($\mathbb{R}$)) of the finite linear combinations of the functions $(g_m\,(x\text{-}k))_{k\in\mathbb{Z}}$, and as before V_j^m is defined by $f\,(x)\in V_j^m \Leftrightarrow f\,(2^{-j}\,x)\in V_0^m$. Now we consider the space S_m of the functions $f\in C^{m-2}$ such that their restrictions to the interval $[k,k+1)$, k entire, is a polynomial of degree at most m-1. In particular, we consider S_1 the space of piecewise constant functions, where the most convenient basis is $\{g_0\,(x\text{-}k)\}_{k\in\mathbb{Z}}$. People can see that

$g_m(x) = \int_0^1 g_{m-1}\,(x\text{-}t)dt$, $V_0^m = \overline{S_m \cap L^2(\mathbb{R})}^{L^2(\mathbb{R})}$ and $g_m\in S_m$. Between the properties of g_m we have: $g_m\,(x) > 0$ if $0 < x < m$, supp $g_m = [0,m]$, $\sum_k g_m\,(x\text{-}k) = 1$, $\forall x$. A more general situation is to consider the space S_m^j of the splines with knot sequences $2^{-1}\,\mathbb{Z}$, $j\in\mathbb{Z}$, and to obtain the chain --- $\subset S_m^{-1} \subset S_m^0 \subset S_m^1 \subset$ --- with $S_m^0 = S_m$. Now we define

$V_j^m = \overline{S_m^j \cap L^2(\mathbb{R})}^{L^2(\mathbb{R})}$ and again --- $\subset V_{-1}^m \subset V_0^m \subset V_1^m \subset$ --- a chain of closed (cardinal) spline subspaces of L^2 ($\mathbb{R}$).

An interesting result is to prove that "$\{V_j^m\}_{j\in\mathbb{Z}}$ is a MRA", where $\{2^{j/2}$ $g_m\,(2^j\,x\text{-}k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for V_j^m. By a similar idea to Haar system, this way also permit us to construct "certain wavelet". In Chui's book [Ch], the splines are presented in their relation with wavelet theory. Splines are one road leading to wavelets.

3. The Shannon Wavelets. In this case the scaling function is $\varphi(x) = \frac{\sin \pi x}{\pi x}$, ie $\hat{\varphi}$ is the characteristic function of $[-\pi,\pi]$, since

$$\varphi(x) = \frac{1}{2\pi} \int e^{ix.\xi}\, \hat{\varphi}\,(\xi) d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix.\xi}\, d\xi = \frac{\sin \pi x}{\pi x}.$$

Then, $\varphi(x)$ and $\varphi(x\text{-}k)$ are orthogonal since

$$\int \varphi(x)\varphi(x-k)dx = \frac{1}{2\pi} \int \hat{\varphi}\,(\xi)\, \overline{\hat{\varphi}(\xi)}\, e^{-ix.\xi}\, d\xi = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix.\xi}\, d\xi = 0,\ k \neq 0.$$

Now, let $f \in L^2(\mathbb{R})$ such that $\hat{f}\,(\xi) = 0$ for $|\xi| > \pi$, then we have the Fourier series $\hat{f}\,(\xi) = \sum_k c_k\, e^{i\xi k}$, $|\xi| \leq \pi$, with $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}\,(\xi)\, e^{-ix.\xi}\, d\xi$. As $f\,(x) = \frac{1}{2\pi} \int \hat{f}\,(\xi)\, e^{ix.\xi}\, d\xi$, $x \in \mathbb{R}$, we see that $c_k = f\,(-k)$, ie. $\hat{f}\,(\xi) = \sum_k f\,(-k) e^{i\xi k}$. Therefore $f\,(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}\,(\xi) e^{ix.\xi} d\xi = \sum_k f\,(-k) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\xi} e^{ix\xi}\, d\xi$, thus we obtain the *Shannon sampling theorem*

$$f\,(x) = \sum_k f\,(-k)\, \frac{\sin \pi\,(x+k)}{\pi\,(x+k)},$$

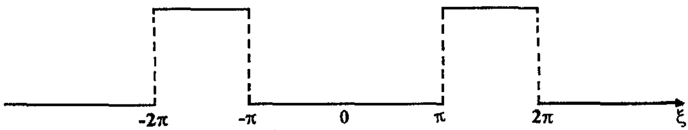
which enable us to recover (the signal) f from its values on the integers. This allow us to construct the linear closed space $V_0 = \{f \in L^2\ /\ \hat{f}\,(\xi) = 0$ for $|\xi| > \pi\} = \{f \in L^2\ /\ f\,(x) = \sum_k f\,(-k)\, \frac{\sin \pi\,(x+k)}{\pi\,(x+k)}\}$. Now, the idea is to consider $V_1 = \{f \in L^2(\mathbb{R})\ /\ \hat{f}\,(\xi) = 0$ for $|\xi| > 2^1 \pi\}$, and in general we obtain the V_j's with "$2^j \pi$", which is again an increasing chain, and the sequence $\{V_j\}_{j \in \mathbb{Z}}$ is a MRA associated with the cited φ.

With this setting, the wavelet ψ is defined by

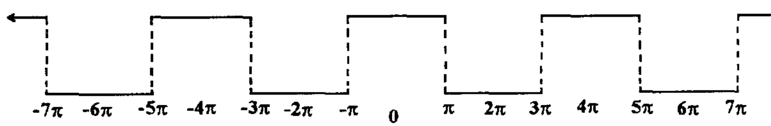
$\hat{\psi}\,(\xi) = e^{-i\frac{\xi}{2}}\, m_0\,(\frac{\xi}{2} + \pi)\, \hat{\varphi}\,(\frac{\xi}{2})$, where m_0 is the 2π-periodic function $m_0\,(\frac{\xi}{2}) = \sum_{k \in \mathbb{Z}} \hat{\varphi}\,(\xi + 4\pi\, k)$.

Also we have $\hat{\varphi}(\xi) = m_0(\frac{\xi}{2})\hat{\varphi}(\frac{\xi}{2})$.

These functions are illustrated in the following figure.



$|\hat{\varphi}(\xi)|$



$m_0(\frac{\xi}{2})$

Observing that supp $\hat{\varphi} \cap$ supp $\hat{\psi} = \phi$, and considering the respective dilation-translation argument, it is obtained an orthonormal sequence $\{\psi_{j,k}\}$ of wavelets.

Before examples say us how to construct the wavelet as soon as we have φ (the scaling function). In general we ask the question: "how to construct the wavelet ψ (in dimension 1) such that $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x-k)$, $j,k\in\mathbb{Z}$, form an orthonormal basis for $L^2(\mathbb{R})$?

The general idea is a follows: for each $j\in\mathbb{Z}$ we define W_j as the orthogonal complement of V_j in V_{j+1}, ie. $V_{j+1} = V_j \oplus W_j$. We then obtain

$$f(x) \in W_j \Leftrightarrow f(x - 2^{-j}k) \in W_j .$$

On the contrary of the V_j 's, the W_j 's are orthogonals and we have the important result:

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j .$$

The closed subspace W_0 is the 'house" where "live" the wavelet ψ to be construct. The following central result answer the above question.

"Let $\{V_j\}_{j\in\mathbb{Z}}$ be a r-regular MRA of $L^2(\mathbb{R})$; then there exists a function $\psi\in W_c$, satisfying $|D^\alpha\,\psi(x)| \leq \frac{C_{M,\alpha}}{(1+|x|)^M}$ $\forall\alpha$, $|\alpha| \leq r$, $\forall\ M \geq 0$ entire. Moreover, $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of W_0 ".

Thus, by dilation property, the family $\{\psi_{j,k}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, as we want. An important corollary says that ψ satisfies $\int x^k\,\psi(x)dx=0$ for $k = 0,1,\ldots,r$. Just, as $\int\psi=0$, the ψ is called a wavelet (a "small wave").

4. Compactly Supported Wavelets.

Of special interest for practical problems, for example, involving time, are the wavelets of compact support, whose existence and properties was established by I. Daubechies [D2] in 1988. Before, Ph. Tchamitchian [Tch 1] constructed two functions φ and ψ, with compact support and prescribed regularity, such that for every $f\in L^2(\mathbb{R})$ we have $f = \sum c_{j,k}\,\varphi_{j,k}$ where $c_{j,k} = \int f(x)\,\overline{\psi_{j,k}(x)}\,dx$. The Daubechies's result is : "there exists a constant $C \geq 0$ such that, for each integer $r \geq 0$, there is MRA of $L^2(\mathbb{R})$, of regularity $r>0$, such that the functions φ and ψ, before seen, are compactly supported, with supports in $[-Cr, Cr]$ ".

We note that $r = 0$ is the Haar case. Also we observe that is imposible to have $\psi\in C_0^\infty \equiv D$ since it would follow, from the fact that the $\{\psi_{j,k}\}$ is an orthonormal basis, that all moments of ψ are zero, which is not possible if ψ is supported in $[-M,M]$ (using the density of the polynomials).

D.The Cotlar and Schur's Lemmas.

The Cotlar's lemma is a useful tool for proving some delicate questions in operator theory. In this way we have the *Zygmund problem* (see [Me 3]): 'prove that the Hilbert transform H: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where (equivalently) $H f(x) = (\frac{1}{\pi}\,p.v.\frac{1}{x})*f$, is a bounded operator without using the Plancherel fórmula".

We note that $\hat{H}f(\xi) = (-i \operatorname{sign} \xi)\ \hat{f}(\xi) \equiv \sigma_H(\xi)\hat{f}(\xi)$, where σ_H is called the symbol of H. In other words, we can write $H = F^{-1}\sigma_H F$, being F the Fourier transform and F^{-1} the anti-Fourier transform. (We look that σ_H is as a "multiplier"). We can observe that $|\sigma_H| = 1$, $\sigma_H^2 = -1$ and $Hf(x)=[\sigma_H\ \hat{f}\,]^{\vee}(x) = (\breve{\sigma}_H * f)(x)$ and therefore $\breve{\sigma}_H(x) = \text{p.v.}\ \frac{1}{x}$ or $\sigma_H(\xi) = (\text{p.v.}\ \frac{1}{x})^{\wedge}(\xi)$.
Also, one can prove that $H^* = -H$ (H^* adjoint of H), and that $-H = H^{-1}$. Therefore H is a unitary operator in $L^2(\mathbb{R})$. In fact, using Plancherel formula we have $\|Hf\|_{L^2} = \|f\|_{L^2}$, ie. H is a bounded operator on $L^2(\mathbb{R})$.

M. Cotlar gave a first answer to Zygmund's problem. Let $\mathcal{H}$ be a hilbert space and $T_j : \mathcal{H} \to \mathcal{H}$ a family of bounded linear operators, where $j \in J$ (an index set). We consider the numbers

$$w(j,k) = \|T_j^* T_k\| \qquad \text{and} \qquad \tilde{w}(j,k) = \|T_j T_k^*\|, \quad \text{with } j,k \in J.$$

Then we have *Cotlar's lemma:* "We suppose that there exist two constans C_0 and C_1 such that

$$\sup_{j\in J} \sum_{k\in J} (w(j,k))^{\frac{1}{2}} \le C_0 \text{ and } \sup_{j\in J} \sum_{k\in J} (\tilde{w}(j,k))^{\frac{1}{2}} \ge C_1 \qquad (8)$$

then we have

$$\| \sum_{j\in J} T_j \| \le (C_0\, C_1)^{\frac{1}{2}},$$

and for each $x \in \mathcal{H}$, the series $\sum_{j\in J} T_j(x)$ is convergent".

Looking the Zygmund's problem, one consider

$$\varphi(x) = \begin{cases} \frac{1}{\pi x} & \text{if } 1 \le |x| \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

and one consider the dilation $\varphi_j(x) = 2^j \varphi(2^j x)$ and also $H_j f = \varphi_j * f$. Thus $H = \sum_{j\in \mathbb{Z}} H_j$. After some considerations it follows that $H_k^* H_j = \varphi_j * \varphi_k = H_K H_j^*$, and some technical arguments give us that $\|\varphi_j * \varphi_k\|_{L^1} \le \frac{C}{2^{|j-k|}}$.

Then by Cotlar's lemma, $\|H\| \le C$, as we want.

Cotlar's lemma also was applied to pseudo differential operators by Calderón-Vaillancourt in their cited paper. We remark that pseudo-differential operators are "almost diagonal" in the "Fourier basis". Let us consider the particular case $\rho = \delta$, with $0 \leq \delta < 1$, and $m = 0$; thus the symbols satisfying

$$|\partial_x^{\beta} \partial_{\xi}^{\alpha} \sigma(x,\xi)| \leq C_{\alpha,\beta} (1+|\xi|)^{\delta(|\beta|-|\alpha|)} .$$

Calderón-Vaillancourt cut the $\sigma(x,\xi)$ into elementary pieces, thus $\sigma(x,\xi)\varphi_j(x,\xi), j \in J$, where (the cut-off functions) $\varphi_j(x,\xi) \in C_0^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ and some other properties. Then they made a Cotlar argument.

If $\delta = 1$ people has problems since the corresponding operator T is not bounded on $L^2(\mathbb{R}^n)$ in general, but it still has the "integral operator representation" $\int k(x,y)f(y)dy$ with k an appropriate kernel. More precisely, T is an integral operator if $T: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator and its kernel k is a signed measure. The idea is that the integral operator remain bounded after suppresing all the cancellations that there exist in k.

In that situation we use the *Schur's lemma:* "let $k(x,y)$ be a non-negative measurable function defined on $\mathbb{R}^n \times \mathbb{R}^n$ and we assume the existence of two positive measurable functions $w(x)$ and $\tilde{w}(x)$, finites everywhere, satisfying

$$\int k(x,y)\, w(y)dy \leq \tilde{w}(x) \text{ a.e. and } \int k(x,y)\, \tilde{w}(x)dx \leq w(y) \text{ a.e.}$$

Then the integral operator $Tf(x) = \int k(x,y)\, f(y)dy$ is bounded on $L^2(\mathbb{R}^n)$ and $\|T\| \leq 1$ ".

The *general plan* is to span the kernel $k(x,y)$ as a series $\sum_{j \in J} k_j(x,y)$, which is not easy to obtain, such that the operators $T_j f(x) = \int k_j(x,y)\, f(y)dy$ are integral operators $\forall j \in J$, and $T_k^* T_j f(x) = \int A_{j,k}(x,y)\, f(y)dy$, and the corresponding non-negative operator defined by the kernel $(A_{j,k}(x,y))$ has a norm-bounded by $w(j,k)$. We do the same for $T_k T_j^*$ (using $\tilde{w}(j,k)$). Moreover, $w(j,k)$ and $\tilde{w}(j,k)$ satisfy (8). Now one applies Schur's lemma. S. Semmes had provided also a solution of the Zygmund's problem, using the Schur's lemma and the Haar basis.

E. Calderón-Zygmund Operators and Matrices.

There exist an interesting relation between bases of wavelets and operator theory. Now we shall try to describe such relations. For more commentaries see [Ja-La]. Let $\{\psi_{j,k}\}$ be a basis of wavelets for $L^2(\mathbb{R}^n)$ and consider operators P_ε such that $P_\varepsilon(\psi_{j,k}) = \varepsilon_{j,k}\,\psi_{j,k}$, where $\varepsilon = (\varepsilon_{j,k})$ with $j,k \in \mathbb{Z}$. More generally, we consider the following class of operators

$$\mathcal{Ç} = \{T \,/\, T(\psi_{j,k}) = \tau_{j,k},\ \ j,k \in \mathbb{Z},\ \ \text{satisfies the } \textit{standard estimates:}$$

$$\text{there exist the constants } C > 0 \text{ and } \alpha > 0 \text{ such that}$$
$$|\tau_{j,k}(x)| \le C\, 2^{j/2}\, w_\alpha(2^j x - k) \text{ and}$$
$$|\tau_{j,k}(x) - \tau_{j,k}(x')| \le C\, 2^{j/2}\, 2^{j\alpha}\, |x-x'|^\alpha\, [w_\alpha(2^j x - k) + w_\alpha(2^j x' - k)] \qquad (9)$$
$$\text{where } w_\alpha(x) = \frac{1}{(1+|x|)^{1+\alpha}}\ \}.$$

Also, by construction, we consider the kernel $k(x,y)$ of T as

$$k(x,y) = \sum_{j,k} \tau_{j,k}(x)\, \overline{\psi_{j,k}(y)},$$

which satisfies the *standard estimates* (or *classical C-Z estimates):* k is a continuous function on $\mathbb{R} \times \mathbb{R} - \{(x,y) \,/\, x = y\}$ and $\exists\ C,\delta$ positive constants such that $|k(x,y)| \le \frac{C}{|x-y|}$ if $x \ne y$ and

$$|k(x,y) - k(x',y)| + |k(y,x) - k(y,x')| \le C \frac{|x-x'|^\delta}{|x-y|^{1+\delta}} \quad \text{if } 2|x-x'| \le |x-y|.$$

If, moreover, *T is continuous on L^2* and Tf is in the sense

$$<Tf,g> = \iint k(x,y)\, f(y)\, g(x)\, dx\, dy \qquad f,g \in \mathcal{D}$$

with disjoint supports, then *T is called a Calderón-Zygmund operator* (*CZO*). It is useful to know that if $\int \tau_{j,k} = 0\ \ \forall j,k \in \mathbb{Z}$, then T is continuous on L^2. In the prove of this result there is a recipe which means to use some related matrices. Thus, as $(\psi_{j,k})$ is a basis for L^2 we have that T is continuous on L^2 $\Leftrightarrow \forall (c_{j,k}) \in l^2$ we have $\|\sum c_{j,k}\,\tau_{j,k}\|_{L^2} \le C\, \|c_{j,k}\|_{l^2}$, which lead us to consider the matrix $(<\tau_{j,k}, \tau_{j',k'}>)_{j,k,j',k'}$.

By convenience one consider a change of notation, thus

$$A = \Big\{\text{numbers } \lambda = \frac{k+\frac{1}{2}}{2^j}\ , \text{j,k}\in \mathbb{Z}, \text{ which are in a 1-1 correspondence with the points } \left(\frac{k}{2^j},\frac{1}{2^j}\right)\Big\}.$$

Therefore, instead $\psi_{j,k}$, $\tau_{j,k}$ we write ψ_λ, τ_λ ...

Now we consider the main class of matrices:

" $\mathcal{M}$ is the space of matrices $M = (m_{\lambda,\lambda'})_{\lambda,\lambda'\in A}$ such that $\exists$ two constants C, $r > 0$ with

$$|m_{\lambda,\lambda'}| \leq C \frac{1}{2^{|j-j'|(\frac{1}{2}+r)}} \left(\frac{\frac{1}{2^j}+\frac{1}{2^{j'}}}{\frac{1}{2^j}+\frac{1}{2^{j'}}+|\lambda-\lambda'|}\right)^{1+r} \text{"}$$

For example, if (τ_λ) is a set of functions satisfying the standard estimates (9), and such that $\int \tau_\lambda = 0\ \forall\lambda$, then the matrix $(<\tau_\lambda, \tau_{\lambda'}>)$ belongs to $\mathcal{M}$. The class $\mathcal{M}$ has interesting properties, like $\mathcal{M}$ is an algebra (ie if $T_1 \in \mathcal{M}$ and $T_2 \in \mathcal{M}$ then $T_1\, T_2 \in \mathcal{M}$); also every matrix in $\mathcal{M}$ defines a continuous operator on $l^1_w(A)$ and on $l^\infty_{\frac{1}{w}}(A)$ where $l^1_w(A) = \big\{(c_\lambda) \,/\, \sum_\lambda |c_\lambda|\, 2^{-j/2} < \infty\}$ and $l^\infty_{\frac{1}{w}}(A) = \{\ (c_\lambda) \,/\, \sup\limits_\lambda |\, c_\lambda|\, 2^{j/2} < \infty\big\}$.

To examine the relations between wavelets, MRA and CZO we shall take MRA leading to ψ, a wavelet compactly supported in $[-M, M]$, and a scaling function φ of regularity $r \geq 1$ with $(r+1)$ moments vanishing. There are two ways to analyse operators

$$T \text{ with wavelets: } \begin{cases} \text{the standard decomposition} & \text{and} \\ \text{the non - standard decomposition.} \end{cases}$$

Before do that, we consider a few of general ideas. We have seen that $\lim\limits_{j\to+\infty} \|P_j f - f\|_{L^2} = 0$, where P_j is the projection $P_j : L^2 \to V_j$. With operators a similar method works. Thus, let T be a continuous linear operator on $L^2(\mathbb{R})$ and one consider the sequence of operators $T_j = P_j\, T\, P_j$, $\forall j \in \mathbb{Z}$, which is a continuous linear operator over V_j. Then we have that $\forall f \in L^2(\mathbb{R})$, $\lim\limits_{j\to+\infty} \| P_j\, T\, P_j\, f - T f \|_{L^2} = 0$, thus informations on T will be obtained from the propierties of T_j. Therefore, the idea is to develop algoritms to study $(T_j)_{j\in\mathbb{Z}}$.

To motive the standard and non-standard (introduced by Beylkin-Coifman-Rokhlin in 1991) decompositions, one consider some general ideas. Let $\mathcal{H}$ be a Hilbert space (for example $L^2(\mathbb{R})$), $(\varepsilon_\lambda)_\lambda$ be an orthonormal basis for $\mathcal{H}$ (for example $(\psi_{j,k})$), and let $T: \mathcal{H} \to \mathcal{H}$ a linear operator (for example, T a CZO). Then:

(i) We have $Tx = \sum\limits_{\lambda} <x,\varepsilon_\lambda> T\varepsilon_\lambda = \sum\limits_{\lambda} \sum\limits_{\lambda} \cdot <T\varepsilon_\lambda, \varepsilon_{\lambda'}> \varepsilon_{\lambda'}$; thus appear an "infinite matrix" whose coefficients are $(T\,\varepsilon_\lambda, \varepsilon_{\lambda'})$, which is useful to depict the operator T (look the form of the coefficients of the matrix M in $\mathcal{M}$). Therefore, as we choose basis in $\mathcal{H}$, the study of those coefficients can reveal certain properties of the operator T.

(ii) We suppose $\mathcal{H} = E_1 \oplus H_1$, where E_1 and H_1 are two non-trivial subspaces of $\mathcal{H}$ (think in $V_{j+1} = V_j \oplus W_j$). Now let the projections $P_{E_1}: \mathcal{H} \to E_1$ and $P_{H_1}: \mathcal{H} \to H_1$ (think in $P_j: L^2 \to V_j$ and $Q_j: L^2 \to W_j$). Then T can be written in the form $T = P_{E_1}\, TP_{E_1} + P_{E_1}\, TP_{H_1} + P_{H_1}\, TP_{E_1} + P_{H_1}\, TP_{H_1}$ thus T is depicted by an "infinite matrix"

$$\begin{pmatrix} P_{E_1}\, TP_{E_1} & P_{E_1}\, TP_{H_1} \\ P_{E_1}\, TP_{H_1} & P_{H_1}\, TP_{H_1} \end{pmatrix} \quad \text{a matrix divided in four blocks.}$$

Now we repeat the argument with H_1, ie. H_1 is written as the orthogonal sum of two non-trivial subspaces and we obtain a decomposition for $P_{H_1}\, TP_{H_1}$, and we iterate the algorithm. Thus we obtain an infinite matrix; the study of its coefficients may give information of the properties of T.

(i) leads us to the standard decomposition, and (ii) to the non-standard decomposition.

The Standard Decomposition. We resume the arguments in the following proposition: "let T be an operator whose matrix is $(m_{\lambda,\lambda'}) = (<T\psi_\lambda, \psi_{\lambda'}>)$, well defined by 0hypothesis. Then we have

[i] $M = (m_{\lambda,\lambda'}) \in \mathcal{M} \Leftrightarrow$ [ii] $T\psi_\lambda = \tau_\lambda$ satisfies the standard estimates (9) and $\int \tau_\lambda = 0, \forall \lambda \Leftrightarrow$

[ii] T is a CZO with $T(1) = 0 = T^*(1)$, where by definition $T(1) = \sum_\lambda \int \sigma_\lambda \psi_\lambda$ and $T^*(1) = \sum_\lambda \int \rho_\lambda \psi_\lambda$ being $\sigma_\lambda(x) \equiv \sigma_{j,k}(x) = \sum_\lambda \beta^j_{l,k} \varphi_{j,l}(x)$ and $\rho_\lambda(x) \equiv \rho_{j,k}(x) = \sum_\lambda \gamma^j_{l,k} \varphi_{j,l}(x)$ ".

The Non-Standard Decomposition. As we know, in a MRA for $L^2(\mathbb{R})$ we have $V_{j+1} = V_j \oplus W_j$ and the respective projections P_j and Q_j on V_j and W_j, where we observe that $P_{j+1} = P_j + Q_j$. As before, we have the telescopic series

$$T = \sum_{j=-\infty}^{\infty} (P_{j+1}\, Tp_{j+1} - P_j\, TP_j) = \sum_{j=-\infty}^{\infty} (Q_j\, TQ_j + Q_j\, TP_j + P_j\, TQ_j)$$

which involves three kinds of scalar products

$$\alpha^j_{k,l} = \langle T\,\psi_{j,k}, \psi_{j,l}\rangle\ ,\ \beta^j_{k,l} = \langle T\,\varphi_{j,k}, \psi_{j,l}\rangle \text{ and } \gamma^j_{k,l} = \langle T\psi_{j,k}, \varphi_{j,l}\rangle. \tag{10}$$

Now we return to some additional commentaries on CZO. In the n-dimensional case, Calderón considered operators

$Tf(x) = \text{p.v.} \int k(x,y)\, f(y)dy$, where the kernel $k(x,y)$ satisfies

$$\left.\begin{array}{l} |k(x,y)| \le \dfrac{C_0}{|x-y|^n} \quad , \quad k(x,y) = -k(y,x) \quad \text{and} \\ \exists\, \delta \in (0,1) \text{ and a constant } C \text{ such that} \\ |k(x',y) - k(x,y)| \le \dfrac{c|x'-x|^\delta}{|x-y|^{n+\delta}} \end{array}\right\} . \text{(C)}$$

We remark that *the claim is* to show how the wavelet based methods permit a complete description of that Calderón class of operators. Since these C-operators are not, in general, bounded on $L^2(\mathbb{R})$, it is amazing how the wavelet analysis leads to a necessary and sufficient conditions for boundedness. For to obtain the goal, we need to consider the *n-dimensional wavelets* ψ_λ .

There are two approaches for constructing wavelets in general dimensions. *The first* consists in generalizing the idea of MRA in $\mathbb{R}^1$ and follows the way of one-dimensional construction case. *The second* method consists in obtaining the wavelets using the 1-dimensional construction. Since the general dimensional case follows from what we do in 2-dimensional case, we explain the 2-dimension.

Thus, let $\{V_j\})_{j\in\mathbb{Z}}$ be a MRA for $L^2(\mathbb{R})$. We construct the subspace $\mathcal{V}_j$ of $L^2(\mathbb{R}^2)$ by $\mathcal{V}_j = V_j \otimes V_j$ (tensorial product), ie

$$\mathcal{V}_j = \overline{\{\text{finite linear combinations of the functions } (2^{\frac{j}{2}}\varphi(2^j x-k)2^{\frac{j}{2}}\varphi(2^j y-l)_{(k,l)\in\mathbb{Z}^2}\}}$$

($\overline{}$ means closure). Then $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ is a MRA for $L^2(\mathbb{R}^2)$.

An orthonormal basis for $\mathcal{V}_j$ is given by $\{\phi_{j,(k_1,k_2)}(x,y)\}_{(k_1,k_2)\in\mathbb{Z}^2}$ where $\phi_{j,(k_1,k_2)}(x,y) = \varphi_{j,k_1}(x)\,\varphi_{j,k_2}(y)$.

(Remember that if $(e_k(x))$ is an orthonormal basis for a Hilbert space $\mathcal{H}$, then $(e_{k_1}(x)\, e_{k_2}(y))$ is an orthonormal basis for $\mathcal{H}\otimes\mathcal{H}$).

Now we construct the wavelets. For this, let $\mathcal{W}_j$ be the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$, ie. $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$. Therefore,

$$\begin{aligned}\mathcal{V}_{j+1} &= V_{j+1}\otimes V_{j+1} = (V_j\oplus W_j)\otimes(V_j\oplus W_j)\\ &= V_j\otimes V_j \oplus (V_j\otimes W_j \oplus W_j\otimes V_j \oplus W_j\otimes W_j)\\ &= \mathcal{V}_j \oplus (\ldots).\end{aligned}$$

Therefore

$$\mathcal{W}_j = V_j\otimes W_j \oplus W_j\otimes V_j \oplus W_j\otimes W_j$$

and *a basis for* $\mathcal{W}_j$ is given by the functions

$\psi^{(i)}_{j,k} = 2^j\,\psi^{(i)}(2^j x\text{-}k)$ where $\psi^{(1)}(x,y) \equiv \psi^{(1,0)}(x,y) = \psi(x)\,\varphi(y)$,

$\psi^{(2)}(x,y) \equiv \psi^{(0,1)}(x,y) = \varphi(x)\,\psi(y)$, and

$\psi^{(3)}(x,y) \equiv \psi^{(1,1)}(x,y) = \psi(x)\,\psi(y)$, where $i\in\{0,1\}^2$ - $(0,0)$.

Thus in the 2-dimensional case we have a basis with three wavelets. In the general case the basis has $2^n - 1$ wavelets. More precisely, if $x = (x_1,\ldots,x_n)$ then $\phi(x) = \varphi(x_1)\,\varphi(x_2)\ldots\varphi(x_n)$ and $\psi_\varepsilon(x) = \psi_{\varepsilon_1}(x_1)\,\psi_{\varepsilon_2}(x_2)\ldots\psi_{\varepsilon_n}(x_n)$ with $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$ in $\{0,1\}^n$, $\varepsilon \neq (0,\ldots,0)$ and $\psi_0(t) = \varphi(t)$, $\psi_1(t) = \psi(t)$.

Now $\lambda \in \Lambda = \mathbb{Z}\times\mathbb{Z}^n\times E$, with $E = \{0,1\}^n - \{(0,\ldots,0)\}$, and $\lambda = (j,k,\varepsilon)$.

The following theorem resumes the non-standard representation for CZO (we return to $n = 1$ case to do notations as simple as possible) where one uses the entries (10). Note the relations between the operator theory and wavelets.

Theorem. " *The kernel k of the operator T,* $Tf(x) = \text{p.v.}\int k(x,y)\,f(y)dy$, *satisfies the conditions* (C)

$$\Leftrightarrow |\alpha^j_{k,l}| \leq \frac{C}{(1+|k-l|)^{1+\delta}}\,,\quad |\gamma^j_{k,l}| \leq \frac{C}{(1+|k-l|)^{1+\delta}}\,,\ \textit{and}$$

$$\alpha^j_{k,l} = -\alpha^j_{l,k}\,,\quad \beta^j_{k,l} = -\gamma^j_{k,l}\,.$$

Moreover, the kernel $k(x,y)$ is given by

$$k(x,y) = \sum_j\sum_k\sum_l \gamma^j_{k,l}\,\varphi_{j,l}(x)\,\psi_{j,k}(y) + \sum_j\sum_k\sum_l \beta^j_{k,l}\,\psi_{j,l}(x)\,\varphi_{j,k}(y)$$
$$+ \sum_j\sum_k\sum_l \alpha^j_{k,l}\,\psi_{j,l}(x)\,\psi_{j,k}(y)\text{ ''.}$$

Under certain conditions we take an operator T in the class considered in the last theorem, defined by $T(\psi_{j,k}) = \lambda_{j,k}\,\varphi_{j,k}$ where the sequence $\lambda_{j,k}$ (of real numbers) characterizes T. It is striking that there exist necessary and sufficient conditions for the L^2 - continuity of T.
For this we need to establish the Carlesson condition. Let $\mathcal{I}$ be the family of all dyadic intervals, and let

$$\lambda(I) = 2^{-j/2}\,\lambda_{j,k}\quad\text{where}\quad I = [\tfrac{k}{2^j}, \tfrac{k+1}{2^j}) \in \mathcal{I}.$$

Then *we say* that $\lambda(I)$, $I\in\mathcal{I}$, satisfies the *Carleson condition* if $\exists\ C$ positive constant such that

$$\sum_{I\subset J} |\lambda(I)|^2 \leq C\,|J| \ldots J\in\mathcal{I}.$$

Thus we arrive to the important result, where it is the spirit of the Zygmund's problem, the motivation of this writing.

Theorem. *The operator* $T(\psi_{j,k}) = \lambda_{j,k}\, \varphi_{j,k}$ *is bounded on* $L^2(\mathbb{R}) \Leftrightarrow \lambda(\mathbb{R}) = 2^{-j/2}\lambda_{j,k}$ *satisfies the Carleson condition.*

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