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STATIONARY SUNSPOT EQUILIBRIUM: A SURVEY

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Abstract

In this paper the main results about existence of Stationary Sunspot Equilibrium are given. Sketch of the proves and examples are commented. The types of economies included in the framework of this work are the intertemporal one-step forward looking economies.

1. Introduction

Rational expectations has been one of the principal constructions made in the recent economic theory for sequential markets. In the weakest version it says that the individuals are abels to know the true distribution of the future states if they know the past states and with this information they maximize their utilities and/or profits vanishing the excess demand.

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Sunspot equilibrium could be seen as particular case of that concept. It explains how the "extrinsics" of the economy can influence the agent's expectations for having equilibrium. The difference between this type of equilibrium and the Arrow-Debreu equilibrium is that in the last one the plans and transactions are decided at the outset taking the prices in the spot markets and future or contingent markets as given. However in actual markets transactions could take place at sucessive calendar dates when the spot and finantial markets open allowing the agents to redistribute wealth.

At this point it is clear that the expectations have a role in modeling this sequential structure. Radner (1972) proposed the perfect foresight hypothesis for making a connection between this type of economy and the Arrow-Debreu economy. In this structure is described the sunspot phenomenon, as an extrinsic uncertainty (Cass and Shell (1983)) which generates random future states. Let us compare both economies. In an Arrow-Debreu world with l goods and m agents we have that the competitive equilibrium are vectors $p \in \mathbb{R}^l_+$ and $\bar{x}_i \in \mathbb{R}^l_+$ $i = 1, \ldots, m$ such that:

a) $\bar{x}_i = \text{Argmax } u_i(x)$ s.t. $px \leq pw_i$ $i = 1, \ldots, m$.

b) $\sum_{i=1}^{m} (\bar{x}_i - w_i) \leq 0$.

Now, suppose that an "extrinsic noise" (sunspot) must be taking in account before accomplishing the transactions. Let $S = \{s_1, \ldots, s_r)$ be the set of sunspot and $\pi_1, \ldots, \pi_r$ the sunspot probabilities. Then a contingent market equilibrium is a vector of random variables $(p, \bar{x}_i, \ldots, \bar{x}_m)$ defined on S (i.e. p: $S \to \mathbb{R}^l_+$, $\bar{x}_i : S \to \mathbb{R}^l_+$ measurable functions) such that:

a) $\bar{x}_i = \text{Argmax} \sum_{k=1}^{r} \pi_k u_i(x(s_k))$ s.t. $\sum_{k=1}^{r} p(s_k)(x(s_k) - w_i) \leq 0$

b) $\sum_{i=1}^{m} (\bar{x}_i(s) - w_i) \leq 0 \quad \forall s \in S$.

Note that the maximization's constrain means that we have a market structure rich enough (complete markets). It is easy to prove that in these circumstances the definitions above coincide if the agents are strictly risk averse.

Theorem. *(Cass and Shell (1983) If u_i is strictly concave function for all i then the functions $p(.)$ and $\bar{x}_i(.)$ are constants (the sunspot do not matter).*

Proof. Let us prove this theorem by contradiction. Suppose that sunspot matter, i.e. there exists an individual j such that Prob$[\bar{x}_j = \mathrm{E}(\bar{x}_j)] < 1$. Consider the allocation $(\mathrm{E}(\bar{x}_i))_{i=1,\dots,m}$. It is easy to see that this allocation is feasible. By the stricty concavity of u_j and the Jensen's inequality we have $u_j(\mathrm{E}(\bar{x}_j)) > \mathrm{E}(u_j(\bar{x}_j))$. Therefore this allocation Pareto-superior to the first one. This contradicts the Pareto optimality of $(\bar{x}_i)_{i=1,\dots m}$.

The proof of this theorem uses strongly the completeness of the markets since for contingent markets the First Welfare Theorem holds. It means that if the sunspot events are not insurables we could have a non-trivial sunspot equilibrium ($p(.)$ and $\bar{x}_i(.)$ are not constants). For example, if sunspot events are "states of mind" or "animal spirits" as argued by Azariadis (1981), they are not verifiables and therefore not insurable.

Finally it is important to remark the close connection between the existence of sunspot equilibrium and the indeterminacy phenomenon. Recent literature (Boldrin and Rustichini (1994), Benhabib and Farmer (1994), Benhabib and Perli (1994)) shows a number of models which exhibit indeterminate equilibrium and therefore the existence of sunspot equilibrium (Woodford (1986a-b)).

2. The framework and some definitions

Let $X \subset \mathbb{R}^n_+$ be the state space (e.g. prices) and $\mathcal{P}(X)$ the probability measures space defined on the borelians of X. We will consider the one-step forward looking case, hence the equilibrium is given by the zeros of the function $\tilde{Z}: X \times \mathcal{P}(X) \to \mathbb{R}^n$. Sometimes this function wil be the stochastic excess demand function.

Definition 2.1 A *temporary equilibrium* is a $(p_0, \mu) \in X \times \mathcal{P}(X)$ such that $\tilde{Z}(p_0, \mu) = 0$.

Here p_0 must be seen as the current price and μ the expectations for the future price. Sometimes we will say that μ rationalizes p_0.

Definition 2.2 *The deterministic excess demand function* $Z: X \times X \to \mathbb{R}^n$ *is defined by* $Z(p_0, p_1) = \tilde{Z}(p_0, \delta_{p_1})$. *It is the excess demand when the current price is* p_0 *and the future price is* p_1 *with probability one.*

Let us define the correspondences $\rho: X \to \mathcal{P}(X)$ and $r: X \to X$ as:

$$\rho(p) = \{\mu \in \mathcal{P}(X);\ \tilde{Z}(p, \mu) = 0\}$$

$$r(p) = \{p' \in X;\ Z(p,p') = 0\}$$

ρ has the *convex valuedness of rationalizing measures* (CVR) property if ρ is convexvalued, i.e. if μ_1 and μ_2 rationalize p_0 then $\alpha\mu_1 + (1 - \alpha)\mu_2$ rationalizes p_0 too.

A *deterministic equilibrium* is a pair $(p_0,p_1) \in X \times X$ such that

$$Z(p_0,p_1) = 0.$$

A *perfect foresight equilibrium* is a sequence $(p_t)_{t\geq 0}$ such that

$$Z(p_t, p_{t+1}) = 0 \text{ for all } t.$$

A *steady state* is a $\overline{p} \in X$ such that $Z(\overline{p}, \overline{p}) = 0$.

A *periodic equilibrium of order k* is a $(p_1, ..., p_k) \in X^k$ such that:

$$Z(p_1,p_2) = Z(p_2,p_3) = ... = Z(p_k,p_1) = 0.$$

A tipical example for this kind of framework is the overlapping generation model. The agents live two periods and there exists a representative agent with separable utility function $V_1(c_1) + V_2(c_2)$, where c_t is the consumption of the unique good in period $t = 1, 2$. Suppose that one unit of good is produced by one unit of the unique productive factor (labor). The agents endowments at each age $t = 1, 2$ are $l_1 > 0$ and $l_2 > 0$. Let us suppose that V_t is twice continuously differentiable, strictly increasing and strictly concave functions. The agent can transfer wealth from period $t = 1$ to $t = 2$ by means of a riskness asset with return z. Let p_1 be the current price of the good and p_2 the (random) price of the good in period $t = 2$. Then the agent must choice the consumption plan c_1 (deterministic), c_2 (stochastic) and the investment m for maximizing:

$$V_1(c_1) + E[V_2(c_2)]$$

with the budgetary constraint:

$$p_1 c_1 + m = p_1 l_1$$

$$p_2 c_2 = p_2 l_2 + mz$$

where the expected value is taking with respect to the probability measure defined by p_2.

The first order condition for this problem is:

$$\frac{1}{p_t} V_1'(l_1 - \frac{m}{p_t}) = E[\frac{z}{p_{t+1}} V_2'(l_2 + \frac{mz}{p_{t+1}}]$$

(we are supposing $m > 0$). If we put:

$$x_t = \frac{m}{p_t}; \quad v_1(x) = x\, V_1'(l_1 - x); \quad v_2(x) = x\, V_2'(l_2 + x)$$

the equilibrium equation results:

$$v_1(x_t) = E[v_2(zx_{t+1})].$$

In this case the function $\tilde{Z}$ is:

$$\tilde{Z}(x,\mu) = v_1(x) - E_\mu[v_2(z\,x')]$$

and the deterministic dynamic is given by:

$$Z(x_t, x_{t+1}) = v_1(x_t) - v_2(z\,x_{t+1}) = 0.$$

Definition 2.3 *A steady state* $\overline{p}$ is *indeterminate* if $\forall\varepsilon > 0$ *there exists (uncountably) infinite set of perfect foresight equilibria* $(p_t)_{t\geq 0}$ *such that* $|p_t - \overline{p}| < \varepsilon$.

A simple form to characterize an indeterminate steady state is noting that in this case the stable manifold associated to $\overline{p}$ for the dynamical system defined by $Z(p_t, p_{t+1}) = 0$ must be non-trivial (at least one-dimensional).

Fact: $\overline{p}$ is an indeterminate steady state if and only if $B = (\partial_0 Z(\overline{p}, \overline{p}))^{-1} \partial_1 Z(\overline{p}, \overline{p})$ has at least one eigenvalue outside the unit disk.

Rational Expectations and Sunspot Equilibrium

Let us suppose that the state variable follows a random process $(\tilde{p}_t)_{t\geq 0}$. We will say that the sequence (p_t, μ_t) is a *rational expectation equilibrium* if:

i) (p_t, μ_t) is a temporary equilibrium for all t.
ii) $\mu_t = \text{Prob}[\tilde{p}_{t+1} \mid \tilde{p}_t = p_t, \ldots, \tilde{p}_0 = p_0]$.

Definition 2.4 A *sunspot equilibrium is a pair* (X_0, Q) *where* $X_0 \subset X$ *and* Q: $X_0 \times \mathcal{P}(X_0)$ $[0,1]$ *is a transition function such that:*

i) $\exists x_0 \in X_0$ *such that* $Q(x_0,.)$ *is truly stochastic.*
ii) $\tilde{Z}(x, Q(x,.)) = 0$ *for all* $x \in X_0$.

The more standard version of sunspot equilibrium can be related with the above definition. Let E be a topological space (the extrinsics). A sunspot equilibrium is a trio (X_0, f, v) where $X_0 \subset X$, f: $E \rightarrow X_0$ (theory) is an homeomorphism and $v : E \times \mathcal{B}(E) \rightarrow [0,1]$ is a transition function (over the extrinsics) such that:

i) $\exists \varepsilon_0 \in E$ such that $v(\varepsilon_0,.)$ is truly stochastic,
ii) $\forall \varepsilon \in E$ $\tilde{Z}(f(\varepsilon), v_{\in}^{f}) = 0$ where $v_{\in}^{f}(A) = v(\varepsilon, f^{-1}(A))$ $\forall \in \mathcal{B}(X_0)$.

This last definition emphasizes the role of the extrinsics (E), the expectations on the evolution of these extrinsics (υ) and their influence over the states of the economy (f). It is clear that every homeomorphic space to X_0 serves as extrinsics for relating the former with the last definition.

Given a sunspot equilibrium (X_0, Q), we will call it *stationary* if there exists an invariant measure for the transition function Q, i.e. if there exists $\mu \in \mathcal{P}(X_0)$ such that:

$$\mu(A) = \int_{X_0} Q(x,A)\, \mu(dx) \quad \forall A \in \mathcal{B}(X_0).$$

3. Some results on the existence of SSE

If X_0 is a finite set and Q is a Markov matrix we will say that (X_0, Q) is a *sunspot equilibrium with finite support* or a finite sunspot equilibrium. For searching such sunspost equilibria some hipotheses must be stated.
Let $X_0 = \{x_1, \ldots, x_k\}$ and $M^i = (m^{i1}, \ldots, m^{ik})$ the conditional probability in X_0 given x_i . $M = [M^1, \ldots, M^k]'$ is a Markov matrix and we denote $Q(x_i,.) = (x_1, \ldots x_k ; M^i)$.

(CD) Hypothesis: (Consistency of derivatives) For all i:

$$\partial_{x_0} \tilde{Z}(x_0, x, \ldots, x; M^i) = \partial_{x_0} Z(x_0, x)$$
$$\partial_{x_j} \tilde{Z}(x_0, x, \ldots, x; M^i) = m^{ij} \partial_x Z(x_0, x).$$

The first condition says that the variation of the stochastic excess demand with respect to the present state when the future state is x almost surely is equal to the variation of the deterministic excess demand with respect to the present state. The second relationship states that the variation of the stochatic excess demand with respect to any of the possibles future states (when it is x almost surely) modifies the variation of the deterministic excess demand with respect to the future state by a factor equal to the probability of such a state. It is easy to see that the model given above satisfies this hypothesis.

The following theorem relate the existence of SSE to the existence of an indeterminate steady state.

Theorem 3.1 (Chiappori, Geoffard and Guesnerie (1992)) *If (CD) holds, the steady state $\overline{p}$ is indeterminate and B has no eigenvalue one then for any neigborhood of $\overline{p}$ there exists stationary sunspot equilibrium with finite support in such a neighborhood.*

The proof of this theorem uses arguments of bifurcation theory. Let us see the sketch of the proof. For $k \in \{2,3,\ldots\}$ consider the map $F: X^k \times \mathcal{M}_k \rightarrow \mathbb{R}^{nk}$ ($\mathcal{M}_k$ is the set of Markov matrices of order k) defined by:

$$F(x^1, \ldots, x^k, M) = (\tilde{Z}(x^i, x^1, \ldots, x^k; M^i))_{i=1,\ldots,k}$$

then $F(\overline{x}, \ldots, \overline{x}, M) = 0 \; \forall M \in \mathcal{M}_k$ and from the (CD) hypothesis:

$$\det(D_{(x^1,\ldots,x^k)} F(\overline{x}, \ldots, \overline{x}, M)) = (\det(\partial_{x_0} Z(\overline{x}, \overline{x})))^k \det(I - M \otimes B)$$

where $\otimes$ is the tensorial product. Therefore F has a bifurcation at $(\overline{x}, \ldots, \overline{x}, M)$ if $M \otimes B$ has a unitary eigenvalue. Since the eigenvalues of $M \otimes B$ are the product of eigenvalues of M and B and B has at least one eigenvalue outside the unit disk, we can find M such that $I\text{-}M \otimes B$ is a singular matrix. Finally we must find a curve M: $(-\varepsilon,\varepsilon) \rightarrow \mathcal{M}_k$ such that $F(x^1,\ldots,x^k,M(s)) = 0$, i.e. $(x^1,\ldots,x^k,M(s))$ is a finite sunspot equilibrium.

Also we have the invariant set argument given by Duffie *et al.* (1994).

Theorem 3.2 *If* ρ *is an upper hemi-continuous correspondence with the (CVR) property, (CD) holds and there exists a compact set K such that each* $p_0 \in K$ *is deterministically rationalized by some* $p \in \overset{\circ}{K}$, *then there exists a stationary sunspot equilibrium with support in K.*

The proof of this theorem consists in finding a measurable selection of the correspondence of rationalizing measures. Blume (1982) gives conditions for existence of such a measurable selection.

The next theorem (Azariadis and Guesnerie (1986)) relates the existence of cycles with sunspot equilibria. Let (p_1,p_2) a cycle or periodic equilibrium of order 2:

$(Z(p_1,p_2) = Z(p_2,p_1) = 0)$, then we have:

$$\tilde{Z}(p_1,p_1,p_2;0,1) = 0 \quad \text{and} \quad \tilde{Z}(p_2,p_1,p_2;1,0) = 0.$$

Let $F(p,p',\alpha,\beta) = (\tilde{Z}(p,p,p';\alpha,(1-\alpha)), \tilde{Z}(p',p,p';(1-\beta),\beta)$, so $F(p_1,p_2,0,0) = 0$, we will say that (p_1,p_2) is a regular periodic equilibrium if:

$$D_{(p,p')}F(p_1,p_2,0,0) = \begin{pmatrix} \partial_0\tilde{Z} + \partial_1\tilde{Z} & \partial_2\tilde{Z} \\ \partial_1\tilde{Z} & \partial_0\tilde{Z} + \partial_2\tilde{Z} \end{pmatrix}$$

is a non-singular matrix. In such a case there exist functions $p(\alpha,\beta)$, $p'(\alpha,\beta)$ defined in a neighborhood of $(\alpha,\beta) = (0,0)$ such that:

$$F(p(\alpha,\beta), p'(\alpha,\beta),\alpha,\beta) = 0$$

for all (α,β) in such a neighborhood, but it means that there exists a SSE with suport $\{p_1,p_2\}$ and Markov matrix:

$$\begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$

Theorem 3.3 *If $(p_1,\ldots,p_k)$ is a regular periodic equilibrium then there exists a finite SSE with support close to $\{p_1,\ldots,p_k\}$.*

If we have two steady states we can also construct a SSE in a similar way. Let $\overline{p}$ and $\overline{\overline{p}}$ be two interior steady states. We will suppose that there exists $\tilde{p} \in X$ and $\pi \in (0,1)$ such that $\tilde{Z}(\tilde{p},\overline{p},\tilde{p},\overline{\overline{p}};\pi,0,1-\pi) = 0$, i.e. there exists a degenerate SSE with transition matrix:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ \pi & 0 & 1-\pi \\ 0 & 0 & 1 \end{pmatrix}.$$

In several models the equation $\tilde{Z}(p,\mu) = 0$ can be written as $\upsilon(p) = E[w(p')]$ where υ is a monotone function. In such cases (if X is a convex set) it is clear the existence of $\tilde{p}$ and π given above. Again we can make the similar construction as in the periodic case (Peck (1988)).

Theorem 3.4 *Let $\overline{p},\overline{\overline{p}}$ be two interior steady states, $\tilde{p}$ and π hold the condition above and $\partial_0 \tilde{Z} + \partial_2 \tilde{Z}$ evaluated at $(\tilde{p},\overline{p},\tilde{p},\overline{\overline{p}};\pi,0,1-\pi)$ is non-singular matrix then there exists a finite SSE with support $\{p_1,p_2,p_3\}$ close to $\{\overline{p},\tilde{p},\overline{\overline{p}}\}$.*

Finally, we will enunciate a theorem which relates the existence of heteroclinic orbit of a particular dynamical system and the existence of SSE (called heteroclinic sunspot equilibrium). Remember that if $F: X \rightarrow X$ is a dynamical system, $\{x_n \in X: n \in \mathbb{Z}\}$ is a heteroclinic orbit if $x_{n+1} = F(x_n)$, $\forall n \in \mathbb{Z}$ and there exist $\overline{x},\overline{\overline{x}} \in X$ such that $x_n \rightarrow \overline{x}$ as $n \rightarrow \infty$ and $x_n \rightarrow \overline{\overline{x}}$ as $n \rightarrow -\infty$. Suppose that the support of the SE we want to find is $X_0 = \{p^s; s \in \mathbb{Z}\}$ with the following conditions:

i) $\text{Prob}[\tilde{p}_{t+1} = p^{s+1} \mid \tilde{p}_t = p^s] = \alpha$
$\text{Prob}[\tilde{p}_{t+1} = p^{s-1} \mid \tilde{p}_t = p^s] = 1 - \alpha,$

ii) $p^s \underset{s\rightarrow+\infty}{\longrightarrow} \overline{p}$ and $p^s \underset{s\rightarrow-\infty}{\longrightarrow} \overline{\overline{p}}$.

This means that we want to find SE with support X_0 such that:

$$\tilde{Z}(p^s, p^{s+1}, p^{s-1}; \alpha, 1-\alpha) = 0 \qquad \forall s \in \mathfrak{Z} \qquad (1)$$

and $\lim_{s\to+\infty} p^s = \overline{p}$ $\lim_{s\to-\infty} p^s = \overline{\overline{p}}$. Let us suppose that equation (1) can be written:

$$p^{s+1} = f(p^{s-1}, p^s, \alpha),$$

if $X^s = (x^{s+1}, x^s)$ we will have the following dynamical system:

$$X^s = F(X^{s-1},) (= (f(x^{s-1}, x^s, \alpha), x^s)),$$

so the problem is to find heteroclinic orbit for this system. For example, Chiappori and Guesnerie (1989) have the following equilibrium equation (1):

$$x^s V'(x^s) = \alpha x^{s+1} + (1 - \alpha)x^{s-1}.$$

In this case, they proved that if $V'(0) > \sqrt{4\alpha(1-\alpha)}$ then there exists heteroclinic SE.

Finally we can note that the main results on the existence of SSE establish the existence of finite SSE and/or let it implicitely determinated. Araujo and Maldonado (1995) show that certain class of dynamical models have SSE with (uncountably) infinite support. Suppose that the equation $Z(p_t, p_{t+1}) = 0$ can be written $p_t = \phi(p_{t+1})$. The following theorem sumarizes their results.

Theorem 3.5 *If* $\phi; K \to K$, $K(\subset X)$ *a compact set, there exists a partition* $(A_i)_{i=1,\dots,N}$ $(N \geq 2)$ *of K such that* $\phi: A_i \to \phi(K)$ *is a bijection and* μ *is a* ϕ-*invariant probability measure on the Borelians of K then there exists SSE with stationary probability measure* μ.

In that paper are showed models where μ is absolutely continuous with respect to the Lebesgue measure, therefore The SSE has an (uncountably) infinite support.

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