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EXAMPLES OF LINEAR CONTROL SYSTEMS ON LIE GROUPS

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1. Introduction

In this work we give some examples for a particular class of control systems: Linear control systems on Lie groups. We start with fundamental results about controllability and observability of linear control systems on the commutative Lie group $\mathbb{R}^n$, [3].

Markus, L. introduces in [4] *the notion of linear control systems on a matrix Lie groups and extendes partially the Kalman's Theorem for controllabillity.*

We define a linear control system Σ *on a connected Lie group G. In this case, the dynamic is given by*

$$\dot{g} = X(g) + \sum_{j=1}^{k} u_j Y^j(g)$$

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where the drift vector field X an infinitesimal automorphism of G, i.e. the one parameter group of X is a subgroup of $Aut(G)$, the Lie group of automorphisms of G. And the control vector fields Y^j, $j = 1,2,...k$ are elements of the Lie algebra $L(G)$ of G, i.e. are right-invariant vector fields on G, $\mathbf{h} \in Hom(G,G_1)$. For this class of systems we establish without proof the following results:

1) The Lie algebra rank condition characterizes transitivity, [2].
2) The rank condition; it is sufficient for controllability, [2].
   In particular, we extend the Kalman's Theorem:
3) The rank condition characterizes controllability when G is an Abelian connected Lie group, [2].
4) The observability rank condition characterizes the local observability, [1]
5) Σ is observable if and only if:
   a) Σ is locall observable,
   b) The intersection of $Ker(h)$ and

$$Fix\,(T) = \{g \in G \mid X_t\,(g) = g, \quad t \geq 0\}$$

is trivial, [1].

We compute some examples and use these results to study controllability and observability.

2. Controllability of linear control systems on Lie groups

2.1 The Case $G = \mathbb{R}^n$

From the theoretical point of view the class of linear control systems is one of the most important class of systems and also for their applications, [5], [6]. It is defined by:

$$\Sigma = (\mathbb{R}^n, \mathcal{D})$$

where $\mathcal{D}$ is the dynamic which is determined by:

$$\dot{x} = Ax + Bu, \qquad (1)$$

here $x \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$, $B \in M_{nxk}(\mathbb{R})$, $u \in U$,

$$U = \{u\colon [0,\infty) \rightarrow \mathbb{R}^k \mid u \textit{ is a piecewise constant function}\}$$

is the class of unrestricted admissible controls. In this context,

$$\mathcal{D} = \{Ax + Bu \mid u \in \mathbb{R}^k\}.$$

For every initial condition $x_0 \in \mathbb{R}^n$ and $u \in U$, the solution of (1) is given by

$$\gamma_t^u(x_0) = e^{tA}\{x_0 + \int_0^t e^{-\tau A} Bu(\tau)d\tau\}.$$

In particular,

$$\gamma_t^u(0) = e^{tA}\int_0^t e^{-\tau A} Bu(\tau)d\tau.$$

The positive orbit $S_\Sigma(0)$ of Σ at the neutral element 0 of $\mathbb{R}^n$ defined by

$$S_\Sigma(0) = \{\gamma_t^u(0) \mid u \in U, t \geq 0\}$$

is a vectorial subspace of $\mathbb{R}^n$. Of course this follows, since the controls are unrestricted.

Definition 2.1.0: *Σ is said to be controllable, if positive orbit $S_\Sigma(0)$ coincide with $\mathbb{R}^n$.*

Let us denote by $A^i B = (A^i b_1\, A^i b_2 ... A^i b_k)$, i=0,1,2,..., the column vectors of the matrix $A^i B$, where A^0 is the identity matrix.

Now we establish the Kalman's controllability theorem which characterize from the algebraic point of view the controllability property by the rank (*ad*-rank) condition.

Theorem 2.1.1 (Kalman, Ho, Narendra, [3]) : *Let $\Sigma = (\mathbb{R}^n, \mathcal{D})$ be a linear control system. Then, Σ* is controllable $\Leftrightarrow$ *rank* $(B \mid AB \mid A^2B \mid ... \mid A^{n-1}B) = n$.

Remarks:

(1) By the rank condition, it is easy to decide the controllability of the system with basic linear algebra. On the other hand, it is very difficult to compute $S_\Sigma(0)$ by integrating the family of differential equation which is given by the elements of $\mathcal{D}$. In general, in the non-linear case this kind of results are looked for.

(2) The rank condition consider A^iB only until $i=n-1$. This is possible due to the Cayley-Hamilton Theorem.

Example 2.1.2 : Let us take the family of differential equations parametrized by $u\in U$ in $\mathbb{R}^2$ such that:

$$\dot{x} = y$$
$$\dot{y} = u.$$

This system is a linear control system on $\mathbb{R}^2$ as follows :

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and it is also a model of many applicable problems, [5]. In particular, for the initial condition (a,b) the solution for $u\in U$ is

$$\gamma_t^u(a,b) = (u\frac{t^2}{2} + bt + a, ut + b).$$

If $a=b=0$, then $\gamma_t^u(0,0) = (u\frac{t^2}{2}, ut)$. This is interesting to see that every points of $\mathbb{R}^2$ are reachable from the origin by using only the constant controls $u=1$ and $u=-1$.

By the Kalman's theorem, this system is controllable. In fact,

$$rank\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2.$$

Example 2.1.3 : Let us take the linear control system Σ which is given by

$$\dot{x} = y$$
$$\dot{y} = -x + u.$$

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

It satisfies the rank condition, thus Σ is controllable. This example is a model of an applicable problem, [5].

2.2 The General Case

As appointed by Markus, a linear control system Σ on $\mathbb{R}^n$ is a particular case of linear control systems on a matrix Lie groups. In fact, we can identify $\mathbb{R}^n$ as a Lie subgroup G of $Gl_{n+1}(\mathbb{R})$ with

$$\mathbb{R}^n \to G \subset GL_{n+1}(\mathbb{R})$$

$$x \rightsquigarrow P = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Then, $\dot{x} = Ax + Bu$ can be identify by :
denote by $b_1,...,b_k$ the column vectors of B, so if we define for each $j=1,...,k$

$$B_j = \begin{pmatrix} 0 & b_j \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

the system Σ is represented by:

$$\dot{P} = XP - PX + \sum_{j=1}^{k} u_j B_j P.$$

In this case the flow of X is given by

$$X_t(P) = e^{tH}.P.e^{-tX}.P.e^{-tX}.$$

In particular, for every $t \in \mathbb{R}$, X_t is an inner automorphism of the Lie group G. In fact, after calculation we have :

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{\bullet} = \begin{pmatrix} 1 & Ax + Bu \\ 0 & 1 \end{pmatrix}.$$

A linear control system on Lie group has the following form :

$$\Sigma = (G, \mathcal{D})$$

for which the state space is a connected real finite dimensional Lie group G and the dynamic D is determined by the family of differential equations on G as follows:

$$\dot{g}(t) = X(g(t)) + \sum_{j=1}^{k} u_j Y^j (g(t)).$$

Here, $X \in \mathcal{I}a\,(G)$, i.e. X is an infinitesimal automorphism of G, i.e. the flow

$$T = \{X_t \mid t \in \mathbb{R}\}$$

generated by the vector field X, is a one-parameter group of G-automorphisms. The control vectors Y^j, $j=1,2,...,k$, belong to the Lie algebra $L(G)$ of G. We shall think of $L(G)$ as the set of right-invariant vector fields. The input functions $u = (u_1, u_2,...,u_k)$ belong to U, the class of unrestricted admissible controls. The elements of U are piece wise constant functions of the form

$$u : [0, \infty) \rightarrow \mathbb{R}^k.$$

$\mathcal{D}$ is the family of vector fields associated with Σ, i.e.

$$\mathcal{D} = \{ X + \sum_{j=1}^{k} u_j Y^j \mid u \in \mathbb{R}^k \}.$$

This class of systems generalizes the linear control systems on $\mathbb{R}^n$. In fact, if Σ is an unrestricted time-invariant linear control system on $\mathbb{R}^n$, then by definition, the dynamic of Σ is given by

$$\dot{x} = Ax + Bu.$$

If $b_1, b_2,...,b_k$ denote the columns of B, then we shall describe Σ by

$$\dot{x} = Ax + \sum_{j=1}^{k} u_j b_j.$$

It is well-known that the constant vector b_j defines a right-invariant vector field Y^j on $\mathbb{R}^n$, given by

$$Y^j(x) = b_j \, , \, x \in \mathbb{R}^n .$$

Moreover, the flow of the linear vector field A given by

$$A_t = e^{tA} \, , \, t \in \mathbb{R}$$

belong to $Gl_n(\mathbb{R})$, the Lie group of all $\mathbb{R}^n$ - automorphisms. Thus

$$\Sigma = (G, \mathcal{D}),$$

where G is the commutative Lie group $\mathbb{R}^n$ and

$$\mathcal{D} = \{ Ax + \sum_{j=1}^{k} u_j Y^j \mid u \in \mathbb{R}^k \}.$$

Let $\Sigma = (G, \mathcal{D})$ be a linear control system determined by

$$\mathcal{D} = \{ X + \sum_{j=1}^{k} u_j Y^j \mid u \in \mathbb{R}^k \}$$

with $X \in \mathcal{?}a(G)$ and $Y^j \in L(G)$, $j=1,2,\ldots,k$.

Σ induces the following objects :
the group

$$G_\Sigma = \{ Z^1_{t_1} \circ Z^2_{t_2} \circ \ldots \circ Z^v_{t_v} \mid Z^j \in \mathcal{D}, t_j \in \mathbb{R}, v \in \mathbb{N} \}$$

and the semi-group

$$S_\Sigma = \{ Z^1_{t_1} \circ Z^2_{t_2} \circ \ldots \circ Z^v_{t_v} \mid Z^j \in \mathcal{D}, t_j \geq 0, v \in \mathbb{N} \}$$

of global diffeomorphisms on G.

Definition 2.2.0 : Σ *is said to be:*

(i) *controllable,* if $S_\Sigma(e) = G$.

(ii) *transitive,* if $G_\Sigma(e) = G$.

Remarks:

1) If $X \in \mathcal{V}a\,(G)$, then $L(G)$ is X-invariant, i.e. $\forall Y \in L(G)$

$$ad\,(X)(Y) := [X,Y] \in L(G),$$

where

$$[X,Y] = XY - YX$$

is the usual Lie bracket on $L\,(G)$, [7].

2) Associated with Σ, we have the Lie subalgebra $\mathcal{H}$ of $L(G)$ generated by the control vectors, i.e.

$$\mathcal{H} = Span_{\mathcal{L.A.}}\{Y^1, \ldots, Y^k\}.$$

Let us define by induction the $ad(X)(\mathcal{H})$-sequence :

$$\mathcal{H}_0 = \mathcal{H}$$
$$\mathcal{H}_i = \mathcal{H}_{i-1} + ad^i\,(X)(\mathcal{H}),\ \ i \in \mathbb{N},$$

where

$$ad^i\,(X)(\mathcal{H}) = \{[X, ad^{i-1}\,(X)\,(Y)] \mid Y \in \mathcal{H}\}.$$

Remark 1, shows that for each i $\mathcal{H}_i$ is a subspace of $L(G)$. In particular, since we suppose $\mathcal{H} \neq 0$ and we consider finite dimensional Lie groups there is an integer p, $0 \leq p < dim\,(G)$ such that the $ad\,(X)(\mathcal{H})$-sequence stabilized at p, i.e.

$$\mathcal{H}_p = \mathcal{H}_{p+q},\ \forall q \in \mathbb{N}.$$

3) Since $\mathcal{H}_p$ is a X-invariant subalgebra of $L(G)$ containing $\mathcal{H}$ it follows immediately that

$$\mathcal{H}_p := < X \mid \mathcal{H} >$$

is the smallest X-invariant subalgebra containing $\mathcal{H}$.

We denote by $L(\Sigma)$ the Lie algebra generated by Σ, i.e.

$$L(\Sigma) = Span_{\mathcal{L.A.}}\{X, Y^1, \ldots, Y^k\}.$$

It is possible to prove the following results :

Theorem 2.2.1, [2]: *Let $\Sigma = (G, \mathcal{D})$ be a linear control system. Then, the Lie algebra $L(\Sigma)$ is the semidirect product of $<X \mid \mathcal{H}>$ and the algebra $L(T)$ of the one-parameter group*

$$T \subset Aut(<X \mid H>),$$

i.e.

$$L(\Sigma) \cong L(T) \oplus <X \mid \mathcal{H}>.$$

In particular, since $X_t(e) = e$, $\forall t \in \mathbb{R}$ we obtain

$$G_\Sigma(e) = <X \mid H>.$$

The Lie algebra rank condition to Σ means

$$dim\ Span_{\mathcal{L.A.}}\{Y^j, ad(X)(Y^j) \mid 1 \leq i < p, j=1,\ldots,k\} = dim(G).$$

Here p is the smallest integer such that $ad(X)(\mathcal{H})$-sequence stabilized at p. In particular, Theorem 2.2.1 says that Σ is transitive if and only if Σ satisfies the Lie algebra rank condition.
The following result due to Markus extends partially the Kalman's Theorem.

Theorem 2.2.2, [4] : *Let G be a Lie subgroup of $Gl_n(\mathbb{R})$. If $\Sigma = (G, \mathcal{D})$ is controllable, then Σ satisfies the Lie algebra rank condition.*

Unfortunately the Lie algebra rank condition does not characterize cotrollability. To show this we give some example on the connected and simply connected Heisenberg Lie group. On the other hand, the *ad*-rank condition to Σ means

$$dim\ Span\{Y^j, ad^i(X)(Y^j) \mid 1 \leq i < p,\ j=1,\ldots,k\} = dim(G).$$

In [2], Ayala and Tirao prove :

Theorem 2.2.3 : *Let $\Sigma(G,\mathcal{D})$ be a linear control system.*

i) *If Σ is controllable, then Σ satisfies the Lie algebra rank condition.*

ii) *If Σ satisfies the ad-rank condition, the Σ is controllable.*

iii) *If G is an Abelian Lie group, then Σ is controllable $\Leftrightarrow$ Σ satisfies the rank condition. In particular, we can decide controllability with the rank condition for any Lie group*

$$G = T^n \times \mathbb{R}^m,\ n \in \mathbb{N},\ m \in \mathbb{N}$$

where $T^n = S^1 \times \ldots \times S^1$, ($n$-times), *is a Torus.*

Examples 2.2.4 :

In this section, we compute some examples. We use the rank condition, Theorem (2.2.3) to study controllability.

(1) Let G be the Heisenberg group of dimension 3,

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a,b,c \in \mathbb{R} \right\},$$

with the Lie algebra

$$L(G) = Span_{\mathcal{L.A.}} \left\{ Y^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

In this case,

$$[Y^1, Y^2] = Y^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We consider the linear control $\Sigma = (G, \mathcal{D})$ with

$$\mathcal{D} = \{X + uY^2 \mid u \in \mathbb{R}\},$$

where the infinitesimal automorphism X is defined by :

$$X(g) = bY^3, \text{ for } g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

A simple computation shows that

$$< X \mid \mathcal{H} > = Span_{\mathcal{L.A.}} \{Y^2, Y^3\}.$$

Then Σ is not controllable. Indeed, Σ is not transitive

$$G_\Sigma(e) = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{R} \right\}.$$

(2) Let $G = SL_2(\mathbb{R})$ be the Lie subgroup of $GL_2(\mathbb{R})$ which has elements are all matrices of determinant 1. The Lie algebra $sl_2(\mathbb{R})$ of G is given by :

$$L(G) = Span_{\mathcal{L.A.}} \left\{ Y^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, Y^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\},$$

where

$$[Y^1, Y^2] = Y^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider the matrix

$$A = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_n(\mathbb{R})$$

and the one-parameter group of inner automorphisms (X_t) of G induced by A, i.e.

$$X_t(g) = e^{tA} \,.\, g \,.\, e^{-tA}, \quad t \in \mathbb{R},\ g \in G.$$

For the linear control system $\Sigma = (G, \mathrm{D})$ with

$$\mathcal{D} = \{X + u\, Y^2 \mid u \in \mathbb{R}\}$$

we have

$$ad\,(X)(Y^2) = Y^3 \quad \text{and} \quad ad^2\,(X)(Y^2) = Y^1 \,.$$

Since Σ satisfies the *ad*-rank condition, then Σ is controllable, i.e.

$$S_\Sigma\,(e) = S\,L_2\,(\mathbb{R}).$$

(3) Let $u \in \mathbb{R}$ and consider the non-linear control system N on $\mathbb{R}^2$ determined by

$$\dot{x} = u$$

$$\dot{y} = x^2 \,.$$

Because, $y \geq 0$.

$$S_N\,(0,0) \underset{\neq}{\subset} \mathbb{R}^2 \,,$$

i.e. N is not controllable. We can write this family of differential equations in the following way :

$$(x, y) = f(x, y) + ug(x, y)$$

where f and g are vector fields on $\mathbb{R}^2$ defined by

$$f = x^2 \frac{\partial}{\partial y} \quad \text{and} \quad g = u \frac{\partial}{\partial x} \,.$$

A simple calculation shows that

$$[f,g]=-2ux\frac{\partial}{\partial y}$$

$$[g,[f,g]]=-2u^2\frac{\partial}{\partial y}.$$

In particular, the vector fields g, $[f,g]$ and $[g, [f,g]]$ generate a nilpotent Lie algebra $\mathcal{G}$ such that the associated Lie group is the Heisenberg group of dimension 3. Of course, in this representation, $ad(f)$ is an infinitesimal automorphism of G and the system $\Sigma = (G, \mathcal{D})$ determined by

$$\dot{w}=ad(f)(w)+ug(w), \quad w\in G,\ u\in \mathbb{R}$$

satisfies the Lie algebra rank condition. But, Σ can notbe controllable. In fact, if not, we have

$$S_\Sigma(e) = G.$$

In particular, with the action on $\mathbb{R}^2$ given by the representation we obtain

$$S_\Sigma(e)\,(0,0) = S_N\,(0,0) = \mathbb{R}^2,$$

which is contradiction. We remark that Σ doesn't satisfy the *ad*-rank condition. Indeed,

$$[f, [f, g]] = 0.$$

3. Observability of linear control systems on Lie groups

3.1 The Case $G = \mathbb{R}^n$

A linear control system $\Sigma = (\mathbb{R}^n, \mathcal{D}, h, \mathbb{R}^s)$ on $\mathbb{R}^n$ with observability function h is given as follows :

$$\dot{x}=Ax+Bu,\ x\in \mathbb{R}^n,$$
$$y=h(x)\in \mathbb{R}^s,$$

$$h: \mathbb{R}^n \to \mathbb{R}^s \text{ is a linear transformation,}$$

where $h(x) = Cx$ and $C \in M_{s\times n}(\mathbb{R})$.

Definition 3.1.0 :

a) *x and y are indistinguishable* $\Leftrightarrow h(\gamma(x,u,t)) = h(\gamma(y,u,t)), \forall u \in U, \forall t \geq 0$.

b) Σ *is said to be (locally) observable, if the equivalent class of x (in a neighborhood) is trivial.*

c) Σ *is said to be (locally) observable, if* Σ is *(locally) observable at x for every x of* $\mathbb{R}^n$.

Remarks :

1) Let us denote by ~ the indistinguishable equivalent relation and by $\tilde{x}$ the equivalent class of $x \in \mathbb{R}^n$. Since

$$\gamma(x,u,t) = e^{tA}\{x + \int_0^t e^{-\tau A} Bu(\tau)d\tau\},$$

then

$$x \sim y \Leftrightarrow h(e^{tA}x) = h(e^{tA}y), \forall t \geq 0$$

$$\Leftrightarrow y - x \in Ker(C.e^{tA}), \forall t \geq 0,$$

¿ From this, it follows :

2) $\tilde{0} = \bigcap_{i=1}^{n-1} Ker(CA^i)$.

3) $\tilde{x} = x + \tilde{0}$.

Theorem 3.1.1, [3] : Σ *is observable* $\Leftrightarrow \Sigma$ *is locally observable* $\Leftrightarrow \tilde{0} = \{0\}$

$$\Leftrightarrow rank \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$

Example : On $\mathbb{R}^2$, we consider the example (2.1.2) in two cases :

a) For $h: \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = \pi_1(x,y) = x$:

$$rank \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2.$$

Thus, this system is observable.

b) For $h: \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = \pi_2(x,y) = y$:

$$rank \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1.$$

Thus, with this observability function the system is not observable.
It is interesting to see the pictures in $\mathbb{R}^2$ in both examples with the constant controls $u = 1$ and $u = -1$.

3.2 The General Case

Let G be a connected Lie group with Lie algebra $L(G)$. A linear control system Σ on G with observability function h is determined by the specification of the following data :

$$\Sigma = (G, \mathcal{D}, h, G_1)$$

where $\mathcal{D}$ is a set of vector fields generated by the family of differential equations :

(1) $$\dot{g}(t) = X(g(t)) + \sum_{j=1}^{k} u_j(t) Y^j(g((t)),$$

$X \in \mathcal{J}a(G)$, $Y^1, \ldots, Y^k \in L(G)$.
The output space G_1 is a Lie group and the output map

$$h : G \to G_1$$

is a homomorphism of Lie groups.
By definition, the indistinguishable equivalence relation on G denoted by "~" is given by :

$$g_1 \sim g_2 \Leftrightarrow h \circ \varphi(g_1) = h \circ \varphi(g_2), \forall \varphi \in S_\Sigma .$$

We denote by $\tilde{g}$ the equivalence class of g.

Remarks : The special form of the solution of (1), [2], allows to obtain :
1) $g_1 \sim g_2 \Leftrightarrow h((X_t(g_1)) = h(X_t(g_2)), \forall t \geq 0.$
2) $\tilde{e} = I = \{g \in G \mid X_t(g) \in Ker(h), \forall t \geq 0\}.$
3) Moreover, for each $g \in G$, $\tilde{g} = Ig$. Indeed

$$g_1 \sim g_2 \Leftrightarrow g_2 \in Ig_1 .$$

In [1], Ayala and Kara prove :

Proposition, [1] : *Let Σ be a linear control system on the Lie group G. Then:*
a) *I is a normal closed Lie subgroup of $Ker(h)$.*
b) *I is G_Σ - invariant, i.e.*

$$I = \{g \in G \mid X_t(g) \in Ker(h), \forall t \in \mathbb{R}\}.$$

We denote by $\mathcal{I}$ and $\mathcal{K}$ the Lie algebras of the Lie groups I and $Ker(h)$, respectively.

Theorem 3.2.1, [1] : *Let n be the dimension of G. Then*

$$\mathcal{I} = \bigcap_{i=0}^{n=1} ad^i(X)^{-1}(\mathcal{K}).$$

This theorem says that $\mathcal{I}$ is the largest X-invariant subalgebra contained in $\mathcal{K}$. As in the linear case on $\mathbb{R}^n$, the Lie algebra $\mathcal{I}$ characterize the local observability. But, it is not enough to characterize observability.

Theorem 3.2.2, [1] : *Σ is locally observable $\Leftrightarrow \mathcal{I} = 0$.*
Theorem 3.3.3, [1] : *Σ is observable $\Leftrightarrow$*
a) $\mathcal{I} = 0$,
b) *$Fix(T) \cap Ker(h) = \{e\}$, where*

$$Fix(T) = \{g \mid X_t(g) = g, \forall t \in \mathbb{R}\}.$$

About observability the first fundamental difference between $\mathbb{R}^n$ and G appear in the last result.
Of course the study of observability depend of the equivalence class of the neutral element of G. But, in general, it is not easy to compute directly the Lie group I. Fortunately, we can give an algorithm to obtain I.
Dual-algorithm :

1. Compute $Ker(h)$
2. Choose a basis $\mathcal{B} = \{Z^1, ..., Z^q\}$ to the Lie subalgebra $\mathcal{K}$.
3. Find the $\mathcal{B}$-dual basis to $K^{\perp}$ in the co-tangent bundle T^*G.

$$\mathcal{B}^{\perp} = \{w^1, ..., w^{n-q}\}$$

To obtain $\mathcal{I}$:

4. Find the B-associated basis to $\mathcal{I}^{\perp}$

$$ad\,(X)\,(\mathcal{B}^{\perp}) = \{ad^{\,i}\,(X)\,(w_j) \mid 0 \leq i < p,\ 1 \leq j \leq n - q\}$$

where

$$ad^{\,0}\,(X) = Id$$
$$ad\,(X)\,(w) = L_X\,(w)$$
$$ad^{\,i}\,(X)\,(w) = ad\,(X)\,(ad^{\,i-1}\,(X)\,(w)),\ i \geq 1.$$

Here, $L_X\,(w)$ denote the Lie derivative of the 1-form w along the vector field X, [7].
Then, we obtain :

Proposition 3.2.3, [1] :
a) $\mathcal{I} = Span\,(ad\,(X)\,(\mathcal{B}^{\perp})\,)^{\perp}$.
b) *I is the Lie group of $\mathcal{I}$.*

Examples :
1) We consider the linear control system Σ as in the example (2.2.5), in two cases :

a) $\Sigma : \begin{cases} \dot{g} = (X + uY^1)g, u \in \mathbb{R} \\ h = \pi_2 \end{cases}$

We have

$$Ker\,(h) = \left\{ \begin{pmatrix} 1 & t & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid t, s \in \mathbb{R} \right\}$$

$$\mathcal{B} = \{X, Y^3\}$$

$$\mathcal{B}^{\perp} = \{Y^{1*}\}$$

A simple carculation shows that

$$[X, X] = 0,\ [X, Y^1] = Y^3, [X, Y^3] = 0.$$

Since

$$L_X(Y^{1*})(.) = < Y^{1*}, [X, .] >$$

it follows that

$$L_X(Y^{1*}) = 0$$

then

$$ad^{i}(X)(\mathcal{B}^{\perp}) \subset \mathcal{K}^{\perp}$$

and

$$\mathcal{I} = \mathcal{K}.$$

Thus, Σ can not be observable.

b) $\Sigma : \begin{cases} \dot{g} = X + uY^3 \\ h = \pi : G \to G / \exp(\mathbb{R}\ Y^1) \end{cases}$

We have

$$Ker(h) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

$$\mathcal{B} = \{Y^1\}$$

$$\mathcal{B}^{\perp} = \{X^*, Y^{3*}\}$$

But

$$L_X(Y^{3*}) = Y^{1*},$$

then

$$ad(X)(\mathcal{B}^{\perp}) = L(G)^* = \mathcal{I}^{\perp},$$

thus

$$\mathcal{I} = 0.$$

Consequently, Σ is locally observable.

But, if $g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in G$, we have:

$$exp(tX) . g = g . exp(tX) \Leftrightarrow b = 0$$

and

$$Fix(T) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a, c \in \mathbb{R} \right\}.$$

Consequently

$$Fix(T) \cap Ker(h) = \{e\}$$

and Σ is globally observable.

2) For each $n \in \mathbb{N}$, we consider the system

$$\Sigma_n = (S^1, \mathcal{D}, h, S^1)$$

defined by

$$\mathcal{D} = \{u \frac{\partial}{\partial \theta} \mid u \in \mathbb{R}\}$$

$$h : S^1 \to S^1, \; h(\theta) = n\theta.$$

In this case, $X = 0$ and we obtain:

$$I_n = \{\theta \mid \theta = \sqrt[n]{1}\}$$

is a discrete subgroup of S^1. Since the Lie algebra $\mathcal{I}_n$ of I_n is trivial, by the Theorem 3.2.2 we can conclude that Σ_n is locally observable $\forall n \in \mathbb{N}$.
On the other hand, since $X_t = Id$, $\forall t \in \mathbb{R}$ follows that

$$Fix\,(T) = S^1$$

and consequently for each $n > 1$

$$\{e\} \underset{\neq}{\subset} Fix\,(T) \cap Ker\,(h) = Ker\,(h).$$

By the Theorem 3.3.3, we obtain :
Σ_n is not observable for each $n > 1$.
We want to finish with some comments. Of course, the generalization that we have explained with examples is interesting from the theoretical point of view. In fact, the proofs use results of Lie groups and Lie algebras theory and so on. On the other hand, general linear control systems allow to understand more complicated dynamic also on $\mathbb{R}^n$.

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