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A NOTE ON Š MULIAN'S THEOREM

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Abstract

An extension of Š mulian's theorem in which neither the linearity nor the local convexity are presupposed is obtained. As a consequence, a non-Archimedean version of Š mulian's theorem is derived. *

Definition *[1] Let $\mathbb{K}$ be a metrizable topological division ring, G a vector space over $\mathbb{K}$, H a set and f a mapping from G×H into $\mathbb{K}$ such taht for each $h \in H$, the mapping*

$$f_h : g \in G \mapsto f_h(g) = f(g,h) \in \mathbb{K}$$

is linear, and such that f separates points of G (that is, for all $g \in G$, $g \neq 0$, there is $h \in H$ with $f_h(g) \neq 0$). The initial topology on g for the family $\mathcal{F}_H = \{f_h : h \in H\}$ will be denoted by $\sigma(G,H)$; G, endowed with $\sigma(G,H)$, is a topological vector space over $\mathbb{K}$.
Similarly, we define $\sigma(H,G)$ as the initial topology on H for the family $\mathcal{F}_G$.

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Example 1 *(a) Let H be a set, G a vector subspace of* $\mathcal{F}(H; \mathbb{K})$ *(the vector space of all mappings from H into* $\mathbb{K}$*) and*

$$f : (g,h) \in G \times H \mapsto g(H) \in \mathbb{K}.$$

It is clear that f satisfies the conditions of the Definition. In this case, $\sigma(G,H)$ *coincides with the topology of simple convergence on G.*

(b) Let $\mathbb{K}$ *be a separated topological field and let G be a topological vector space over* $\mathbb{K}$ *with the Hahn-Banach property ([6], p.7). Let*

$$f : (x,h) \in G \times G' \mapsto h(x) \in \mathbb{K},$$

where G' denotes the topological dual of G. Then f satisfies the conditions of the Definition, the fact that f separates points of G being a consequence of Lemma I.1.3.1. of [6].

As particular cases of Example 1(b) we have the separated locally convex spaces over $\mathbb{R}$ or $\mathbb{C}$ [3] and the polar separated locally convex spaces over a non-Archimedean non-trivially valued complete field [8]. On the other hand, if $0 < p < 1$, then l^p endowed with the metric $d(x,y) = \sum_{n=1}^{\infty} |x_n - x_n|^p$ is a topological vector space with the Hahn-banach property which is not locally convex ([7], Chap.3, Exercise 5(d)).

The next result has Šmulian's theorem [9] and a non-Archimedean version for this theorem (see the Corollary) as particular cases.

Theorem *Let* $\mathbb{K}$*, G, H and f be as in the Definition. If* $(H, \sigma(H,G))$ *is separable, then every* $\sigma(G,H)$*-(relatively)- countably compact subset of G is* (G,H)*-(relatively)-sequentially compact.*

Proof: Since $(H, \sigma(H,G))$ is separable and $\mathbb{K}$ is a separated uniform space in which the countably compact subsets are sequentially compact ($\mathbb{K}$ is metrizable), it follows from the Corollary of Proposition 5 of [2] that every countably compact subset of $C_s((H,\sigma(H,G)), \mathbb{K})$ is sequentially compact (where $C_s((H,\sigma(H,G)), \mathbb{K})$ denotes the space of all continuous functions from $(H,\sigma(H,G))$ into $\mathbb{K}$ endowed with the vector topology of pointwise convergence, τ_s).

Now consider the function

$$F : g \in G \mapsto f_g \in C((H, \sigma(H,G), \mathbb{K}),$$

where $f_g(h) = f(g,h)$ for all $h \in H$. By the fact that f separates points of G, F is injective. Therefore F is a homeomorphism from $(G,\sigma(G,H))$ onto $M = F(G)$, M endowed with the vector topology induced by τ_s.

As the countably compact subsets and the sequentially compact subsets of M coincide, we conclude that every $\sigma(G,H)$-(relatively)- countably compact subset of G is $\sigma(G,H)$-(relatively)-sequentially compact, as was to be shown.

Remark 1 *If $\mathbb{K}$ is an arbitrary topological division ring, then the Theorem holds if and only if every countably compact subset of $\mathbb{K}$ is sequentially compact.*

Example 2 *In this example, we will see that for each locally compact separated topological division ring $\mathbb{K}$ there is a topological vector space G over $\mathbb{K}$ such that the canonical bilinear form*

$$(x,\varphi) \in G \times G' \mapsto \varphi(x) \in \mathbb{K}$$

separates points of G (so that the Definition is satisfied). Moreover, there is a $\sigma(G',G)$-compact subset of G' which is not $\sigma(G',G)$-sequentially compact.

Indeed, by Theorem 1.44 of [5], there is an absolute value $|.|$ on $\mathbb{K}$ which defines its topology.

Let l^∞ be the vector space over $\mathbb{K}$ of all bounded sequences of elements of $\mathbb{K}$, endowed with its usual norm. Then $(l^\infty)'$ endowed with the norm

$$\|\varphi\| = \sup_{\lambda \neq 0} |\varphi(\lambda)| / \|\lambda\|$$

is a Banach space over $\mathbb{K}$.

For each $n \geq 1$, let $\varphi_n((\lambda_j)_{j\geq 1}) = \lambda_n$; then $\varphi_n \in (l^\infty)'$ and $\|\varphi_n\| = 1$. $\{\varphi_n : n \geq 1\}$ is $\sigma((l^\infty)', l^\infty)$-relatively compact, being a subset of the closed unit ball of $(l^\infty)'$, which is $\sigma((l^\infty)', l^\infty)$- compact ([4], Lemma 3.1). We claim that $\overline{\{\varphi_n : n \leq 1\}}^{\sigma((l^\infty)', l^\infty)}$ is not $\sigma((l^\infty)', l^\infty)$-sequentially compact.

For otherwise, there would be a subsequence $(\varphi_{n_j})_{j\geq 1}$ of $(\varphi_n)_{n\geq 1}$ and $\varphi \in (l^\infty)'$ such that $(\varphi_{n_j}(\lambda))_{j\geq 1} \to \varphi(\lambda)$, for all $\lambda \in l^\infty$.

But for $\lambda = (\lambda_i)_{i\geq 1} \in l^\infty$ *given by*

$$\begin{cases} \lambda_i = 0, \text{ if } i \neq n_j & \text{for all } j \geq 1, \\ \lambda_i = 0, \text{ if } i = n_j & \text{with } j \text{ even}, \\ \lambda_i = 0, \text{ if } i \neq n_j & \text{with } j \text{ odd}, \end{cases}$$

we have that $(\varphi_{n_j}(\lambda))_{j\geq 1}$ *does not converge.*

Corollary *Let* $\mathbb{K}$ *be a non-Archimedean non-trivially valued complete field an E a strongly polar separated locally convex space over* $\mathbb{K}$. *If E'is* $\sigma(E',E)$*- separable, then every* $\sigma(E,E')$*- (relatively)-countably compact subset of E is (relatively)- sequentially compact.*

Proof: Let τ be the given topology of E. By Proposition 4.11 of [8], every $\sigma(E,E;)$-convergent sequence is τ- convergent. Consequently, the $\sigma(E,E')$- (relatively)- sequentially compact subsets of E and the τ-(relatively)- sequentially compact subsets of E coincide. Thus it is sufficient to apply the Theorem to finish the proof.

Remark 2 *If we consider in Example 2* $\mathbb{K}$ *as a non-Archimedean non-trivially valued locally compact field, then we obtain an example of a locally convex space over* $\mathbb{K}$ *for which the Corollary is not true.*

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