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# A SEPARATION AXIOM BETWEEN SEMI-$T_0$ AND SEMI-$T_1$*

*Miguel* **Caldas**

*Dedicated to Professor Chaim S. Hönig*
*on the occasion of his 70th birthday*

***Abstract***

*The author introduces a new separation axiom and studies some of their basic properties. The implication of these new separation axiom among themselves and with the well known axioms semi-$T_2$ semi-$T_1$ and semi-$T_0$ are obtained.*

## 1. Introduction[1]

Semi-open sets where introduced and investigated by N. Levine [6] in 1963. In 1975, S.N. Maheshwari and R. Prasad [7] used semi-open sets to define and investigate three new separation axioms, called semi-$T_2$, semi-$T_1$ and semi-$T_0$ . Moreover, they have shown that the following implications hold.

✏ *Universidade Federal Fluminense IMUFF, RJ-Brasil.*

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[1] *Key Words: Topology, semi-open sets, map semi-continuous map irresolute, sg-closed sets.*

$$\begin{array}{ccc} T_2 & \rightarrow & semi\text{ - }T_2 \\ \downarrow & & \downarrow \\ T_1 & \rightarrow & semi\text{ - }T_1 \\ \downarrow & & \downarrow \\ T_0 & \rightarrow & semi\text{ - }T_0 \end{array}$$

Later, in 1982 P.Bhattacharyya and B.K. Lahiri [1] used semi-open sets to define the axiom semi-$T_{1/2}$ and further investigated the separation axioms semi-$T_2$, semi-$T_1$ and semi-$T_0$ . For other properties of semi-$T_{1/2}$, see [4]. The purpose of this paper is to introduce a new separation axiom semi-$D_1$ which is strictly between semi-$T_0$ and semi-$T_1$, and discuss its relations with the axioms mentioned above.

Listed below are definitions that will be utilized. We identify the separation axioms with the class of topological spaces satisfying these axioms.

**Definition 1.1** *If* $(X,\tau)$ *is a topological space and* $A \subset X$*, then A is called semi-open* [6] *if, there exists* $O \in \tau$ *such that* $O \subset A \subset Cl(O)$.
*The family of all semi-open set will denoted by* $SO(X,\tau)$.

**Definition 1.2** *Let* $(X,\tau)$ *be a topological space and let* $A,B \subset X$*. Then A is semi-closed* [2] *if, its complement* $A^c$ *(or* $X - A$*) is semi-open and the semi-closure of B, denoted by* $sCl(B)$*, is the intersection of all semi-closed sets containing B.*

**Definition 1.3** *A mapping* $f: (X,\tau) \rightarrow (Y,\sigma)$ *is said to be irresolute* [5] *if, for every semi-open set A its inverse image* $f^{-1}(A)$ *is also semi-open.*

**Definition 1.4** *A topological space* $(X,\tau)$ *is semi-*$T_0$ *(resp. semi-*$T_1$ *)* [7] *if, for* $x,y \in X$ *such that* $x \neq y$ *there exists a semi-open set containing x but not y or (resp., and) a semi-open set containing y but not x.*

**Definition 1.5** *A topological space* $(X,\tau)$ *is semi-*$T_2$ [7] *if, for* $x,y \in X$ *such that* $x \neq y$ *there exist semi-open sets* $O_1$ *and* $O_2$ *such that* $x \in O_1$ *,* $y \in O_2$ *and* $O_1 \cap O_2 = \emptyset$.

**Definition 1.6** *If* $(X,\tau)$ *is a topological space and* $A\subset X$, *then A is sg-closed* [1] *if,* $sCl(A)\subset O$ *holds whenever* $A\subset O$ *and* $O\in SO(X.\tau)$.

**Definition 1.7** *A topological space* $(X,\tau)$ *is semi-*$T_{1/2}$ [1] *if, every sg-closed set in* $(X,\tau)$ *is semi-closed in* $(X,\tau)$.

## 2. The separation axiom semi-$D_1$

**Definition 2.1** *Let X be a topological space. A subset* $S\subset X$ *is called a semi-Difference set (in short sD-set) if there are two semi-open sets* $O_1$ , $O_2$ *in X such that* $O_1 \neq X$ *and* $S = O_1 \setminus O_2$ .

A semi-open set $O \neq X$ is sD-set since $O = O \setminus \varnothing$.

If we replace semi-open sets in the usual definitions of semi-$T_0$, semi-$T_1$, semi-$T_2$ with sD-sets, we obtain separation axioms semi-$D_0$, semi-$D_1$ , semi-$D_2$ respectively as follows.

**Definition 2.2** *A topological space* $(X,\tau)$ *is semi-*$D_0$ *(resp. semi-*$D_1$*) if, for* $x,y \in X$ *such that* $x\neq y$ *there exists a sD-set of X containing x but not y or (resp., and) a sD-set containing y but not x.*

**Definition 2.3** *A topological space* $(X,\tau)$ *is semi-*$D_2$ *if, for* $x,y\in X$ *such that* $x\neq y$ *there exist disjoint sD-sets* $S_1$ *and* $S_2$ *such that* $x\in S_1$ *and* $y\in S_2$ .

**Remark 1.** *(i) If* $(X,\tau)$ *is* $T_i$, *then* $(X,\tau)$ *is semi-*$T_i$, $i$=0,1,2, *and the converse is false* [2].
*(ii) If* $(X,\tau)$ *is semi-*$T_i$, *then it is semi-*$T_{i-1}$, $i$=1,2, *and the converse is false* [2].
*(iii) Obviously, if* $(X,\tau)$ *is semi-*$T_i$ , *then* $(X,\tau)$ *is semi-*$D_i$, $i$=0,1,2.
*(iv) If* $(X,\tau)$ *is semi-*$D_i$, *then it is semi-*$D_{i-1}$ , $i$=1,2.

**Theorem 2.4** *Let* $(X,\tau)$ *be a topological space. Then,*
*(i)* $(X,\tau)$ *is semi-*$D_0$ iff, $(X,\tau)$ *is semi-*$T_0$ .
*(ii)* $(X,\tau)$ *is semi-*$D_1$ *iff,* $(X,\tau)$ *is semi-*$D_2$ .

**Proof.** (i) The sufficiency is Remark 2.1(iii). To prove necessity. Let $(X,\tau)$ semi-$D_0$ . Then for each distinct pair $x,y\in X$, at least one of $x,y$, say $x$, belongs to a sD-set $S$ but $y \notin S$. Let $S = O_1 \setminus O_2$ where $O_1 \neq X$ and

$O_1, O_2 \in SO(X,\tau)$. Then $x \in O_1$, and for $y \notin S$ we have two cases: (a) $y \notin O_1$; (b) $y \in O_1$ and $y \in O_2$.

In case (a), $O_1$ contains $x$ but does not contain $y$;

In case (b), $O_2$ contains $y$ but does not contain $x$.

Hence $X$ is semi-$T_0$.

(ii) Sufficiency. Remark 1(iv).

Necessity. Suppose $X$ semi-$D_1$. Then for each distinct pair $x, y \in X$, we have sD-sets $S_1$, $S_2$ such that $x \in S_1$, $y \notin S_1$; $y \in S_2$, $x \notin S_2$. Let $S_1 = O_1 \setminus O_2$, $S_2 = O_3 \setminus O_4$. From $x \notin S_2$ we have either $x \notin O_3$ or $x \in O_3$ and $x \in O_4$. We discuss the two cases separately.

(1) $x \notin O_3$. From $y \notin S_1$ we have two subcases:

(a) $x \notin O_1$. From $x \in O_1 \setminus O_2$ we have $x \in O_1 \setminus (O_2 \cup O_3)$ from $y \in O_3 \setminus O_4$ we have $y \in O_3 \setminus (O_1 \cup O_4)$. It is easy to see that

$$(O_1 \setminus (O_2 \cup O_3)) \cap (O_3 \setminus (O_1 \cup O_4) = \varnothing$$

(b) $y \in O_1$ and $y \in O_2$. We have $x \in O_1 \setminus O_2$, $y \in O_2$. $(O_1 \setminus O_2) \cap O_2 = \varnothing$.

(2) $x \in O_3$ and $x \in O_4$. We have $y \in O_3 \setminus O_4$, $x \in O_4$. $(O_3 \setminus O_4) \cap O_4 = \varnothing$.

From the discussion above we know that the space $X$ is semi-$D_2$. □

**Theorem 2.5** *If $(X,\tau)$ is semi-$D_1$, then it is semi-$T_0$.*

**Proof.** Remark 1(iv) and Theorem 2.4(i). □

We give an example which shows that the converse of Theorem 2.5 is false.

**Example 2.6** *Let $X = \{a,b\}$ with topology $\tau = \{\varnothing, \{a\}, X\}$. Then $(X,\tau)$ is semi-$T_0$ since is $T_0$ but not semi-$D_1$ since there is not a sD-set containing b but not a.*

The following example shows that semi-$D_1$ space need not be semi-$T_1$.

**Example 2.7** *Let $X = \{a,b,c,d\}$ with topology*

$$\tau = \{\varnothing, \{a\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}, X\}.$$

*Then we have*

$$SO(X,\tau) = \{\varnothing, \{a\}, \{a,d\}, \{a,b\}, \{a,c\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c\}, X\}.$$

Since $\{b\} = \{a,b,d\} \setminus \{a,d\}, \{c\} = \{a,c,d\} \setminus \{a,d\}, \{d\} = \{a,d\} \setminus \{a\}$, *hence $(X,\tau)$ is semi-$D_1$ space but not semi-$T_1$ since each semi-open set containing b contains a.*

**Definition 2.8** *Let $x_o$ be a point in a topological space X. If $x_o$ has not semi-neighborhood other than X, then call $x_o$ a sc.c (common to all semi-closed sets point).*

The name comes from the fact that $x_o$ belongs to each semi-closed set in $X$.

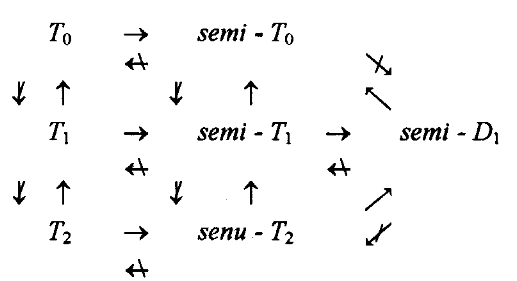
**Theorem 2.9** *A semi-$T_0$ space X is semi-$D_1$, iff there is not sc.c point in X.*

**Proof.** Necessity. If $X$ is semi-$D_1$ then each point $x \in X$, belongs to a sD-sets $S = O_1 \setminus O_2$ , hence $x \in O_1$ . Since $O_1 \neq X$, $x$ is not *sc.c* point.
Sufficiency. If $X$ is semi-$T_0$ , then for each distinct pair of points $x,y \in X$, at least one of $x,y$, say $x$ has a semi-neighborhood $U$ such that $x \in U$ and $y \notin U$, hence $U \neq X$ is a sD-set. If $X$ has not *sc.c* point, then $y$ is not a *sc.c* point, hence there exists a semi-neighborhood $V$ of $y$ such that $V \neq X$. Now $y \in V \setminus U$, $x \notin V \setminus U$ and $V \setminus U$ is a sD-set. Therefore $X$ is semi-$D_1$ . □

**Corollary 2.10** *A semi-$T_0$ space X is not semi-$D_1$ iff, there is a unique sc.c point in X.*

**Proof.** Only the uniqueness of the *sc.c* point is to be proved. If $x_o$ , $y_o$ are two *sc.c* points in $X$ then since $X$ is semi-$T_0$, at least one of $x_o$, $y_o$, say $x_o$, has a semi-neighborhood $U$ such that $x_o \in U$, $y_o \notin U$, hence $U \neq X$, $x_o$ is not a *sc.c* point, contradiction. □

**Remark 2.** *From* ([7], *Examples 3.1; 4.1; and 5.1), Example 2.6 and Example 2.7, we have the following diagram:*




The following example shows that semi-$D_1$ space does not imply $D_1$ space in the sense of J.Tong [8].

**Example 2.11** *If* $X = \{a,b,c,d\}$ *and let* $\tau = \{\varnothing,\{a\},\{b\},\{a,b\}, X\}$ *be the topology on X. Then X is a semi-*$D_1$ *space which is not* $D_1$ .

**Definition 2.12** *A topological space* $(X,\tau)$ *will be termed semi-symmetric* [4] *if, for x and y in X, then* $x \in sCl(\{y\})$ *implies that* $y \in sCl(\{x\})$.

**Theorem 2.13** [4]. *Let* $(X,\tau)$ *be a topological space. Then,*
*(i)* $(X,\tau)$ *is semi-symmetric iff,* $\{x\}$ *is sg-closed for each x in X,*
*(ii) If* $(X,\tau)$ *is a semi-*$T_1$ *space, then* $(X,\tau)$ *is semi-symmetric,*
*(iii)* $(X,\tau)$ *is semi-symmetric and semi-*$T_0$ *iff,* $(X,\tau)$ *is semi-*$T_1$ .

**Theorem 2.14** *Let* $(X,\tau)$ *be a semi-symmetric space. Then the following are equivalent.*
*(i)* $(X,\tau)$ *is semi-*$T_0$ ,
*(ii)* $(X,\tau)$ *is semi-*$T_{1/2}$ ,
*(iii)* $(X,\tau)$ *is semi-*$T_1$ .

**Proof.** ([4], Theorem 3.2). □

From Remark 1 (iii), Theorem 2.5, Theorem 2.13 (iii) and Theorem 2.14, we have the following theorem.

**Theorem 2.15** *Let* $(X,\tau)$ *be a semi-symmetric space then the following are equivalent,*
*(i)* $(X,\tau)$ *is semi-*$T_0$
*(ii)* $(X,\tau)$ *is semi-*$D_1$,
*(iii)* $(X,\tau)$ *is semi-*$T_{1/2}$ ,
*(iv)* $(X,\tau)$ *is semi-*$T_1$ .

**Theorem 2.16** *If* $f: X \to Y$ *is a irresolute and surjective mapping, S is a sD-set in Y, then* $f^{-1}(S)$ *is a sD-set in X.*

**Proof.** Let $S$ be a sD-set in $Y$. Then there are semi-open sets $O_1$ and $O_2$ in $Y$ such that $S = O_1 \setminus O_2$ and $O_1 \neq Y$. By the irresolute of $f$, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are semi-open in $X$. Since $O_1 \neq Y$, we have $f^{-1}(O_1) \neq X$. Hence $f^{-1}(S) = f^{-1}(O_1) \setminus f^{-1}(O_2)$ is a sD-set. □

**Theorem 2.17** *If* $(Y,\sigma)$ *is semi-*$D_1$ *and* $f:(X,\tau)\rightarrow(Y,\sigma)$ *is irresolute and bijective, then* $(X,\tau)$ *is semi-*$D_1$ .

**Proof.** Suppose that $Y$ is a semi-$D_1$ space. Let $x$ and $y$ be any pair of distinct point in $X$. Since $f$ is injective and $Y$ is semi-$D_1$ , there exist sD-sets $S_x$ and $S_y$ of $Y$ containing $f(x)$ and $f(y)$ respectively, such that $f(y)\notin S_x$ and $f(x)\notin S_y$ . By Theorem 2.16, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are sD-sets in $X$ containing $x$ and $y$ respectively. This implies that $X$ is a semi-$D_1$ space. □

We finish our article giving a other characterization of semi-$D_1$ spaces.

**Theorem 2.18** *A space X is semi-*$D_1$ *iff, for each pair of distinct points x and y in X, there exists a irresolute, and surjective mapping f of a space X into a semi-*$D_1$ *space Y such that* $f(x)\neq f(y)$.

**Proof.** Necessity. For every pair of distinct point of $X$, it suffices, to take the identity mapping on $X$.
Sufficiency. Let $x$ and $y$ be any pair of distinct points in $X$. By hipothesis, there exists a irresolute, surjective mapping $f$ of a space $X$ into a semi-$D_1$ space $Y$ such that $f(x)\neq f(y)$. Therefore, there exist disjoint sD-sets $S_x$ and $S_y$ in $Y$ such that $f(x)\in S_x$ and $f(y)\in S_y$ . Since $f$ is irresolute and surjective, by Theorem 2.16, $f^{-1}(S_x)$ and $f^{-1}(S_y)$ are disjoint sD-sets in $X$ containing $x$ and $y$, respectively. Hence and Theorem 2.4 (ii), $X$ is semi-$D_1$ space. □

**Remark 3.** *Theorem 2.17, can be obtained as a consequence from Theorem 2.18* .

**Corollary 2.19** *Let* $\{X_\alpha/\alpha\in I\}$ *be any family of Topological spaces. If* $X_\alpha$ *is semi-*$D_1$ *, for each* $\alpha\in I$*, then, the product space* $\Pi X_\alpha$ *is semi-*$D_1$ .

**Prof.** Let $(x_\alpha)$ and $(y_\alpha)$ be any pair of distinct points in $\Pi X_\alpha$ . Then there exists an index $\beta\in I$ such that $x_\beta\neq y_\beta$. The natural projection $P_\beta:\Pi X_\alpha\rightarrow X_\beta$ is continuous and open (therefore irresolute [5]) and $P_\beta((x_\alpha))\neq P_\beta((y_\alpha))$. Since $X_\beta$ is semi-$D_1$ , by Theorem 2.18, $\Pi X_\alpha$ is semi-$D_1$ . □

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*Miguel Caldas Cueva*
*gmamccs@vm.uff.br*
*Universidade Federal Fluminense*
*Departamento de Matemática Aplicada - IMUFF*
*Rua São Paulo S/N°24020-005, Niteroi, RJ-Brasil*