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CONTROLLABILITY OF LINEAR SYSTEMS ON NON-ABELIAN COMPACT LIE GROUPS

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Abstract

In this paper, we shall deal with a linear control system Σ defined on a Lie group G with Lie algebra L(G). We prove that, if G is a compact connected Lie group, then the vector fields associated to dynamic of Σ are conservative, and that if G is also non-Abelian then, by using Poincare Theorem, Σ is transitive if and only if it is controllable.

1. Introduction

We will obtain a controllability property for a particular class of systems, linear control systems of the form

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$$\sum = (G,\mathcal{D})$$

for which the state space is a finite dimensional Lie group G and the dynamic $\mathcal{D}$ is determinated by the family of differential equations on G

$$\dot{x} = X(x) + \sum_{j=1}^{k} u_j Y^j(x).$$

Here, the drift vector field X is an element of the normalizer of $L(G)$ in the Lie algebra $X(G)$ of all C^{∞} vector fields on G. The control vectors Y^j, $j = 1, 2,\dots,k$, belong to the Lie algebra $L(G)$ of G. We shall think of $L(G)$ as the set of left invariant vector fields. The input functions $u = (u_1, u_2,\dots u_k)$ belong to $\mathcal{U}$, the class of unrestricted admissible controls. The elements of $\mathcal{U}$ are piecewise constant functions of the form

$$u:[0,\infty) \to \mathbb{R}^K$$

$\mathcal{D}$ is the family of vector fields associated with $\sum$, i.e.

$$\mathcal{D} = \{X + \sum_{j=1}^{k} u_j Y^j \mid u \in \mathbb{R}^K\}.$$

In a Lie group G a natural measure assigned to the tangent space at the identity element e can be transferred by multiplication on the left to any point of G. Such a measure is invariant under multiplication on the left, this is the measure arising from it by integration. It is important to know whether this measure is invariant under right multiplication as well. It may be mentioned that a one-sided invariant measure exists in any locally compact group (Haar measure); if the group is compact, it is two-sided invariant, [3]. By using this fact and theorem 2.12 of [2], we show that if G is compact and connected then the vector fields associated to dynamic of system are conservative.

Theorem of Poincare says that the Poisson stable points of a dynamical system generated by a conservative vector field are dense. Up to now it is knowing that the controllability of linear control systems on $\mathbb{R}^n$ and Abelian Lie groups are characterized by their transitivity. By using the result of Poincare, we will prove:

Theorem 2.4. *Let G be compact connected non-Abelian a Lie group and let $\sum = (G, \mathcal{D})$ be a linear control system with normalizer. Then*

$$\textstyle\sum \text{ is transitive} \Leftrightarrow \sum \text{ is controllable.}$$

2. Controllability

The elements of dynamic $\mathcal{D}$ of system $\sum$ induces the group

$$G_\Sigma = \{\varphi^{\nu}_{t_\nu} \circ \ldots \circ \varphi^{2}_{t_2} \circ \varphi^{1}_{t_1} \,|\, \varphi^j \in \mathcal{D},\ t_j \in \mathbb{R},\ \nu \in \mathbb{N}\}$$

and the semi-group

$$S_\Sigma = \{\varphi^{\nu}_{t_\nu} \circ \ldots \circ \varphi^{2}_{t_2} \circ \varphi^{1}_{t_1} \,|\, \varphi^j \in \mathcal{D},\ t_j \geq 0,\ \nu \in \mathbb{N}\}$$

of global diffeomorphisms on G.

Definition 2.1 *A linear control system* $\sum = (G, \mathcal{D})$ *is said to be*:

a) *controllable if* $S_\Sigma(x) = G \quad \forall x \in G,$
b) *transitive if* $G_\Sigma(x) = G \quad \forall x \in G.$

Definition 2.2 *Let G be a Lie group. A conservative vector field is a vector field for which the natural measure in G is invariant under the action of dynamical system associated to it.*

Theorem 2.3 *Let G be a compact connected Lie group and let* $\sum = (G, \mathcal{D})$ *be a linear control system with normalizer. Then the vector fields associated to dynamic* $\mathcal{D}$ *are conservative.*

Proof. Since $\sum$ is a linear control system with normalizer, the dynamic $\mathcal{D}$ is determined by

$$\dot{x} = X(x) + \sum_{j=1}^{k} u_j Y^j(x),$$

where the drift vector field X is an element of the normalizer of $L(G)$ in $X(G)$ and the control vectors $Y^j, j = 1, 2,\dots, k$, belong to $L(G)$. We shall think of $L(G)$ as the set of left-invariant vector fields.

From theorem 2.12 in [2], an element $X \in \mathrm{norm}_{X(G)}(L(G))$ can be characterized by the pair $(W, Y^j) \in \partial(L(G)) \otimes L(G)$ for $j = 1, 2,\dots, k$ where $\partial L(G)$ is the Lie algebra of derivations of $L(G)$. In this case, for constant controllers $u_j = 1$, the dynamic $\mathcal{D}$ reduces to the form.

$$\mathcal{D} = \{W + \sum_{j=1}^{k} Y^j \mid W \in \partial(L(G)), Y^j \in L(G)\}.$$

Since the sum part of $\mathcal{D}$ consists of left invariant vector fields, $\mathcal{D}$ is left invariant. And since G is compact, this sum is left measure invariant.

The map $W : L(G) \rightarrow L(G)$ is defined by

$$W[X, Y] = [WX, Y] + [X, WY] \quad \forall X, Y \in L(G)$$

which preserve left invariance, so it preserves left measure invariance. Hence $\mathcal{D}$ consists of conservative vector fields. Therefore, the proof is complete.

The well-known theorem of Poincare, which is mentioned in the part of introduction of this paper, is the main trick in the proof of theorem 2.4.

Recall that a point is Poisson stable if and only if for every neighborhood N_x of x and every positive t, there exists t_1 and t_2 greater than t, such that $X_{t_1}(x)$ and $X_{t_2}(x)$ are in N_x.

Theorem 4.2. *Let G be a compact connected non-Abelian Lie group and let* $\sum = (G, D)$ *be a linear control system with normalizer. Then*

$$\sum \text{ is transitive } \Leftrightarrow \sum \text{ is controllable}$$

Proof. If $\sum$ is controllable then from definition 2.1, the transitivity of system is clear. So, we will show other implication:

For $x, y \in G$ there exist $i \in \mathbb{N}$, $t_1 \in \mathbb{R}^+$ such that

$$y = \varphi^{p}_{t_p} \circ \dots \circ \varphi^{j}_{t_j} \circ \dots \circ \varphi^{1}_{t_1}(x)$$

The theorem in [5] which is given with the state "if the system is transitive, for every $x \in G$ the interior points of the positive orbit $S^+_\Sigma(x)$ are dense in the positive orbit" is true if we replace $S^+_\Sigma(x)$ by $S^-_\Sigma(x)$. Let $\bar{y}$ be an interior point of $S^-_\Sigma(y)$ and let $N_{\bar{y}}$ be an open neighborhood of $\bar{y}$ such that $N_{\bar{y}} \subset S^-_\Sigma(y)$. Since the system is transitive, there exist $i \in \mathbb{N}$, $s_j \in \mathbb{R}$ such that

$$\bar{y} = \varphi^{p}_{s_p} \circ \dots \circ \varphi^{j}_{s_j} \circ \dots \circ \varphi^{1}_{s_1}(x)$$

Assume for simplicity that

$$\bar{y} = \varphi^{2}_{s_2} \circ \varphi^{1}_{s_1}(x),$$

Where s_1 is positive and s_2 is negative. Let $g = \varphi^{1}_{s_1}(x)$. Then the set

$$N_g = \varphi^{2}_{-s_2}(N_{\bar{y}})$$

is an open neighborhood of g, and it has a non-empty intersection with $S^+_\Sigma(x)$. Hence it contains a point z which is Poisson stable for φ^2. Let N_z be a neighborhood of z such that $N_z \subset N_g \cap S^+_\Sigma(x)$. By the definition of Poisson stable point, there exists a real number s greater than $|s_2|$ such that point φ^2_s belongs to N_z. Then

(i) the number $s + s_2$ is positive,

(ii) the point $\varphi^2_{s+s_2}(z)$ belongs to $N_{\bar{y}}$.

The point y is in the positive orbit of any point in $S_{\Sigma}^{-}(y)$. Thus y belongs to $S_{\Sigma}^{+}\varphi_{s+s_2}^{2}(z)$. It completes the proof of theorem 2.4.

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