Pro Mathematica Vol. XII, Nos. 23-24, 1998

A DISCRETE ANALOGUE OF MACLAURIN SERIES

Mumtaz ***Ahmad K.***

Abstract

In the present paper a (p,q)-analytic function analogue of Maclaurin's series has been obtained.

1. Introduction

In 1993, the present author [1] defined and studied (p,q)-analytic functions. Some of its definitions required in this paper are as given below:

Definition 1. *The discrete plane Q' with respect to some fixed point $z' = (x', y')$ in the first quadrant, is defined by the set of lattice points,*

$$Q' = \{(p^m x', q^n y'); m, n \in Z \text{ the set of integers}\}.$$

Definition 2. *Two lattice points, $z_i, z_{i+1} \in Q'$ are said to be ' adjacent' if z_{i+1} is one of (px_i, y_i), $(p^{-1} x_i, y_i)$ (x_i, qy_i) or $(x_i, q^{-1} y_i)$.*

✏ *Department of Applied Mathematics. Faculty of Engineering, U.P., India.*

Definition 3. *A 'discrete curve' C in Q' connecting* z_0 *to* z_n *is denoted by the sequence*

$$C \equiv < z_0, z_1, z_2, \ldots, z_n >,$$

where z_i, z_{i+1}; $i = 0, 1, 2, \ldots, (n-1)$, *are adjacent points of* Q'.

If the points are distinct ($z_i \neq z_j$; $i \neq j$) then the discrete curve C is said to be '*simple*'.

Definition 4. *A 'discrete closed curve' C in Q' is given by the sequence* $< z_0, z_1, z_2, \ldots, z_n >$ *where* $< z_0, z_1, \ldots, z_{n-1} >$ *is simple and* $z_0 = z_n$.

Denote by $\overline{C}$ the continuous closed curve formed by joining adjacent points of the discrete closed curve C. Then $\overline{C}$ encloses certain points of Q', denoted by $Int(C)$.

Definition 5. *A 'finite discrete domain' B is defined as*

$$B = \{z \in Q';\ z \in C \overset{\bullet}{U}\ Int(C)\}.$$

Definition 6. *Functions defined on the points of a discrete domain B are said to be 'discrete functions'.*

Definition 7. *The p-difference and q-difference operators* $D_{p,x}$ *and* $D_{q,y}$ *are defined as follows.*

$$D_{p,x}\left[f(z)\right] = \frac{f(z) - f(px, y)}{(1-p)x} \qquad \ldots(1.1)$$

$$D_{q,y}\left[f(z)\right] = \frac{f(z) - f(x, qy)}{(1-q)iy} \qquad \ldots(1.2)$$

where $f(z) \equiv f(x, y)$, $z = x + iy$.

The two operators (1.1) and (1.2) involve a '*basic triad*' of points denoted by

$$T(z)=\{(x,y),(px,y),(x,qy)\}. \qquad ...(1.3)$$

Let B be a discrete domain. Then a discrete function f is said to be '*(p,q)-analytic*' at $z \in B$ if

$$D_{p,x}[f(z)] = D_{q,y}[f(z)]. \qquad ...(1.4)$$

If in addition (1.4) holds for every $z \in B$ such that $T(z) \subseteq B$ then f is said to be (p,q) - *analytic* in B. ...(1.5)

For $0 \leq q < 1$, we also have

$$(q^{\alpha})_n \equiv (1-q^{\alpha})_{n,q} = (1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\alpha+2})...(1-q^{\alpha+n-1});\ (q^{\alpha})_0 = 1 \qquad ...(1.6)$$

From (1.1), we have

$$D_{p,x}[x^n] = \frac{1-p^n}{1-p}x^{n-1} \qquad ...(1.7)$$

and

$$D^j_{p,x}[x^n] = \begin{cases} (1-p^{n-j+1})(1-p^{n-j+2})...(1-p^{n-1})(1-p^n)x^{n-j}; & j \leq n \\ 0, & j > n \end{cases} \qquad ...(1.8)$$

The continuation operator C_y and the convolution operator $*$ were studied by Khan and Najmi in their papers [2] and [3] respectively. To use them in this paper, their definitions are reproduced here. The continuation operator C_y is defined as

$$C_y \equiv \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}}(iy)^j D^j_{p,x} \qquad ...(1.9)$$

and $f(z) = C_y[f(x,0)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}}(iy)^j D^j_{p,x}[f(x,0)]$ ...(1.10)

Similarly,

$$C_x[f(0,y)]=\sum_{j=0}^{\infty}\frac{(1-p)^j}{(1-p)_{j,p}}x^j D_{q,y}^j[f(0,y)] \qquad ...(1.11)$$

A discrete analogue $z^{(n)}$ of the classical function z^n was defined by Khan and Najmi [4] as

$z^{(n)} \equiv C_y\,(x^n)$; n a non-negative integer

which by virtue of (1.8) and (1.9) becomes

$$z^{(n)}=\sum_{j=0}^{n}\frac{(1-p)_{n,p}(1-q)^j}{(1-p)_{n-j,p}(1-q)_{j,q}(1-p)^j}(iy)^j x^{n-j} \qquad ...(1.12)$$

The convolution operator ✱ is defined as

$$(f ✱ q)(z) \equiv C_y\,[f(x,0)\,g(x,0)]$$

$$=\sum_{j=0}^{\infty}\frac{(1-q)^j}{(1-q)_{j,q}}(iy)^j D_{p,x}^j[f(x,0)g(x,0)] \qquad ...(1.13)$$

for all $z \in R'$ such that the series converges, where the '*discrete rectangular domain*' R' is defined by

$$R' = \{(p^m x', q^n y'); \; m = 0, 1, 2,...; \; n = 0, 1, 2,...\}. \qquad ...(1.14)$$

If X^+, Y^+ are defined by

$$X^+ \equiv \{(p^m x', 0); \; m = 0, 1, 2,...\}, \qquad ...(1.15)$$

$$Y^+ \equiv \{(0, q^n y'); \; n = 0, 1, 2,...\}, \qquad ...(1.16)$$

then the extended rectangular domain $\overline{R}$ is defined as

$$\overline{R} = R' \cup X^+ \cup Y^+. \qquad ...(1.17)$$

The following theorem from the author's earlier paper [5] is also required:

Theorem 1.1. *If lim sup* $|a_j|^{\frac{1}{j}} = a$, *then the series*

$$f(z) = \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} a_j z^{(j)}$$

converges absolutely for all z *such that*

$$\| (1-p)x + (1-q)iy \| < \frac{1}{a}.$$

2. Discrete Maclaurin Series

To include the point (0,0), extend the definition of $\overline{R}$ (from (1.17)) as follows:

$$\overline{R}_o \equiv \overline{R} \cup (0,0) \qquad \text{...(2.1)}$$

A discrete function f is said to be *(p,q)-analytic* on $\overline{R}_0$ if it is *(p,q)-analytic* on $\overline{R}$ and, in addition, $\lim\limits_{(x,y)\to(0,0)} D^j\left[f\left(x,y\right)\right]$ exists. The limit is denoted by $D^j f(0,0)$. ...(2.2)

Under certain conditions the discrete Maclaurin series can be shown to represent a *(p,q)-analytic* function, provided the series converges. For example the following theorem holds:

Theorem 2.1. *Let* f *be (p, q)-analytic in* $\overline{R}_o$. *If* $f(z) = C_y\ [f(x,0)] = C_x\ [f(0,y)]$, *the series representations of* C_y, C_x *being uniformly and absolutely convergent in* $\overline{R}_o$, *then*

$$f(z) = \sum_{j=O}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} D^j\left[f(0,0)\right] z^{(j)},$$

the series being absolutely convergent for all $z \in \overline{R}_o$.

Proof: $f(z) = C_x\,[f(0, y)]$,

whence by (1.11),

$$f(z)=\sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} x^j D_{q,y}^j [f(0,y)]$$

Hence,

$$f(x,0)= \lim_{y\to 0} \sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} x^j D_{q,y}^j [f(0,y)]$$

and by uniform convergence,

$$f(x,0)=\sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} x^j \lim_{y\to 0} D_{q,y}^j [f(0,y)].$$

By (2.2),

$$f(x,0)=\sum_{j=0}^{\infty} \frac{(1-p)^j}{(1-p)_{j,p}} x^j D^j [f(0,0)]$$

Now $f(z)$ is also given by

$$f(z) = C_y\,[f(x,0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} (iy)^j D_{p,x}^j [f(x,0)]$$

and so by above,

$$f(z) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} (iy)^j D_{p,x}^j \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)_{k,p}} x^k D^k [f(0,0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_{j,q}} (iy)^j \sum_{k=0}^{\infty} \frac{(1-p)^k}{(1-p)_{k,p}} x^k D^{k+j} f(0,0).$$

By absolute convergence, the summation can be rearranged to give

$$f(z)=\sum_{j=0}^{\infty}\frac{(1-q)^{j}}{(1-q)_{j,q}}D^{j}f(0,0)\sum_{k=0}^{j}\frac{(1-q)_{j,q}(1-p)^{k}x^{k}(iy)^{j-k}}{(1-q)_{j-k,q}(1-q)^{k}(1-p)_{k,p}}$$

$$=\sum_{j=0}^{\infty}\frac{(1-q)^{j}}{(1-q)_{j,q}}D^{j}\left[f(0,0)\right]z^{(j)}.$$ This proves the theorem.

Theorem 1.1 provides as a direct consequence a condition for convergence of the discrete Maclaurin series as follows:

Theorem 2.2. *If* $\limsup\left\{\left|D^{j}f(0,0)\right|^{\frac{1}{j}}\right\}=a$, *then the series*

$$\sum_{j=0}^{\infty}\frac{(1-q)^{j}}{(1-q)_{j,q}}D^{j}\left[f(0,0)\right]z^{(j)}$$

converges absolutely for all z such that

$$\left\|(1-p)x+(1-q)iy\right\|<\frac{1}{a}.$$

References

[1] Khan, M. A. *(p,q) - Analytic functions and Discrete 'bibasic' Hypergeometric series.* Math. Notae, Argentina, vol. 37, pp. 43-56, 1993-94.

[2] Khan, M. A. and Najmi, M. *Discrete analytic continuation of a (p,q)-analytic function.* Communicated for publication.

[3] Khan, M. A. and Najmi, M. *On a convolution operator for (p,q)-analytic functions.* Communicated for publication.

[4] Khan, M. A. and Najmi, M. *On Discrete powers and Polynomials.* Communicated for publication.

[5] Khan, M.A. *A note on discrete exponential and trigonometric functions.* Communicated for publication.

Mumtaz Ahmad Khan
Department of Applied Mathematics,
Faculty of Engineering
A.M.U., ALIGARH-202002, U.P., INDIA