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PROBLEMS IN INCOMPRESSIBLE LINEAR ELASTICITY INVOLVING TANGENTIAL AND NORMAL COMPONENTS OF THE DISPLACEMENT FIELD

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Abstract

We consider the linear system $-\Delta \mathbf{u} + \mathbf{grad}\, p = \mathbf{f}$ plus the divergence-free condition $\mathbf{div}\, \mathbf{u} = 0$, in a bounded and connected but non simply connected open set Ω of $\mathbb{R}^3$, with a boundary Γ of C^∞ class. Using orthogonal decompositions of the Hilbert space of square integrable vector fields on Ω, we show well posedness for two boundary value problems involving normal or tangential components of the displacement field on Γ.

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1. Introduction

In [5], the method of orthogonal projections on the space $\{L^2(\Omega)\}^3$ of square integrable vector fields on Ω, is suggested to study some constrained problems in elasticity theory. In this work, we are placed on the special case of divergence-free constraint in linear elasticity.

In the sequel, we denote $L^2(\Omega)^3 := L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ with its usual norm

$$|(v_1, v_2, v_3)|^2_{L^2(\Omega)^3} = \int_\Omega \{(v_1)^2 + (v_2)^2 + (v_3)^2\} dx.$$

The divergence-free constraint, $\mathbf{div}\, \boldsymbol{u} = 0$, implies

$$-\Delta = \mathbf{curl\ curl} = -\mathbf{div\ grad}$$

from the classical identify $\mathbf{curl\ curl} = -\mathbf{div\ grad} + \mathbf{grad\ div}$.

Hodge's decompositions of a given vector field $f \in L^2(\Omega)^3$ ([2] Corollaries 5' and 6), give us $f = \mathbf{grad}\, p + \mathbf{curl}\, w$. The isomorphisms of the **curl** operator are used to solve the two problems by similar arguments in [1].

2. Terminology and notations

The results of this section in more detailed form can be found in [2,3,4].

Let Ω a bounded and connected open subset of $\mathbb{R}^3$ with boundary Γ, which is an regular (of C^∞ class) oriented surface in $\mathbb{R}^3$, with an exterior normal vector field n.

i. Ω is not necessarily simply connected and Γ is an union of connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ (Γ_0 being the boundary of the unbounded connected component of the complement Ω^c of Ω in $\mathbb{R}^3$).

ii. There exists a cut surface of Ω, that is, a nonoverlapping union of regular surfaces $\Sigma = \Sigma_1 \cup ... \cup \Sigma_N$, with Σ_i (cut surfaces) contained in Ω and transversal to the components Γ_j of Γ. N is the minor positive integer such that $\Omega_\Sigma = \Omega \backslash \Sigma$ became a simply connected, lipschitzian open subset of $\mathbb{R}^3$. Thus, Ω_Σ has the boundary $\Gamma_\Sigma = \Gamma \cup \Sigma$. Associated to any Σ_i we consider Σ_i^+ and Σ_i^-, respectively, the two opposites sides of Σ_i and we still denote by n the normal vector field on Σ_i that is directed from Σ_i^+ to Σ_i^-. If there exists the restrictions $\varphi|_{\Sigma_i^+}$ and $\varphi|_{\Sigma_i^-}$, for a given function φ on Ω_Σ, the jump of φ on Σ_i is denoted by

$$[\varphi]_{\Sigma_i} = \varphi|_{\Sigma_i^+} - \varphi|_{\Sigma_i^-}.$$

For instance, we can think of Ω in $\mathbb{R}^3$ as a three-dimensional torus (non simply connected) or a simply connected open region interior to two concentric spheres Γ_0 of radius r_0 and Γ_1 of radius r_1 $(r_1 < r_0)$.

Traces Theorems and Green Identities

The trace operator of $H^1(\Omega)$

$\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma), \varphi \mapsto \gamma_0\varphi$ is a continuous linear surjective operator which is denoted by $\gamma_0\varphi = \varphi|_\Gamma$. The norm on $H^{\frac{1}{2}}(\Gamma)$ is given for ϕ by $\|\phi\|_{H^{\frac{1}{2}}(\Gamma)} = inf_{u \in \gamma_0^{-1}(\phi)} \|u\|_{H^1(\Omega)}$. It's topological dual is $H^{-\frac{1}{2}}(\Gamma)$, the duality product $<\ ,\ >_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}$, is denoted by $<\ ,>_\Gamma$. The norm of a functional ℓ in $H^{-\frac{1}{2}}(\Gamma)$ is given by $\|\ell\|_{H^{-\frac{1}{2}}(\Gamma)} = sup_{\|u\|_{H^1(\Omega)}=1} <\ell, u|_\Gamma>_\Gamma$.

***The normal trace in** H (**div**, Ω)*

We consider H (**div**, Ω) = $\left\{u \in L^2(\Omega)^3 : \mathbf{div}\, u \in L^2(\Omega)\right\}$ with the scalar product

$$\left(u,v\right)_{H(\mathbf{div},\Omega)} = \left(u,v\right)_{L^2(\Omega)^3} + \left(\mathbf{div}\ u, \mathbf{div}\ v\right)_{L^2(\Omega)}.$$

The normal trace γ_n is the continuous linear surjective operator

$$\gamma_n : H(\mathbf{div}, \Omega) \to H^{-\frac{1}{2}}(\Gamma)$$

which is the continuous extension of the operator γ_n: $u \mapsto u_{|\Gamma} \cdot n$ defined on $D\left(\overline{\Omega}\right)^3$, where $D\left(\overline{\Omega}\right) = \{ \phi_{|\Omega} : \phi \in D(\mathbb{R}^3)\}$.. We will denote γ_n by $u \cdot n_{|\Gamma}$.

We have $\forall \phi = \left(\phi_0, \dots, \phi_m\right) \in H^{\frac{1}{2}}(\Gamma)$,

$$< u \cdot n_{|\Gamma}, \phi >_\Gamma = < u \cdot n_{|\Gamma_0}, \phi_0 >_{\Gamma_0} + \dots + < u \cdot n_{|\Gamma_m}, \phi_m >_{\Gamma_m}.$$

Consequently, $u \cdot n_{|\Gamma} = 0 \Leftrightarrow u \cdot n_{|\Gamma_i} = 0,\ 0 \le i \le m.$

In particular,

$$< u \cdot n_{|\Gamma}, 1 >_\Gamma = < u \cdot n_{|\Gamma_0}, 1 >_{\Gamma_0} + \dots + < u \cdot n_{|\Gamma_m}, 1 >_{\Gamma_m},$$

where we are taking $\phi \equiv \in H^{\frac{1}{2}}(\Gamma)$. Usually, for all $i, 0 \le i \le m$, we denote

$$< u \cdot n_{|\Gamma_i}, \phi_{|\Gamma_i} > := \int_{\Gamma_i} u \cdot n \phi, \qquad \text{for all } \phi \in H^{\frac{1}{2}}(\Gamma).$$

Green identify in H **(div,** Ω**)**

$\forall u \in H\left(\mathbf{div}, \Omega\right), \forall \varphi \in H^1(\Omega),$

$$\left(\varphi, \mathbf{div}\, u\right)_{L^2(\Omega)} + \left(\mathbf{grad}\varphi, u\right)_{L^2(\Omega)^3} = < u \cdot n_{|\Gamma}, \varphi_{|\Gamma} >_\Gamma.$$

In particular, for $u \in H(\mathbf{div}, \Omega)$ we have

$$\int_\Omega \mathbf{div}\, u = \langle u \cdot n_{|\Gamma}, 1 \rangle_\Gamma .$$

The tangential trace in H (curl, Ω)

It's the continuous linear operator

$$\gamma_t : H(\mathbf{curl}, \Omega) \rightarrow H^{-\frac{1}{2}}(\Gamma)^3$$

which is the continuous extension of the map $u \in D(\overline{\Omega})^3 \rightarrow u \wedge n_{|\Gamma} \in D(\Gamma)^3$, where we are using the notation $\gamma_t(u) = u \wedge n_{|\Gamma}$ and

$$H(\mathbf{curl}, \Omega) = \{ u \in L^2(\Omega)^3 : \mathbf{curl}\, u \in L^2(\Omega)^3 \},$$

has the scalar product

$$(u, v)_{H(\mathbf{curl}, \Omega)} = (u, v)_{L^2(\Omega)^3} + (\mathbf{curl}\, u, \mathbf{curl}\, v)_{L^2(\Omega)^3} .$$

If $\psi \in H^{\frac{1}{2}}(\Gamma)^3$ for $\psi = (\psi_0, \ldots, \psi_m)$, then

$$\langle u \wedge n_{|\Gamma}, \psi \rangle_\Gamma = \langle u \wedge n_{|\Gamma_0}, \psi_0 \rangle_{\Gamma_0} + \ldots + \langle u \wedge n_{|\Gamma_m}, \psi_m \rangle_{\Gamma_m} .$$

Green identity in H (curl, Ω)

$$\forall u \in H(\mathbf{curl}, \Omega), \quad \forall \psi \in H^1(\Omega)^3,$$

$$(\psi, \mathbf{curl}\, u)_{L^2(\Omega)^3} - (\mathbf{curl}\, \psi, u)_{L^2(\Omega)^3} = \langle u \wedge n_{|\Gamma}, \psi_{|\Gamma} \rangle_\Gamma .$$

The Isomorphisms of the Curl Operator

Let Σ be a cut surface for Ω. The spaces **curl** $(H^1(\Omega)^3) := H^\Gamma(\mathbf{div}0; \Omega)$ and $\mathbf{curl}\left(H_0^1(\Omega)^3\right) := H_0^\Sigma(\mathbf{div}0; \Omega)$ are closed vector subspaces of $L^2(\Omega)^3$. They have the following characterization:

$$u \in H^{\Gamma}\left(\mathbf{div}0;\,\Omega\right) \Leftrightarrow \mathbf{div}\,u \;=\; 0, < u \cdot n_{|\Gamma_i}\,, 1 >_{\Gamma_i} \;= 0\left(0 \le i \le m\right)$$

and

$$u \in H_0^{\Sigma}\left(\mathbf{div}0;\Omega\right) \Leftrightarrow \mathbf{div}\,u = 0,\, u \cdot n_{|\Gamma} = 0, < u \cdot n_{|\Sigma_j}\,, 1 >_{\Sigma_j} = 0\left(1 \le j \le N\right).$$

Using the notations:

$$H_{t0}^1(\Omega)^3 = \left\{u \in H^1(\Omega)^3 : u \wedge n_{|\Gamma} = 0\right\},\, H_{n0}^1(\Omega)^3 = \left\{u \in H^1(\Omega)^3 : u \cdot n_{|\Gamma} = 0\right\}$$

we have the following

Proposition 1 *In the diagram*:

$$\begin{array}{ccc} H_{n0}^1(\Omega)^3 \cap H_0^{\Sigma}\left(\mathbf{div}0;\Omega\right) & \xrightarrow{\mathbf{curl}} & H^{\Gamma}\left(\mathbf{div}0;\Omega\right) \\ \downarrow & & \uparrow \\ H_0^{\Sigma}\left(\mathbf{div}0;\Omega\right) & \xleftarrow{\mathbf{curl}} & H_{t0}^1(\Omega)^3 \cap H^{\Gamma}\left(\mathbf{div}0;\Omega\right) \end{array}$$

the arrows **curl** *represent isomorphisms. The domains in each case are closed subspaces of* $H^1(\Omega)^3$. *The vertical arrows represent compact and dense immersions.*

3. The results

In the follows Ω is an open set and Σ is a cut surface for Ω.

Proposition 2 *Given* $f \in L^2(\Omega)^3$, *there exists an unique* $u \in H^1(\Omega)^3$ *and there exists* $p \in H^1(\Omega)$, *unique up to additive constant, such that*

$$\begin{cases} -\Delta u + \mathbf{grad}\, p = f, & \text{in } \Omega \\ \mathbf{div}\, \mathrm{u} = 0, & \text{in } \Omega \\ u \wedge n_{|\Gamma} = 0 & \\ \mathbf{curl}\, u \cdot n_{|\Gamma} = 0 & \\ < u \cdot n_{|\Gamma_i}, 1 >_{\Gamma_i} = 0, & 0 \le i \le m. \end{cases}$$

Moreover, if $f \in H(\mathbf{div};\ \Omega)$*, there exists a positive constant* c *which depends only on* Ω *such that*

$$\|u\|_{H^1(\Omega)^3} + \|p\|_{L^2(\Omega)} \le c\left\{\|f\|_{H(\mathbf{div};\Omega)} + \left\|f\cdot n_{|\Gamma}\right\|_{H^{-\frac{1}{2}}(\Gamma)}\right\}. \quad (1)$$

Proof. We can give for f an unique decomposition $f = \mathbf{grad}\ p + \mathbf{curl}\ w$ from ([2] Corollary 5') with $p \in H^1(\Omega)$ unique up to additive constant and $w \in H^1(\Omega)^3$ with $n\cdot \mathbf{curl}\, w_{|\Gamma} = 0$, unique belonging to $H_0^{\Sigma}(\mathbf{div}0;\Omega)$. Thus, $w \in H^1_{n0}(\Omega)^3 \cap H_0^{\Sigma}(\mathbf{div}0;\Omega)$.

From Proposition 1 there exists an unique $u \in H^1_{t0}(\Omega)^3 \cap H^{\Gamma}(\mathbf{div}0;\Omega)$ such that $\mathbf{curl}\ u = w$.

From this, $f = \mathbf{grad}\, p + \mathbf{curl}\ \mathbf{curl}\ u$ or, $-\Delta u + \mathbf{grad}\ p = f$ in Ω.

As a consequence of arguments in the proof, we can see that the vector field u satisfies $\mathbf{div}\ u = 0$ in Ω, $u \wedge n = 0$ on Γ and $\int_{\Gamma_i} u\cdot n d\Gamma = 0$, for $i = 0, \ldots, m$.

Again from Proposition 1 there exist positive constants c_0 and c_1 such that

$$\|u\|_{H^1(\Omega)^3} \le c_0 \|w\|_{L^2(\Omega)^3} \text{ and } \|w\|_{H^1(\Omega)^3} \le c_1 \|\mathbf{curl}\, w\|_{L^2(\Omega)^3}.$$

This imply $\|u\|_{H^1(\Omega)^3} \le c_0\, c_1 \|\mathbf{curl}\, w\|_{L^2(\Omega)^3} = c_0 c_1 \|f - \mathbf{grad}\, p\|_{L^2(\Omega)^3}$.

Then, we have

$$\|u\|_{H^1(\Omega)^3} + \|p\|_{L^2(\Omega)^3} \le c'\left\{\|f\|_{L^2(\Omega)^3} + \|p\|_{H^1(\Omega)}\right\}, \text{with } c' = max\ \{c_0 c_1,\ 1\}.$$

We have in particular

$$\begin{cases} \Delta p = \mathbf{div} f, & \text{in } \Omega \\ \dfrac{\partial p}{\partial n}_{|\Gamma} = f\cdot n_{|\Gamma} & \end{cases}$$

and by well known result about continuous dependence on initial data for Neumann problem, see ([3] Proposition 1.2).

$$\|p\|_{H^1(\Omega)} \leq c'' \left\{ \|\mathbf{div}\, f\|_{L^2(\Omega)} + \|f \cdot n_{|\Gamma}\|_{H^{-\frac{1}{2}}(\Omega)^3} \right\}.$$

From this, with $c = max\,\{c', c''\}$ we have finally

$$\|u\|_{H^1(\Omega)^3} + \|p\|_{L^2(\Omega)^3} \leq c \left\{ \|f\|_{L^2(\Omega)^3} + \|\mathbf{div}\, f\|_{L^2(\Omega)} + \|f \cdot n_{|\Gamma}\|_{H^{-\frac{1}{2}}(\Omega)^3} \right\}.$$

Remark 1 *The potential function p in the Proposition 1 can be taken in the form* $p = p_0 + p_1$ *where*

$$\Delta p_0 = \mathbf{div}\, f \ \ in\, \Omega, \qquad p_{0_{|\Gamma}} = 0$$

and

$$p_1 \in H^1(\Omega)^3, \Delta p_1 = 0 \ \ in\, \Omega \ \ and \ \ p_{1_{|\Gamma}} = constant\ (i = 0, ..., m).$$

Proposition 3 *Given* $f \in L^2(\Omega)^3$*, there exists an unique* $u \in H^1(\Omega)^3$ *and there exists an unique* $\vec{p} \in L^2(\Omega)^3$*, such that*

$$\begin{cases} -\Delta u + \vec{p} & = & f, & in\ \Omega \\ \mathbf{div}\, u & = & 0, & in\ \Omega \\ u \cdot n_{|\Gamma} & = & 0 & \\ \mathbf{curl}\, u \wedge n_{|\Gamma} & = & 0 & \\ < u \cdot n_{|\Sigma_j}, 1 >_{\Sigma_j} & = & 0, & 1 \leq j \leq N. \end{cases}$$

The vector $\vec{p}$ *has the form* $\vec{p} = \mathbf{grad}\, p + h$ *with* $p \in H^1(\Omega)$ *and* $h \in L^2(\Omega)^3$ *is a vector field satisfying*

$$\mathbf{div}\, h = 0, \ \mathbf{curl}\, h = 0 \ \text{and} \ h \cdot n_{|\Gamma} = 0.$$

Proof. First of all, we consider an arbitrary cut surface Σ for Ω. We can write the unique decomposition f = **grad** p + h + **curl** w from ([2] Corollary 6). In this decomposition, we have $p \in H^1(\Omega)$, unique up to additive constant, **curl** $h = 0$, **div** $h = 0$, $h \cdot n_{|\Gamma} = 0$, and an unique $w \in H_0^1(\Omega)^3$ with $w \wedge n_{|\Gamma} = 0$ and such that $< w \cdot n_{|\Gamma_i}, 1 >_{\Gamma_i} = 0$ for $(0 \le i \le m)$ and **div** $w = 0$.

By construction $w \in H_{t0}^1(\Omega)^3 \cap H_0^\Sigma(\mathbf{div}0;\Omega)$. Using Proposition 1 we deduce that there exists an unique $u \in H_{n0}^1(\Omega)^3 \cap H_0^\Sigma(\mathbf{div}0;\Omega)$ such that **curl** $u = w$. That is

$$f = \mathbf{grad}\, p + h + \mathbf{curl\ curl}\, u$$

or

$$-\Delta u + \mathbf{grad}\, p + h = f \quad \text{in } \Omega$$

and this u satisfies

$$\mathbf{div}\, u = 0 \text{ in } \Omega,\ u \cdot n_{|\Gamma} = 0,\ \mathbf{curl}\, u \wedge n_{|\Gamma} = 0$$

and

$$\int_{\Sigma_j} u \cdot n \, d\Sigma = 0 \qquad (j = 1, \dots, N).$$

Remark 2 *The vector field h in this Proposition is a gradient in the classical sense of a local potential q of C^∞ class on Ω_Σ (In fact $\Delta q = 0$ in Ω_Σ, in the classical sense). We have h =* **grad** *q* with *$q \in H^1(\Omega_\Sigma)$ ($q \notin H^1(\Omega)$) solution of the transmission problem*

$$\begin{cases} \Delta q = 0, & \text{in } \Omega_\Sigma \\ \dfrac{\partial q}{\partial n}_{|\Gamma} = 0 & \\ [q]_{\Sigma_i} = \textit{constant}, & i = 1, \dots, N \\ \left[\dfrac{\partial q}{\partial n}\right]_{\Sigma_i} = 0, & i = 1, \dots, N. \end{cases}$$

For more of details, see for instance ([2] Proposition 2).

Now we suppose Ω simply connected. Next result follows immediately from Propositions 2 and 3.

Corollary 1 *Given* $f \in L^2(\Omega)^3$, *there exists an unique* $u \in H^1(\Omega)^3$ *and there exists* $p \in H^1(\Omega)$, *unique up to additive constant, such that*

$$\begin{cases} -\Delta u + \mathbf{grad}\, p &= f, \quad \text{in } \Omega \\ \mathbf{div}\, u &= 0, \quad \text{in } \Omega \\ u \cdot n_{|\Gamma} &= 0, \\ \mathbf{curl}\, u \wedge n_{|\Gamma} &= 0. \end{cases}$$

Moreover, if $f \in H(\mathbf{div};\, \Omega)$, *there exists a positive constant* c *which depends only on* Ω *such that*

$$\|u\|_{H^1(\Omega)^3} + \|p\|_{H^1(\Omega)} \leq c\left\{\|f\|_{H(\mathbf{div};\,\Omega)} + \left\|f \cdot n_{|\Gamma}\right\|_{H^{-\frac{1}{2}}(\Gamma)}\right\}.$$

4. Conclusion

The solutions for these problems depend on the topology of the open set Ω. For instance, as Proposition 3 shows, if Ω is not simply connected, the $\vec{p}$ vector field corresponding to the solution of the Stokes problem having only tangential component on the boundary, is not a global gradient in Ω.

5. References

[1] Bossavit, A. *Les deux isomorphismes du rotationnel et les deux formes du problème de la magnétostatique dans un domaine borné.* EDF-Bulletin de la Direction des Études et Recherches, Série C, Mathematiques, Informatique, N° 1, pp 5-20, 1986.

[2] Cessenat, M. Chap. 9(*Exemples en Electromagnétisme et en Physique Quantique) in Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques.* (R. Dautray et J-L. Lions, eds), Masson (Paris), 1985.

[3] Girault, V., Raviart, P-A. *Finite Element Methods for Navier Stokes Equations, Theory and Algorithms.* Springer -Verlag, 1986.

[4] Teman, R. *Navier-Stokes Equations, Theory and Numerical Analysis.* North-Holland, 1979.

[5] Weyl, H. *The method of orthogonal projection in potential theory.* Duke Math. J., 7, pp 411-444, 1940.

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