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MAXIMA AND MINIMA BEFORE CALCULUS

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Introduction

The need for a general method for finding maxima and minima of functions was one of the main driving forces of the development of Calculus. Apparently, many textbooks authors have inferred from this that maximization and minimization cannot be treated at all at the precalculus level. Very few precalculus textbooks present such problems, even in an unsystematic manner. This is a pity; the pursuit of this goal should start well before any calculus course.

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Outside the classroom, everybody faces situations in which one has to optimize results while satisfying certain constraints ("build a box of maximum capacity with a given amount of material", to give a schematic example). However, optimization is almost never treated at the precalculus level or before, and all attention is given to computation and root-solving, and to applications which are but lightly disguised versions of these. Such an approach not only bores the students, but also misleads them; they acquire an unnecessarily restricted idea of the range of applicability of mathematics.

Quite often, maxima and minima can be found through the use of classical inequalities, e.g., the inequality between the arithmetic and geometric means, supplemented with some ad-hoc machinery. These techniques can be put within the reach of precalculus students, and, while they may suffer from a certain lack of generality and systematicity, this very lack has the virtue of spurring the inventiveness of students. Nevertheless, it is a lack, hence we will provide an alternative. We will show how to introduce more general techniques that foreshadow some of the concepts of differential calculus, without using the concept of limit, much less that of derivative. Moreover, we will show how students can use calculators to facilitate the use of these techniques, and even to discover them under guidance.

Some of our treatment follows the same lines as the methods of Pierre de Fermat (1601-1665), which antecede Calculus (Andersen 1983). The present work is thus one more example of the usefulness of a partially genetic approach, which exploits the fact that the basic outline of historical mathematical developments is often one of the most natural paths in the learning process.

1. The Tent Problem

Consider prism-like tents whose vertical walls are isosceles triangles (Figure 1). What should be the angle at the vertex of the triangle for the tent to enclose the maximum volume? Since the length of the tent is fixed, we should try to maximize the area A of the triangle. Many students would undoubtedly start dealing with a particular case, for instance an isosceles

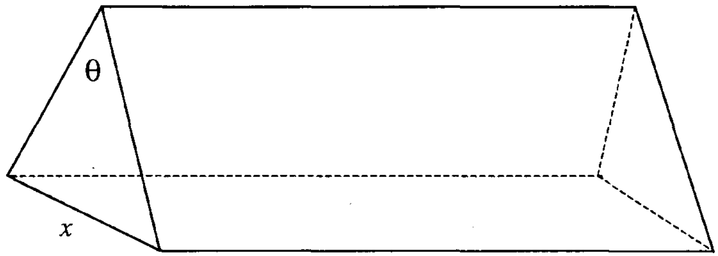

Figure 1

triangle with equal sides 3 m. long. In this case $A(x) = \frac{x}{2}\sqrt{9-\frac{x^2}{4}}$ is the area of the triangle as a function of the bottom side. Using a graphics calculator (say, a TI 82, 83, or 85) one can graph $A(x)$ and use the *Calc* command to find the approximate point where it attains its maximum, namely 4.2426424[1]. (Figure 2). Then $\theta = 2\sin^{-1}\left(\frac{2.1215}{3}\right) = 90.00004626$ degrees. Changing once or twice the length of the equal sides of the triangle we arrive at a value very close to 90° for θ. The next step is to develop a short program in order to handle particular cases with great speed.

Program Tent (TI-82/TI-83/TI-85)
: Input "LENGTH?", L
: fMax $\left(0.5x\sqrt{\left(L^2 - 0.25x^2\right)}, x, 0, 2L\right) \rightarrow R$
$:\sin^{-1}\left(\frac{R}{2L}\right) \rightarrow S$
: Clrhome
: Output (1, 1, 2S)

[1] *This is an approximation. It may vary slightly depending on calculator settings*

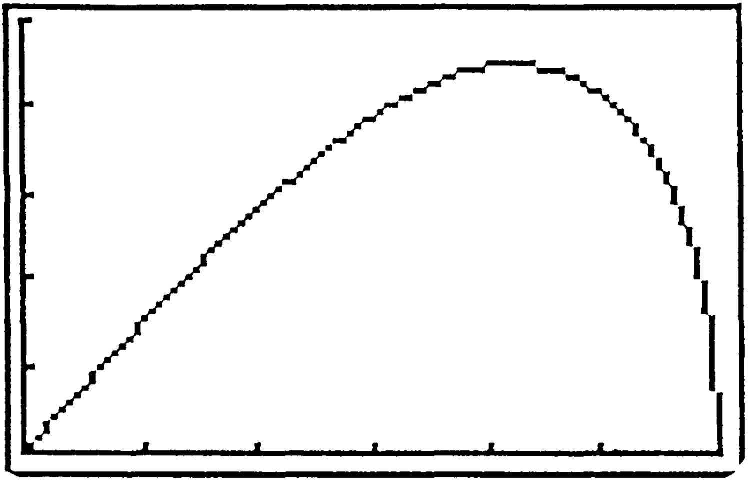

Figure 2

Note: The fMax command requires the specification of a right bound. Since $L^2 - 0.25x^2 > 0$ we get $4L^2 > x^2$, i.e. $x < 2L$. So, we can safely assume the value $2L$ for it. All the other details of the program are self-explanatory.

Once students understand how the program is built, they can load it in their calculator and then check that for any conceivable value of L the output is always a number very close to 90°. Thus we conjecture that the answer to the problem is to build the tent with a right angle at the top. Can we find a mathematical proof of this fact? Let us complete of squares:

$$\begin{aligned} A^2 &= \frac{x^2}{4}\left(L^2 - \frac{x^2}{4}\right) \\ &= -\frac{1}{16}\left(x^4 - 4L^2x^2\right) \\ &= -\frac{1}{16}\left(\left(x^2 - 2L^2\right)^2 - 4L^4\right) \\ &= -\frac{1}{16}\left(x^2 - 2L^2\right)^2 + \frac{L^4}{4}. \end{aligned}$$

This expression has a maximum value for $x^2 = 2L^2$, i.e., for $x = \sqrt{2}L$. Clearly, the value of x that maximizes $A^2(x)$ also maximizes $A(x)$. Thus, if θ is the angle between the equal sides that makes the area maximal, $sin\left(\frac{\theta}{2}\right) = \frac{\frac{\sqrt{2}}{2}L}{L} = \frac{\sqrt{2}}{2}$. Hence $\frac{\theta}{2} = sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$, i.e., $\theta = 90°$, as we wished to prove.

An even simpler proof goes as follows: drop a perpendicular from one of the vertices at the base onto the opposite side of length L. Then $A = \frac{Lh}{2}$, where h is the length of the perpendicular. But $sin\,\theta = \frac{h}{L}$. So $A(\theta) = 0.5\ L^2\ sin\ \theta$. Since the sine of an angle attains its maximum value (namely 1) when the angle equals 90°, we can assert that $A(\theta)$ will reach its maximum when θ, the angle at the top, is a right angle.

Due to the simplicity of both the algebraic and geometric proofs, it may not be clear why we spent our time building a program. We did so as a way of illustrating a methodology to be used later. In the next two problems, a straight proof-oriented approach may not be the best path to follow at the beginning. Testing a conjecture through a program is a sensible first step in the search for a mathematical proof.

2. The Cylinder with Minimum Surface Area

Given a right cylinder with fixed volume V, what should be the height and the radius of the base for the lateral surface S to be minimal? We have $V = \pi x^2 y$, and $S = 2\pi x^2 + 2\pi xy$ where x is the radius and y the height. Thus, we have to minimize the function $S(x) = 2\pi x^2 + 2V/x$. Let us start by considering a particular case, namely $V = 100$. Using a graphics calculator we can graph the function $S(x)$ and find the point where it attains its minimum, namely $x = 2.515398$ (Figure 3). Then $S(x) = 100/(\pi \cdot 2.515398^2) = 5.030795975$, approximately twice the value of the corresponding x. Is this true in general? The following program will help us answer this question.

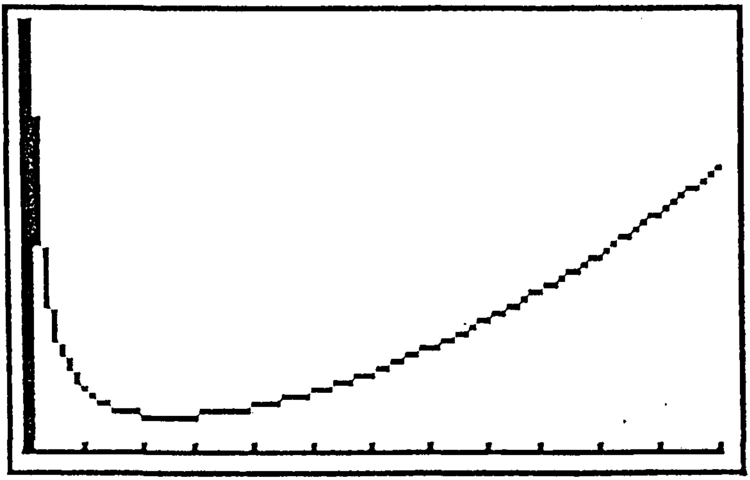

Figure 3

Program MinSurf (on a TI-82/TI-83/TI-85)

```
: Input "Volume?", V
: Input "Right?", K
: fMin (2πx² + 2V)/x, x, 0, K) → R
: V/(π R²) → T
: Clrhome
: Output (1, 1, 2R)
: Output (2, 2, T)
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For example, if $V = 35$ and $K = 10$, the program outputs $2R = 3.545355116$ and $T = 3.545346432$. The parameter K has to be chosen sufficiently large for a given V. Otherwise the *fMin* command will not do its job properly since the minimum of $S(x)$ could be attained outside the interval $[0, K]$. (Thus $K = 2$, $V = 190$ give the spurious output $2R = 3.999992331$, $T = 15.11977757$). How large does K have to be for a given V?

We know K is a safe choice if the function S is strictly increasing on the interval $[K, \infty)$. For all x, $\delta > 0$ we have:

$$\begin{aligned} S(x+\delta)-S(x) &= 2\pi(x+\delta)^2+\frac{2V}{x+\delta}-\left(2\pi x^2+\frac{2V}{x}\right) \\ &= 2\pi\left(2\delta x+\delta^2\right)-\frac{2V\delta}{x^2+x\delta} \\ &> 4\pi\delta x-\frac{2V\delta}{x^2} \\ &= \frac{2\delta}{x^2}\left(2\pi x^3-V\right). \end{aligned}$$

So $S(x + \delta) - S(x) > 0$ for $x \geq \left(\frac{V}{2\pi}\right)^{1/3}$. In other words, S is strictly increasing on the interval $[\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}, \infty)$. Hence $\left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$ or any greater number can be safely chosen for K.

Students can execute the program for many pairs (V, K) and realize that $2R$ is practically equal to T. We must now find a mathematical proof of this fact. Guided by the preceding argument and by the graph of $S(x)$ for $V = 100$, it is natural to try to show that $S(x)$ is strictly decreasing on the interval $(0, \left(\frac{V}{2\pi}\right)^{1/3})$. Thus, we will try to prove that $S(x - \delta) - S(x) > 0$ for $x < \left(\frac{V}{2\pi}\right)^{1/3}$ and any $0 < \delta < x$. We have

$$\begin{aligned} S(x-\delta)-S(x) &= 2\pi(x-\delta)^2+\frac{2V}{x-\delta}-\left(2\pi x^2+\frac{2V}{x}\right) \\ &= 2\pi\left(-2\delta x+\delta^2\right)+\frac{2V\delta}{x^2-\delta x} \\ &> 2\pi\left(-2\delta x+\delta^2\right)+\frac{2V\delta}{x^2} \\ &> -4\pi\delta x+\frac{2V\delta}{x^2} \end{aligned}$$

$$= \left(-2\pi x^3 + V\right)\frac{2\delta}{x^2}.$$

This last expression is bigger than zero provided that $x < \left(\frac{V}{2\pi}\right)^{1/3}$. Since $S(x)$ is strictly decreasing on $(0, \left(\frac{V}{2\pi}\right)^{1/3})$ and strictly increasing on $(\left(\frac{V}{2\pi}\right)^{1/3}, \infty)$, we can conclude that $S(x)$ attains its minimum at $\left(\frac{V}{2\pi}\right)^{1/3}$. As expected, at the point $x = \left(\frac{V}{2\pi}\right)^{1/3}$ (where $S(x)$ attains its minimum) we have $y = 2x$. Notice how the need to run the program properly led to a proof of the minimization problem at hand. The effort involved in working with technology has provided us with a proof, a possibility that is often overlooked by students. Of course, the distinction between experimentation –as exemplified by the graphics calculator program– and a valid mathematical proof must be made clear to the students, as Hung-Hsi Wu quite forcefully argues (Wu 1994).

Another, radically different proof makes use of the inequality between the geometric and arithmetic means of positive numbers:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$. There is a proof of this inequality for the special case $n = 3$ at the reach of precalculus students (Beckenbach and Bellman 1961).

We have

$$\left(2\pi x^2 \cdot \pi xy \cdot \pi xy\right)^{1/3} \leq \frac{1}{3}\left(2\pi x^2 + \pi xy + \pi xy\right)$$

for any radius of length x and any height of length y, with equality holding true if and only if $2\pi x^2 = \pi xy$, i.e. $2x = y$. That is to say, $3\left(2\pi V^2\right)^{1/3} \leq S$. Furthermore, S attains its minimum value if and only if $y = 2x$, in which case

$x = \left(\dfrac{V}{2\pi}\right)^{1/3}$. This proof is shorter because we are using a powerful inequality. However, it is not as "natural" as the preceding proof because at first sight it may not be obvious how to choose the three positive numbers when applying the inequality between the geometric and arithmetic means.

3. The Rectangular Box with Maximum Volume

Suppose that we have a rectangular closed box whose base is a square. The surface area of the box is a given number L. What are the dimensions of the box of maximum volume with the given lateral surface? Let $L = 50$, and denote by x the side of the base and by y the height of the box. Then $4xy + 2x^2 = 50$. Hence $y = \dfrac{50-2x^2}{4x}$. The volume V can thus be written as

$$V(x) = x^2 \cdot \frac{50-2x^2}{4x} = \frac{x}{4}\left(50-2x^2\right).$$

Using a graphics calculator one can obtain a graph (figure 4) and then use the *Calc* command in order to find the approximate point where $V(x)$ attains its maximum. The calculator will give as an answer $x = 2.8867524$ or thereabouts. A simple calculation leads to $y = \dfrac{50-2\cdot(2.8867524)^2}{4\cdot(2.8867524)} =$ 2.886749238. In other words, the cube with side of length around 2.8867 has the maximum volume among all closed boxes with surface area $L = 50$. A short program will allow us to test whether the general answer is a cube.

Program MaxVol (TI-82/TI-83/TI-85)

: Input "Surface?", L

: fMax $\left(0.25x\left(L-2x^2\right), x, 0, \sqrt{L/2}\right) \rightarrow R$

: $(L - 2R^2) / 4R \rightarrow S$

: Clrhome

: Output (1, 1, R)

: Output (2, 2, S)

Note : Since $L - 2x^2 > 0$ we must have the inequality $x < \sqrt{L/2}$. That is why we chose $\sqrt{L/2}$ as the right bound in the fMax command.

If $L = 5$ we will get on the screen the two numbers 0.9128693155 and 0.9128741565. These two numbers are almost equal, as predicted by our conjecture. By experimenting with many different values of L we conjecture that the height of the box of maximum volume should be equal to the length of the side of its base. Since $2x^2 + 4xy = L$ and y would be equal to x, we would get $x = \sqrt{\frac{L}{6}}$ as the common side of the cube.

Is it possible to find an acceptable mathematical proof that does not use Calculus? We could use the same method as in the previous section, that is, prove that $V(x)$ is strictly increasing on $\left[0, \sqrt{\frac{L}{6}}\right]$ and strictly decreasing on $\left[\sqrt{\frac{L}{6}}, \sqrt{\frac{L}{2}}\right]$. In other words, we could prove that

$$V\left(x+\delta\right)-V(x)<0 \quad \text{for} \quad x\geq\sqrt{\frac{L}{6}}, \delta>0, x+\delta\leq\sqrt{\frac{L}{2}}$$

$$V\left(x-\delta\right)-V(x)<0 \quad \text{for} \quad x\leq\sqrt{\frac{L}{6}}, \delta>0, x-\delta\geq 0.$$

In fact, it suffices to prove that

$$V\left(\sqrt{\frac{L}{6}}+\delta\right)-V(x)<0 \qquad \text{for} \quad \delta>0, \sqrt{\frac{L}{6}}+\delta\leq\sqrt{\frac{L}{2}}$$

$$V\left(\sqrt{\frac{L}{6}}-\delta\right)-V(x)<0 \qquad \text{for} \quad \delta>0, \sqrt{\frac{L}{6}}+\delta\geq 0.$$

This is just another way of stating that V takes its maximum at $\sqrt{\frac{L}{6}}$.

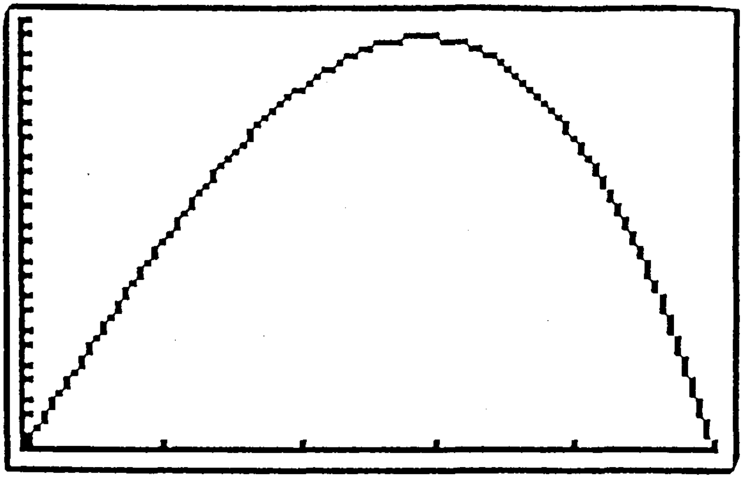

Figure 4

We have

$$V\left(\sqrt{\frac{L}{6}}-\delta\right)-V\left(\sqrt{\frac{L}{6}}\right)=\frac{1}{4}\left(\sqrt{\frac{L}{6}}-\delta\right)\left(L-2\left(\sqrt{\frac{L}{6}}-\delta\right)^2\right)-$$

$$\frac{1}{4}\sqrt{\frac{L}{6}}\left(L-2\left(\sqrt{\frac{L}{6}}\right)^2\right)$$

$$=\frac{1}{4}\left(\sqrt{\frac{L}{6}}-\delta\right)\left(L-2\left(\frac{L}{6}-2\delta\sqrt{\frac{L}{6}}+\delta^2\right)\right)-$$

$$\frac{1}{4}\sqrt{\frac{L}{6}}\left(L-2\left(\sqrt{\frac{L}{6}}\right)^2\right)$$

$$=\frac{1}{4}(-\delta)\left(L-2\left(\frac{L}{6}-2\delta\sqrt{\frac{L}{6}}+\delta^2\right)\right)+$$

$$\frac{1}{4}\sqrt{\frac{L}{6}}\left(4\delta\sqrt{\frac{L}{6}}-2\delta^2\right)$$

$$=\left(-\frac{L}{4}+\frac{L}{12}+\frac{L}{6}\right)\delta-\frac{3}{2}\sqrt{\frac{L}{6}}\delta^2+\frac{1}{2}\delta^3$$
$$=-\frac{3}{2}\sqrt{\frac{L}{6}}\delta^2+\frac{1}{2}\delta^3.$$

Since $\delta\leq\sqrt{\frac{L}{6}}, \sqrt{\frac{L}{6}}\delta^2\geq\delta^3$, and therefore $-\frac{3}{2}\sqrt{\frac{L}{6}}\delta^2+\frac{1}{2}\delta^3<0$. By the same token,

$$V\left(\sqrt{\frac{L}{6}}+\delta\right)-V\left(\sqrt{\frac{L}{6}}\right) = \frac{1}{4}\left(\sqrt{\frac{L}{6}}+\delta\right)\left(L-2\left(\sqrt{\frac{L}{6}}+\delta\right)^2\right)$$
$$-\frac{1}{4}\sqrt{\frac{L}{6}}\left(L-2\left(\sqrt{\frac{L}{6}}\right)^2\right).$$
$$=\left(-\frac{L}{4}+\frac{L}{12}+\frac{L}{6}\right)(-\delta)-$$
$$\frac{3}{2}\sqrt{\frac{L}{6}}(-\delta)^2+\frac{1}{2}(-\delta)^3$$
$$=-\frac{3}{2}\sqrt{\frac{L}{6}}\delta^2-\frac{1}{2}\delta^3.$$

Since $\delta > 0$, this is less than zero, as was to be proven. Hence the maximum is attained at $x = \sqrt{\frac{L}{6}}$. Notice that this proof would have been difficult to find if our experiments had not made us suspect the solution: we had to know to examine $V\left(\sqrt{\frac{L}{6}}+\delta\right)-V\left(\sqrt{\frac{L}{6}}\right)$ and not, say, $V(2+\delta) - V(2)$ or $V(\pi+\delta)-V(\pi)$.

There is an alternative proof, using the inequality between arithmetic and geometric means:

$$V^2 = \frac{1}{16}x^2\left(L-2x^2\right)$$

$$
\begin{aligned}
&= x^2\left(\frac{L}{4}-\frac{x^2}{2}\right)\left(\frac{L}{4}-\frac{x^2}{2}\right) \\
&\leq \left(\left(\frac{1}{3}\right)\left(x^2+\frac{L}{4}-\frac{x^2}{2}+\frac{L}{4}-\frac{x^2}{2}\right)\right)^3 \\
&= \frac{1}{27}\left(\frac{L^3}{8}\right).
\end{aligned}
$$

Thus, V^2 will attain its maximum when $x^2 = L/4 - x^2/2$, i.e., at $x=\sqrt{L/6}$. Of course, this is the same point where V will attain its maximum. For this value of x we get $y=\left(L-2x^2\right)/4x=\sqrt{L/6}$.

The more general problem of the maximum volume of a rectangular box when the lengths of the bottom sides are not necessarily equal is posed at the end as an activity for the students.

4. The Cylinder with Maximum Volume

Given a right cylinder with fixed lateral surface S, what should be the dimensions of the height and radius of the base for volume V to be maximal? This problem is closely related to the second one we presented, in which the volume was held fixed and the surface area was minimized; nevertheless, it must be solved separately.

We have to maximize the function

$$
\begin{aligned}
V(x) &= \pi x^2\left(\frac{S-2\pi x^2}{2\pi x}\right) \\
&= \frac{x}{2}\left(S-2\pi x^2\right).
\end{aligned}
$$

Proceeding as we did before, we have that, if V attains its maximum at $x = x_0$, then

$$V\left(x_0+\delta\right)-V\left(x_0\right)=\frac{x_0+\delta}{2}\left(S-2\pi\left(x_0+\delta\right)^2\right)-\frac{x_0}{2}\left(S-2\pi x_0^2\right)$$
$$=\frac{\delta}{2}\left(S-2\pi\left(x_0+\delta\right)^2\right)+\frac{x_0}{2}\left(-2\pi\left(2\delta x_0+\delta^2\right)\right) \qquad (1)$$
$$=\left(-3\pi x_0^2+\frac{S}{2}\right)\delta+\left(-3\pi x_0\right)\delta^2-\pi\delta^3$$

must be negative for $\delta \neq 0$, $0 \leq x_0+\delta \leq \sqrt{\frac{S}{2\pi}}$.

It follows immediately that for $\delta > 0$,

$$V\left(x_0+\delta\right)-V\left(x_0\right)<\left(-3\pi x_0^2+\frac{S}{2}\right)\delta,$$

so $V\left(x_0+\delta\right)-V\left(x_0\right)<0$ provided that $-3\pi x_0^2+\frac{S}{2}<0$, i.e., $V\left(x_0+\delta\right)-V\left(x_0\right)<0$ if $x_0>\sqrt{\frac{S}{6\pi}}$. However, it is not quite so easy to prove that $V\left(x_0-\delta\right)-V\left(x_0\right)<0$ when $\delta > 0$ and $x_0<\sqrt{\frac{S}{6\pi}}$. We will adopt a different approach to the problem. We notice that when $x_0=\sqrt{\frac{S}{6\pi}}$ the coefficient of δ in

$$\left(-3\pi x_0^2+\frac{S}{2}\right)\delta+\left(-3\pi x_0\right)\delta^2-\pi\delta^3$$

becomes zero. This is not an isolated case, as the reader may check in the problem of the rectangular box with maximum volume discussed before; the coefficient of δ is zero there, too. We have the following general result:

Theorem 1. *Let $f(x)$ be a polynomial in x such that, for any $\delta \neq 0$, $f\left(x+\delta\right)-f(x)=a_1(x)\delta+a_2(x)\delta^2+\cdots+a_n(x)\delta^n$. If $a < x_0 < b$ is such that $a_1(x_0) \neq 0$, then there are $a < x_1, x_2 < b$ such that $f(x_1) < f(x_0) < f(x_2)$. Thus f attains neither a minimum nor a maximum at x_0.*

Proof. Choose[2] $\delta \neq 0$ such that $|\delta| < 1$ and $|\delta| < \dfrac{|a_1(x_0)|}{|a_2(x_0)| + \cdots + |a_n(x_0)|}$.

Then

$$\begin{aligned} |a_2(x_0)\delta^2 &+ a_3(x_0)\delta^3 + \ldots + a_n(x_0)\delta^n| \\ &\leq |a_2(x_0)||\delta|^2 + |a_3(x_0||\delta|^3 + \ldots + a_n(x_0)||\delta|^n \\ &< |a_1(x_0)||\delta|. \end{aligned}$$

Consequently[3]

$$a_1(x_0)\,\delta + a_2(x_0)\delta^2 + \ldots + a_n(x_0)\,\delta^n$$

and $a_1(x_0)\delta$ have the same sign. Since $|-\delta| = |\delta|$, we can conclude similarly that

$$a_1(x_0)\,(-\delta) + a_2(x_0)(-\delta)^2 + \ldots + a_n(x_0)\,(-\delta)^n$$

and $a_1(x_0)$ $(-\delta)$ have the same sign. But $a_1(x_0)\delta$ and $a_1(x_0)$ $(-\delta)$ obviously have opposite signs. Hence $f(x_0 + \delta) - f(x_0)$ and $f(x_0 - \delta) - f(x_0)$ have opposite signs. In other words,

$$f(x_0 + \delta) - f(x_0) > 0 \text{ and } f(x_0 - \delta) - f(x_0) < 0 \text{ or}$$
$$f(x_0 + \delta) - f(x_0) < 0 \text{ and } f(x_0 - \delta) - f(x_0) > 0.$$

Thus $f(x_0 - \delta) < f(x_0) < f(x_0 + \delta)$ or $f(x_0 + \delta) < f(x_0) < f(x_0 - \delta)$. (If either $x_0 - \delta$ or $x_0 + \delta$ lie outside (a, b), we need only make δ smaller). Finally, we choose $x_1 = x_0 - \delta$ and $x_2 = x_0 + \delta$ or $x_1 = x_0 + \delta$ and $x_2 = x_0 - \delta$. □

Thus, when we search for the maximum or the minimum of a polynomial in $[a, b]$, we need not to look at any points except a, b, and those x in (a, b) for which $a_1(x) = 0$. There are thus only a finite number of points where the values of f have to be compared. (That a continuous function on a

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2 We are assuming that $|a_2(x_0)| + \ldots + |a_n(x_0)| > 0$. If $a_2(x_0) = \ldots = a_n(x_0) = 0$, then $f(x_0 + \delta) - f(x_0) = a_1(x_0)\delta$. Let us recall that $a_1(x_0) \neq 0$. Then it is very easy to show that f attains neither a minimum nor a maximum at x_0.

3 We are using the elementary fact that for any non-zero numbers r, s, if $|r| > |s|$ then $r + s$ and r have the same sign.

closed interval $[a, b]$ has a maximum and a minimum can be taken to be intuitively clear.) Thus we have reduced the problem of finding the maximum (minimum) of a polynomial to the problem of finding the roots of a polynomial.

Let us finish the problem of the cylinder. The maximum must be attained at 0, at $\sqrt{\frac{S}{2\pi}}$ or at one of the roots of $-3\pi x_0^2+\frac{S}{2}=0$, that is, at $\sqrt{\frac{S}{6\pi}}$. (We eliminate $-\sqrt{\frac{S}{6\pi}}$ for being out of $\left[0,\sqrt{\frac{S}{2\pi}}\right]$.) Since $V(0) = 0$, $V\left(\sqrt{\frac{S}{2\pi}}\right)=0$ and $V\left(\sqrt{\frac{S}{6\pi}}\right)=\frac{S}{3}\sqrt{\frac{S}{6\pi}}$, we conclude that V attains its maximum at $\sqrt{\frac{S}{6\pi}}$.

Let us now reexamine the tent problem. We want to maximize the function $f(x)=0.5x\sqrt{\left(L^2-0.25\cdot x^2\right)}$, or, what is the same, maximize its square $0.25x^2(L^2-0.25\cdot x^2)$, in the interval $[0, 2L]$. We have

$$\begin{aligned} f^2(x+\delta)-f^2(x) &= \tfrac{1}{4}(x+\delta)^2\left(L^2-\tfrac{1}{4}\cdot(x+\delta)^2\right)-\tfrac{1}{4}x^2\left(L^2-\tfrac{1}{4}x^2\right) \\ &= \tfrac{1}{2}\left(L^2-0.5x^2\right)x\delta+\cdots, \end{aligned}$$

where we have not bothered to write down the coefficients of δ^2, δ^3 and δ^4, as they are of no relevance to our method. The only root of $\frac{1}{2}\left(L^2-0.5x^2\right)$ within $[0, 2L]$ is $x=\sqrt{2}\,L$; comparing $f^2(0)$, $f^2(2L)$ and $f^2\left(\sqrt{2}\,L\right)$, we find that maximum is attained at $\sqrt{2}\,L$, as we had already determined by ad-hoc methods. Thus, problems that before seemed completely disparate can now all be solved in the same way. Notice also that the need to explore maxima and minima beforehand with the calculator has disappeared for polynomial functions. We can find the maximum, not merely confirm it.

5. The Limitations of the δ – Method

In all the problems that we have discussed so far, the δ–method can be used to present a mathematical proof. However, when the function involves the sum of two square roots, the δ–method becomes unmanageable. The derivation of the law of refraction from Fermat's principle of least time is a case in point. Suppose a ray of light travels from a point (N, O) in air to a point (P, Q) in water. According to Fermat's principle of least time, the path chosen by the ray is such that the time to traverse it is the smallest. So, we have to find the point where the function.

$$f(x)=\frac{1}{30}\sqrt{(x-N)^2+O^2}+\frac{1}{22.5}\sqrt{(P-x)^2+Q^2}$$

attains its minimum (the velocity of light in air is 30 cm/nanoseconds, while the velocity of light in water is 22.5 cm/nanoseconds; one nanosecond is 10^{-9} seconds). The problem is to derive the law of refraction, namely $\frac{\sin\alpha}{\sin\beta}=\frac{30}{22.5}$ where α is the angle of incidence and β is the angle of refraction. After strenous effort, Fermat succeeded in proving the derivation by a procedure similar to the δ–method. His proof is long and complicated (Sabra 1981). Modern available elementary (non-calculus) proofs can hardly be considered simple (e.g. Schiffer and Bowden 1984, Niven 1981). A graphics calculator program may then play part of the role of a proof as a convincing empirical argument (Helfgott 1998).

Another example is the derivation of the law of reflection for parabolic mirrors from Fermat's principle of least time. Given two points (N, O) and (P, Q) inside a parabolic mirror described by the equation $y = x^2$, we have to find the point where the function

$$g(x)=\sqrt{(x-N)^2+(x^2-O)^2}+\sqrt{(x-P)^2+(x^2-Q)^2}$$

attains a local minimum. Again the δ–method cannot be applied with ease. Calculations become cumbersome and lengthy. A better alternative is to provide an empirical justification through a graphics calculator program and then wait for a calculus course wherein a proof can be given for any "smooth" mirror (Helfgott and Simonsen 1998).

In both these problems, the graphics calculator takes the center stage due to the lack of a mathematical proof that is both simple and elementary.

Writing a correct program requires - and thus leads to - understanding of the task at hand, and sometimes can give the basic idea for a rigorous solution later on.

6. Activities for Students

6.1 Activity 1. Among all rectangles with fixed area, which one has the smallest perimeter? Students should start with a particular value for the area, graph the perimeter function $p(x)$, and then find the point where it attains its minimum. This process will allow them to make a conjecture. Most students will be able to conjecture that the answer is the square. Thereafter they should write a program that works for any area value:

: Input "Area?", A
: Input "Right?", K
: fMin $\left(2x+\frac{2A}{x}, x, 0, K\right) \rightarrow R$
: Clrhome
: Output (1, 1, R)
: Output (2, 1, A/R)

What right bound K should we choose for a given input A? We have

$$p(x+\delta)-p(x)=\frac{2\delta x^2+2\delta^2 x-2A\delta}{x(x+\delta)}>\frac{2\delta(x^2-A)}{x(x+\delta)}.$$

Thus $p(x+\delta)-p(x)>0$ provided that $x^2-A>0$, i.e., $x>\sqrt{A}$. So, $p(x)$ is strictly increasing on $[\sqrt{A}, \infty)$. Consequently, we can safely choose any value bigger than $\sqrt{A}$ for K. Say $K=2\sqrt{A}$.

Once students run the program several times, checking that both outputs are practically the same, all that is left to do is prove that $p(x-\delta)-p(x)>0$ for $x<\sqrt{A}$. For this purpose we notice that

$$p(x-\delta)-p(x)=\frac{-2\delta x^2+2\delta^2 x+2A\delta}{x(x-\delta)}$$

$$> \frac{2\delta\left(-x^2+A\right)}{x^2}.$$

Hence the perimeter function attains its minimum at $x = \sqrt{A}$. Obviously $y=\frac{A}{\sqrt{A}}=\sqrt{A}$, where y is the other side of the rectangle. Thus we reach a square as the answer!

6.2. Activity 2. Given a closed box with non-square base, show that there exists a closed square-based box with the same lateral surface and larger volume. In other words, prove that given positive numbers x, y, z ($x \neq y$) and lateral area $2L$, i.e. $xy + yz + zx = L$, there exist positive numbers u, v, w, with $u = v$, such that $uv + vw + wu = L$ and $uvw > xyz$. This result will play a crucial role in the next activity.

Most students will need a hint in order to develop a proof. The choice $u = v = \sqrt{xy}$, $w=\frac{L-uv}{u+v}$ works. (The value of w is determined by the fact that $uv + (u + v)w$ has to be equal to L.)

6.3. Activity 3. This activity deals with problem of the maximum volume of a rectangular box when the lengths of the bottom sides are not necessarily equal. Students are asked to prove that if A is a non-cubic box with given lateral surface and B is the cubic box with the same lateral surface, then $V(B) > V(A)$, where V stands for the volume function.

Probably the instructor will have to set the path to be followed. If A has a square base, then, as we showed in Section 3, $V(B) > V(A)$. So we should only worry about the case when A has a non-square base. By the result shown in the previous activity, there exists a closed box C with square base and the same lateral surface as A such that $V(C) > V(A)$. Then consider the two possibilities, namely $C = B$ and $C \neq B$. For the latter case, the result obtained in Section 3 leads to $V(B) > V(C)$. So $V(B) > V(A)$.

The problem of finding the dimensions of the box of maximum volume with given surface area could be approached starting with the general case from the very beginning. This is the approach used in some works (Polya 1954, Beckenbach and Bellman 1961). Students can profit by analyzing this type of solution, an interesting activity in its own right. However, it seems more natural to start considering the particular case of boxes with square bases. An experimental approach to the general case, i.e., writing a program, involves many steps because we would be discussing an optimization problem in two variables. Writing such a program may well be beyond the reach of students who have not done some programming before; for others, it would be a beneficial experience, making them conscious of the great computational costs of solving optimization problems in two variables by brute force. This should convince them of the need for the techniques here developed - if they are not convinced yet. All students should be conscious that tools such as *fMax* are not magic spells, and that, for complex problems, they may be utterly impractical.

6.4. Activity 4. Among all the rectangles inscribed in a circle, which one encloses the greatest area? Let us start with a circle of radius 1 and draw the coordinate axes with center at the origin (Figure 5). Denoting by (x, y) the vertex of the rectangle that lies in the first quadrant, the problem is to maximize the area function $4xy$ subject to the constraint $x^2 + y^2 = 1$. Thus, we have to find where the function $A(x) = 4x\sqrt{1-x^2}$ attains its maximum. This problem is quite similar to the Tent Problem. Working with A^2 instead of A, we can complete squares to get the answer $x = \frac{\sqrt{2}}{2}$. Then

$$y=\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}=\frac{\sqrt{2}}{2}.$$

Thus the answer is the square. Next we can work with a circle of any radius L. Instead of $A(x) = 4x\sqrt{1-x^2}$ we will have to deal with $A(x) = 4x\sqrt{L^2-x^2}$. Once again, we can complete squares in the expression $16x^2(L^2 - x^2)$. As expected we get $x=\frac{L\sqrt{2}}{2}$,

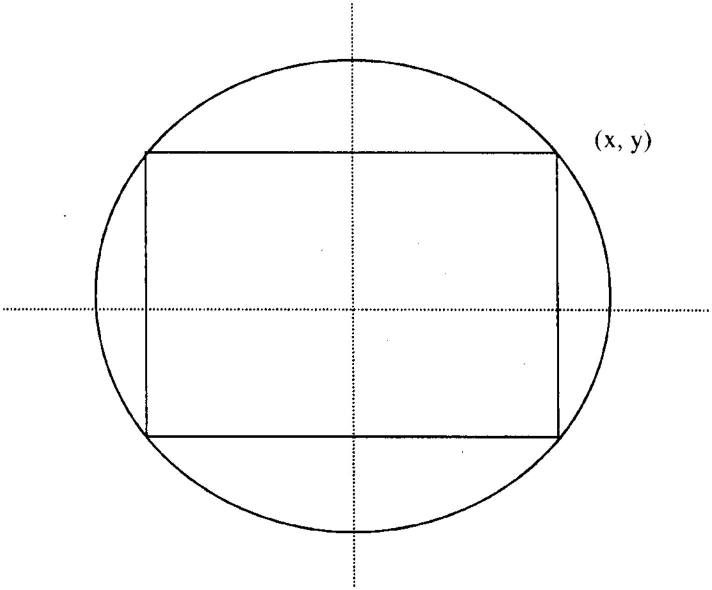

Figure 5

$$y = \sqrt{L^2 - \left(\frac{L\sqrt{2}}{2}\right)^2} = \sqrt{L^2 - \frac{1}{2}L^2} = \frac{L\sqrt{2}}{2}.$$

Although a graphics calculator program is not essential in this problem, students could write one before trying to do a proof:

: Input "L?", L

: fMax $\left(4x\sqrt{\left(L^2 - x^2\right)}, x, 0, L\right) \to R$

: $\sqrt{L^2 - R^2} \to S$

: Output (1, 1, R)

: Output (2, 1, S)

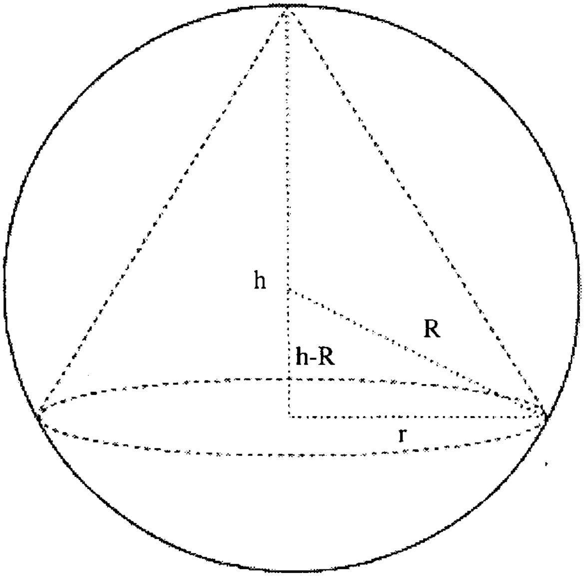

Figure 6

The values of R and S on the screen will always be the same, no matter what radius L is chosen for the circle. A third approach is to use Theorem 1, applied to the function $A^2(x) = 16x^2(L^2 - x^2)$. We reach the same answer, namely $x = \dfrac{L\sqrt{2}}{2}$.

6.5 Activity 5. We wish to inscribe a right circular cone of maximum volume in a sphere of radius R. Find the radius and height in terms of R (Figure 6). This problem will lead us to a third degree polynomial, for whose analysis Theorem 1 is ideally suited. From the figure we notice that $V = \frac{1}{3}\pi r^2 h$ and $(h - R)^2 + r^2 = R^2$. So

$$V(h) = \frac{1}{3}\pi\left(R^2 - (h - R)^2\right)h,$$

$$= \frac{1}{3}\pi\left(2Rh^2 - h^3\right).$$

Then

$$V(h+\delta) - V(h) = \left(\frac{4}{3}\pi Rh - \pi h^2\right)\delta + \cdots$$

Thus, according to Theorem 1, we have to solve the equation $\frac{4}{3}\pi Rh - \pi h^2 = 0$, whose roots are $h = 0$ and $h = \frac{4R}{3}$, both of which are within the allowed interval $[0, 2R]$. Since $V(0) = V(2R) = 0$ and $V\left(\frac{4R}{3}\right) = \frac{8\pi}{27}R^3 > 0$ we can conclude that V takes its maximum at $h = \frac{4R}{3}$.

Obviously $r^2 = R^2 - \left(\frac{4R}{3} - R\right)^2 = R^2 - \frac{1}{9}R^2 = \frac{8}{9}R^2$, i.e., $r = \frac{2\sqrt{2}}{3}R$.

7. Final Considerations

There are many other problems on maximization or minimization that could be discussed at the precalculus level. Some relevant examples are the reflection principle (Niven 1981), Dido's problem for a triangle (Beckenbach and Bellman 1961), the problem of Archimedes about spherical segments (Tikhomirov 1990) and the problem of the square prism of given volume at minimum cost (Levenson 1967). Students of Calculus can also profit from the approach presented in this article. They will become acquainted with alternative techniques to the standard procedures of optimization learned in a calculus class. It is of obvious benefit to any student to know more than one way of solving a problem.

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