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AN APPLICATION FOR THE GAUSS-BONNET THEOREM

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Abstract

The principal aim of this paper is to give an example of the Gauss-Bonnet Theorem together with its anew structure by using connection and curvature matrices with stereographic projection on the unit 2-sphere, S^2. We determine an orthonormal basis by applying stereographic projection on S^2 and we obtain the area of the unit 2-sphere S^2 computing connection and curvature matrices.

1. Introduction

One way to define a system of coordinates for the sphere, given by $x^2 + y^2 + z^2 = 1$, is to consider the so-called stereographic projection

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$\pi_1 : S^2 \setminus \{N\} \to R^2$ which carries a point $P=(x,y,z)$ of the sphere S^2 minus the north pole $N=(0,0,1)$ onto the intersection of the xy-plane with the straight line which connects N to P, [1].

By the expression in this coordinates of unit 2-sphere, S^2, we determine an orthonormal basis and calculate connection and curvature matrices. Now, we give some basic notions which are used in this paper.

1.1 Definition: An exterior form of degree 1 in R^3 is a map w that associates to each $p \in R^3$ an element $w(p) \in (R_p^3)^*$, where $(R_p^3)^*$ is dual space of the tangent space R_p^3 at the point p; w can be written as

$$w(p) = a_1(p)(dx_1)_p + a_2(p)(dx_2)_p + a_3(p)(dx_3)_p$$

or

$$w = \sum_{i=1}^{3} a_i dx_i$$

where a_i are real functions in R^3. If the functions a_i are differentiable, w is called a differential form of degree 1.

Now, let $\Lambda^2(R_p^2)^*$ be the set of maps $\varphi : R_p^3 x R_p^3 \to R$ that are bilinear and alternate (i.e. $\varphi(v_1, v_2) = -\varphi(v_2, v_1)$). Denoting the element $(dx_i)_p \Lambda (dx_j) \in \Lambda^2(R_p^3)^*$ by $(dx_i \Lambda dx_j)_p$, $\Lambda^2(R_p^3)^*$ has properties that

$$(dx_i \Lambda dx_j)_p = -(dx_j \Lambda dx_i)_p$$

and

$$(dx_i \Lambda dx_i)_p = 0.$$

1.2 Definition: An exterior form of degree 2 in R^3 is a correspondence that associates to each $p \in R^3$ an element $w(p) \in \Lambda^2(R_p^3)^*$; w can be written in the form

$$w(p) = a_{12}(p)(dx_1 \Lambda dx_2)_p + a_{13}(p)(dx_1 \Lambda dx_3)_p + a_{23}(p)(dx_2 \Lambda dx_3)_p$$

or

$$w=\sum_{i<j} a_{ij}\, dx_i \Lambda dx_j \qquad i=1,2; \qquad j=2,3,$$

where a_{ij} are real functions in R^3. When the functions a_{ij} are differentiable, w is a differential form of degree 2. Note that we can generalize the notion of differential form to R^n, [2].

2. Calculation of the connection and curvature matrices

Let $\pi_1(x, y, z) = (u, v)$ where $(x, y, z) \in S^2 \setminus \{N\}$ and $(u, v) \in R^2$. The expression in the local coordinates of unit 2-sphere, S^2 which is obtained by stereographic projection from the north pole N and south pole S are respectively

$$\varphi(u,v)=\pi_1^{-1}(u,v)=(2u/(u^2+v^2+1),\ 2v/(u^2+v^2+1),\ 1-2/(u^2+v^2+1)),$$

$$\varphi(u,v)=\pi_1^{-1}(u,v)=(2u/(u^2+v^2+1),\ 2v/(u^2+v^2+1),\ 2/(u^2+v^2+1)\ -1).$$

We will consider this study only for the north pole N, since it is the same thing for the south pole S.

Since,

$$\varphi_u=((-2u^2+2v^2+2)/(u^2+v^2+1)^2,\ -4uv/(u^2+v^2+1)^2,\ 4u/(u^2+v^2+1)^2)$$

and

$$\varphi_v=(-4uv/(u^2+v^2+1)^2,(2u^2-2v^2+2)/(u^2+v^2+1)^2,4v/(u^2+v^2+1)^2),$$

the induced metric in this coordinates on S^2 is

$$G=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix} 4/(u^2+v^2+1)^2 & 0 \\ 0 & 4/(u^2+v^2+1)^2\end{pmatrix}.$$

Hence, one can define an orthonormal basis given by

$$w_1=2/(u^2+v^2+1)\,du, \qquad w_2=2/(u^2+v^2+1)\,dv$$

with $ds^2 = \sum g_{ij}\, dx^i \otimes dx^j$ and so we find

$$dw_1 = v\, w_1 \wedge w_2$$

$$dw_2 = -u\, w_1 \wedge w_2 .$$

Now, let us choose real valued functions a_{ijk} so that

$$dw_k = \sum_{ij} a_{ijk}\, w_i \wedge w_j .$$

If we set

$$a_{ijk} = b_{ijk} + c_{ijk}$$

with

$$b_{ijk} = 1/2\,(a_{ijk} + a_{jik} - a_{kij} - a_{kji} + a_{jki} + a_{ikj})$$

which is symmetric in i, j, and

$$c_{ijk} = 1/2\,(a_{ijk} - a_{jik} + a_{kij} + a_{kji} - a_{jki} - a_{ikj})$$

which is skew-symmetric in j, k then we obtain

$$dw_k = \sum_{ij} c_{ijk}\, w_i \wedge w_j .$$

The 1-forms

$$w_{kj} = \sum_i c_{ijk}\, w_i$$

constitute the unique skew-symmetric matrix with

$$dw_k = \sum_j w_{kj}\, w_j \quad .$$

Since we are in the special case of a 2-dimensional oriented Riemannian manifold, with respect to an oriented local orthonormal basis w_1, w_2 for 1-forms the connection and curvature matrices are of the form

$$\begin{pmatrix} 0 & w_{12} \\ -w_{12} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & F_{12} \\ -F_{12} & 0 \end{pmatrix}$$

with $dw_{12} = F_{12}$, [3].

Now we can compute these in our special case as follows: First, we note that

$$a_{121} = -a_{211} \quad (a_{121} = v/2),$$

$$a_{122} = -a_{212} \quad (a_{122} = -u/2).$$

Moreover, since

$$c_{i11} = 0 \quad \text{and} \quad c_{i12} = 1/2(a_{i12} - a_{1i2} + a_{2i1} + a_{21i} - a_{12i} - a_{i21})$$

we have

$$w_{12} = vw_1 - uw_2 .$$

Thus, we find connection and curvature matrices in the following:

$$\begin{pmatrix} 0 & vw_1 - uw_2 \\ -vw_1 + uw_2 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & dv\Lambda w_1 + vdw_1 - du\Lambda w_2 - udw_2 \\ -dv\Lambda w_1 - vdw_1 + du\Lambda w_2 + udw_2 & 0 \end{pmatrix}.$$

Finally, we obtain

$$F_{12} = dw_{12} = -(w_1 \Lambda w_2).$$

Although this 2-form dw_{12} is independent of the choice of oriented orthonormal basis, these computations appear to be new. The form dw_{12} is called the Gauss-Bonnet 2-form as a well-defined global 2-form on oriented manifold (in our case S^2). We can set $dw_{12} = -Kw_1 \Lambda w_2$ where K is scaler function called the Gaussian curvature, [2]. By the Gauss-Bonet Theorem we obtain the area of the unit 2-sphere

$$\iint dw_{12} = 4\pi .$$

In this paper, it is given an explicit application of Gauss-Bonnet Theorem on 2-dimensional oriented manifold S^2. We hope this approach might be a good starting point for all the other 2-dimensional oriented manifolds.

3. References

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