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WEIERSTRASS FORMULA FOR MINIMAL SURFACES IN HEISENBERG GROUP

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Abstract

In this paper we study the Gauss map of minimal surfaces in the Heisenberg group, $\mathcal{H}_3$. We obtain a representation formula for minimal surfaces in $\mathcal{H}_3$ by means of the Gauss map. As consequence we conclude that: The Gauss map of a minimal surface of $\mathcal{H}_3$ is antiholomorphic if the minimal surface is a plane.

1. Introduction

The purpose of this paper is to study the Gauss map of minimal surfaces in the Heisenberg group, $\mathcal{H}_3$, analytically. By the existence of isothermal

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coordinates and considering the unit 2-sphere as the Riemann sphere, the Gauss map, for surfaces in a Lie group, is a complex mapping. We now review the contents of the paper.

In section 2 we present the basic Riemannian geometry of $\mathcal{H}_3$ equipped with a left-invariant metric and a relationship between the Gauss map and the extrinsic geometry of surfaces in $\mathcal{H}_3$. In the same section we describe, in charts, the tension field of minimal surfaces in $\mathcal{H}_3$ and get some consequence.

In section 3 we shall prove that the Gauss map of a minimal immersion in $\mathcal{H}_3$ must satisfy a first order differential equation of Beltrami type.

Section 4 carry out a representation formula for minimal surfaces in $\mathcal{H}_3$ by means of the Gauss map.

Finally, in section 5 we shall show that the Gauss map of an arbitrary minimal surface in $\mathcal{H}_3$ satisfies a second order differential equation which is a complete integrability condition of the above obtained representation formula. In the same section we give some consequences of this representation.

2. Basic Riemannian Geometry of $\mathcal{H}_3$

The Lie algebra, h_3, of $\mathcal{H}_3$ is isomorphic to $\mathbb{R}^3$ with the Lie product:

$$\begin{cases} [e_1, e_2] = e_3 \\ [e_i, e_3] = 0, \ i = 1, 2, 3. \end{cases}$$

where $\{e_i\}$ is the canonical basis in $\mathbb{R}^3$.

The exponential map, $\exp : h_3 \to \mathcal{H}_3$, is given by:

$$\exp (A) = I + A^2 + A^3$$

and it is a diffeomorphism which induces on h_3, by the Campbell-Hausdorff formula, the group structure on $\mathcal{H}_3$:

$$x_1 * x_2 = x_1 + x_2 + \frac{1}{2}[x_1, x_2]. \qquad (1)$$

where $x = xe_1 + ye_2 + ze_3$. Notice that the 1-parameter subgroups are straight lines.

From now on, modulo the identification given by exp, we consider $\mathcal{H}_3$ as $\mathbb{R}^3$ with the product given in (1). Using $\{e_i\}$ as the orthonormal frame at the identity, we have an orthonormal basis of left-invariant vector fields:

$$
\begin{aligned}
E_1 &= \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} \\
E_2 &= \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} \\
E_3 &= \frac{\partial}{\partial z}
\end{aligned}
$$

and the left-invariant metric, induced by the Euclidean metric at the identity, is given by

$$
ds^2 = dx^2 + dy^2 + \left(\frac{y}{2}dx - \frac{x}{2}dy + dz\right)^2 \tag{2}
$$

Then the Riemann connection of ds^2, in terms of the basis $\{E_i\}$, is given by:

$$
\begin{aligned}
\nabla_{E_1} E_2 &= \frac{1}{2}E_3 &&= -\nabla_{E_1} E_2 \\
\nabla_{E_1} E_3 &= -\frac{1}{2}E_2 &&= \nabla_{E_3} E_1 \\
\nabla_{E_2} E_3 &= \frac{1}{2}E_1 &&= \nabla_{E_3} E_2 \\
\nabla_{E_i} E_i &= 0.
\end{aligned}
$$

Let M be an 2-dimensional connected Riemannian manifold and $f : M \to \mathcal{H}_3$ an isometric immersion of M into $\mathcal{H}_3$. At a neighborhood of any point of M we shall use an isothermal coordinate $z = u + iv$,

$$
\begin{tikzcd}
M \arrow[r, "f"] & \mathcal{H}_3 \\
U \arrow[u] \arrow[ur, "X"'] &
\end{tikzcd}
$$

and making use of it, the first fundamental form is now written by $ds^2 = \lambda^2 |dz|^2$, $\lambda > 0$. The coordinate fields, $X_u = f_*\left(\frac{\partial}{\partial u}\right)$ and $X_v = f_*\left(\frac{\partial}{\partial v}\right)$, are given by:

$$X_u = x_u E_1 + y_u E_2 + \alpha E_3$$
$$X_v = x_v E_1 + y_v E_2 + \beta E_3$$

where we set $\alpha = \frac{y}{2}x_u - \frac{x}{2}y_u + z_u$ and $\beta = \frac{y}{2}x_v - \frac{x}{2}y_v + z_v$. Hence, it follows that

$$< X_u,\ X_u > \ = \ < X_v,\ X_v > \ = \ \lambda^2; \quad < X_u,\ X_v > \ = \ 0. \qquad (3)$$

A unit normal vector field of the immersion f is given by:

$$\eta = \frac{1}{\lambda^2}\left[(\beta y_u = \alpha y_v) E_1 + (\alpha x_v - \beta x_u) E_2 + (x_u y_v - x_v y_u) E_3\right]$$

where we will denote the coordinates of η, in the basis $\{ E_i \}$, by (a, b, c). Then the tension field of the immersion f is given by:

$$\tau(f) = \lambda^{-2}(\nabla x_u X_u + \nabla x_v X_v) = 2\boldsymbol{H}$$

where $\boldsymbol{H}$ is the mean curvature vector. If f is minimal, $\boldsymbol{H} = 0$, we have:

$$\Delta x = -(\alpha y_u + \beta y_v)$$
$$\Delta y = \alpha x_u + \beta x_v \qquad (4)$$
$$\frac{y}{2}\Delta x - \frac{x}{2}\Delta y + \Delta z = 0.$$

Remark 1. *Let us make he following comments of the above system. The third equation of the system of (4) is equivalent to:*

$$\alpha_u + \beta_v = 0. \qquad (5)$$

If the coordinates x and y of the minimal immersion f are harmonic functions, we have that

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

Recall that $(x_u y_v - x_v y_u) = \lambda^2 c$, *hence we have two cases: If* $c \neq 0$ *we have* $\alpha = \beta = 0$, *that is, the Gauss map is constant and equal to the North Pole, but this a contradiction, because there is no minimal surface in* $\mathcal{H}_3$ *with this property (see* [1]). *Now, if* $c = 0$ *we have that the image of the Gauss map is the Equator and the rank of the Gauss map is one, which again can not happen (see* [1]).

Finally we recall in the Euclidean case the differential of the Gauss map is just the second fundamental form for surfaces in $\mathbb{R}^3$. This fact can be generalized for hypersurfaces in any Lie Group. The following theorem (see [3]) establishes a relationship between the Gauss map and the extrinsic geometry of S.

Theorem 1. *Let* S *be an orientable hypersurfaces of a Lie group. Then*

$$dL_p \circ d\gamma_p (v) = -A_\eta (v) + \alpha_{\bar{\eta}}), \qquad v \in T_p S,$$

where A_η *is the Weingarten operator,* $\alpha_{\bar{\eta}}(v) = \nabla_v \bar{\eta}$ *and* $\bar{\eta}$ *is a left invariant vector field such that* $\eta(p) = \bar{\eta}(p)$.

3. The Beltrami Equation

In this section we shall prove that the Gauss map of any minimal immersion in $\mathcal{H}_3$ satisfies a Beltrami equation.

We indicate the matrices of the Weingarten operator and $\alpha_{\bar{\eta}}$ by (h_{ij}) and $(\hat{h}_{ij})$, respectively, in the basis $\{X_u, X_v\}$. If we set $(\gamma_{ij}) = (h_{ij} + \hat{h}_{ij})$, by Theorem (1), we have

$$d L_p \circ d\gamma_p = -(\gamma_{ij}).$$

In particular, the coefficients of $\alpha_{\bar{\eta}}$ is given by:

$$\lambda^2 \alpha_{\bar{\eta}} = \lambda^2 (\hat{h}_{ij}) = \begin{pmatrix} \alpha(bx_u - ay_u) & \frac{\lambda^2}{2} + \beta(bx_u - ay_u) \\ \alpha(bx_u - ay_v) - \frac{\lambda^2}{2} & \beta(bx_v - ay_v) \end{pmatrix} \qquad (6)$$

Now we compute the derivatives of a, b and c with respect to u:

$$\begin{array}{lcl} a_u & = & -\gamma_{11}x_u - \gamma_{21}x_v \\ b_u & = & -\gamma_{11}y_u - \gamma_{21}y_v \\ c_u & = & -\gamma_{11}\alpha - \gamma_{21}\beta \end{array} \tag{7}$$

and the derivatives of a, b and c with respect to v:

$$\begin{array}{lcl} a_v & = & -\gamma_{12}x_u - \gamma_{22}x_v \\ b_v & = & -\gamma_{12}y_u - \gamma_{22}y_v \\ c_v & = & -\gamma_{12}\alpha - \gamma_{22}\beta. \end{array} \tag{8}$$

Let S^2 be the unit sphere in $h_3 \simeq T_o\mathcal{H}_3$ and we consider S^2as the standard Riemann sphere: We cover S^2 by the union of the two open sets U_i, where we set $U_1 = S^2 - \{\text{north pole}\}$ and $U_2 = S^2 - \{\text{south pole}\}$. Let ψ_i be the coordinate functions on U_i. Then we know

$$\psi_1(x) = \frac{x_1 + ix_2}{1 - x_3}, \qquad \text{if } x \in U_1$$

$$\psi_2(x) = \frac{x_1 - ix_2}{1 + x_3}, \qquad \text{if } x \in U_2.$$

We consider, for any surface in $\mathcal{H}_3$, the following sequence of mappings:

$$M \xrightarrow{f} f(M) \xrightarrow{Gauss\ Map} S^2 \xrightarrow{\psi} w-\ plane$$

The composed map, which will be also called the Gauss map of M,

$$\psi : M \rightarrow \text{Riemman sphere}$$

is considered as a complex mapping of a 1-dimensional complex manifold M to the Riemann sphere.

Lemma 2. *Under the above notations, we have*

$$\frac{\partial \psi_1}{\partial \bar{z}} = -\frac{\Theta}{2}(1+\psi_1\overline{\psi}_1)^2\left(\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right)$$

where $\Theta = \dfrac{i(\hat{h}_{12} - \hat{h}_{21})}{2}$.

Proof. We know that

$$\psi_1(z)=\frac{a(z)+ib(z)}{1-c(z)}.$$

Since we put $\dfrac{\partial\psi_1}{\partial\bar z}=\dfrac{1}{2}\left(\dfrac{\partial\psi_1}{\partial u}+i\dfrac{\partial\psi_1}{\partial v}\right)$ we have, using (7),

$$\frac{\partial\psi_1}{\partial u}=\frac{1}{(1-c)^2}\{[(-x_u+y_v)+i(y_u+x_v)]\gamma_{11}-[(x_v+y_u)+i(y_v-x_u)]\gamma_{21}\}.$$

By similar way, using (8), we have that

$$i\frac{\partial\psi_1}{\partial v}=\frac{1}{(1-c)^2}\{[(x_v+y_u)+i(y_v-x_u)]\gamma_{12}+[(-x_u+y_v)-i(x_v+y_u)]\gamma_{22}\}.$$

Observe that $(-x_u+y_v)-i(x_v+y_u)=-i[(x_v+y_u)+i(y_v-x_u)]$. Then, substituting in the above two equation and summing up we obtain that

$$\frac{\partial\psi_1}{\partial\bar z}=\frac{1}{2(1-c)^2}\{(\gamma_{12}-\gamma_{21})-i(\gamma_{11}+\gamma_{22})[(x_v+y_u)+i(y_v-x_u)]\}.$$

Notice that $\gamma_{11}+\gamma_{22}=2H+(\hat h_{11}+\hat h_{22})=0$ because $H=0$ (minimal immersion) and $\hat h_{11}+\hat h_{22}$ is the trace of the matrix $\alpha_{\bar\eta}$, which in $\mathcal{H}_3$ is equal to zero. Now, a real part is $\gamma_{12}-\gamma_{21}=\hat h_{12}-\hat h_{21}$, because the Weingarten operator is symmetric in the basis $\{X_u, X_v\}$. Then,

$$\frac{\partial\psi_1}{\partial\bar z}=\frac{\hat h_{12}-\hat h_{21}}{2(1-c)^2}\{(x_v+y_u)+i(y_v-x_u)\}$$

Now $(x_v+y_u)+i(y_v-x_u)=-2i\left(\dfrac{\partial x}{\partial\bar z}+i\dfrac{\partial y}{\partial\bar z}\right)$ and using the fact that

$$(1+\psi_1\bar\psi_1)(1-c)=2,$$

follow the result. ■

Remark 2. *Note that Θ is a function of ψ. In fact, by using (6) we can see that $\Theta=\frac{i}{2}(c^2)$. Then we have*

$$\Theta = \frac{i}{2}\left(\frac{\psi\overline{\psi} - 1}{1 + \psi\overline{\psi}}\right)^2 \tag{9}$$

and, by remark (1), $\Theta \neq 0$.

We define here the following functions:

$$\Phi = \frac{1}{2}(h_{11} - h_{22}) - ih_{12}; \quad \hat{\Phi} = \frac{1}{2}(\hat{h}_{11} - \hat{h}_{22}) - \frac{i}{2}(\hat{h}_{12} + \hat{h}_{21})$$

Lemma 3. *Under the above notations, we have*

$$\frac{\partial \psi_1}{\partial z} = -\frac{(\Phi + \hat{\Phi})}{2}(1 + \psi_1\overline{\psi}_1)^2 \left(\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right)$$

Proof. Since we have $\frac{\partial \psi_1}{\partial z} = \frac{1}{2}\left(\frac{\partial \psi_1}{\partial u} - i\frac{\partial \psi_1}{\partial v}\right)$, we can prove the Lemma 3 in the same way as its of Lemma 2. ■

We can calculate the norms of these complex vectors

Corollary 4. *Let* ψ *be the Gauss map of an arbitrary minimal surface in* $\mathcal{H}_3$. *Then we have*

$$\left|\frac{\partial \psi}{\partial \bar{z}}\right| = \frac{\lambda}{2}(1 + \psi\overline{\psi})|\Theta|$$

$$\left|\frac{\partial \psi}{\partial z}\right| = \frac{\lambda}{2}(1 + \psi\overline{\psi})\left|\Phi + \hat{\Phi}\right|.$$

Proof. Firstly we prove that

$$4\left|\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right|^2 = \lambda^2(1 - c)^2.$$

In fact,

$$\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}[(x_u - y_v) + i(x_v + y_u)].$$

Then, using (3), we have

$$4\left|\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right|^2 = 2\lambda^2(1-c) - (\alpha^2 + \beta^2).$$

In other hand

$$\lambda^2 (a^2 + b^2) = (\alpha^2 + \beta^2).$$

Substituting in the above equation and using the fact that $a^2 + b^2 + c^2 = 1$, follows the result. ■

Now we establishes and prove the main result of this section,

Theorem 5. *The Gauss map ψ of a minimal surface in $\mathcal{H}_3$ satisfies a Beltrami equation*:

$$\left[\Phi + \hat{\Phi}\right]\frac{\partial \psi}{\partial \bar{z}} = \Theta\frac{\partial \psi}{\partial z} \tag{10}$$

Proof. By Lemmas 2 and 3 we obtain $\left[\Phi + \hat{\Phi}\right]\frac{\partial \psi_1}{\partial \bar{z}} = \Theta\frac{\partial \psi_1}{\partial z}$ in U_1. On $U_1 \cap U_2$ we have also $\left[\Phi + \hat{\Phi}\right]\frac{\partial \psi_2}{\partial \bar{z}} = \Theta\frac{\partial \psi_2}{\partial z}$ by virtue of $\psi_1\psi_2 = 1$. By the continuity we have the same formula on U_2. ■

4. The Weierstrass Formula

In this section we shall give a Weierstrass formula for minimal surfaces in $\mathcal{H}_3$. Since $\psi_1\psi_2 = 1$, we have

$$\begin{aligned}
1 + \psi_1\overline{\psi}_1 &= \overline{\psi}_1(\overline{\psi}_2 + \psi_1) \\
1 + \psi_2\overline{\psi}_2 &= \psi_2(\overline{\psi}_2 + \psi_1) \\
\psi_1\frac{\partial \psi_2}{\partial \bar{z}} + \psi_2\frac{\partial \psi_1}{\partial \bar{z}} &= 0.
\end{aligned}$$

This, together with Lemmas 2 and 3, yields the following equation,

$$\Theta\psi_1(\psi_2)^2(\overline{\psi}_2+\psi_1)^2\left(\frac{\partial x}{\partial \bar{z}}-i\frac{\partial y}{\partial \bar{z}}\right)+\Theta\psi_2(\overline{\psi}_1)^2(\overline{\psi}_2+\psi_1)^2\left(\frac{\partial x}{\partial \bar{z}}+i\frac{\partial y}{\partial \bar{z}}\right)=0.$$

Since $(\overline{\psi}_2+\psi_1)^2\neq 0$ and $\Theta\neq 0$, we have

$$\left[\frac{\partial x}{\partial \bar{z}}-i\frac{\partial y}{\partial \bar{z}}+(\overline{\psi}_1)^2\left(\frac{\partial x}{\partial \bar{z}}+i\frac{\partial y}{\partial \bar{z}}\right)\right]=0. \tag{11}$$

Lemma 6. Let $f: M \to \mathcal{H}_3$ be a minimal immersion of M into $\mathcal{H}_3$ and $\psi: M \to S^2$ be the Gauss map of M into S^2 considered as the Riemann sphere. Then we have, on U_1,

$$\begin{aligned}\frac{\partial x}{\partial \bar{z}} &= \frac{2i(1-\overline{\psi}_1^2)}{(\psi_1\overline{\psi}_1-1)^2}\frac{\partial \psi_1}{\partial \bar{z}}\\ \frac{\partial y}{\partial \bar{z}} &= \frac{2(1+\overline{\psi}_1^2)}{(\psi_1\overline{\psi}_1-1)^2}\frac{\partial \psi_1}{\partial \bar{z}}\\ \frac{\partial \xi}{\partial \bar{z}} &= \frac{-4\overline{\psi}_1}{(\psi_1\overline{\psi}_1-1)^2}\frac{\partial \psi_1}{\partial \bar{z}}\end{aligned} \tag{12}$$

where ξ is such that $\xi_u=-\beta$ and $\xi_v=\alpha$.

Proof. From (11) we have

$$(1+\overline{\psi}_1^2)\frac{\partial x}{\partial \bar{z}}=i(1-\overline{\psi}_1^2)\frac{\partial y}{\partial \bar{z}}. \tag{13}$$

Since $1+\overline{\psi}_1^2\neq 0$, by virtue of Lemma 2 and equation (13), we have

$$(1+\overline{\psi}_1^2)\left[\frac{-2}{\Theta(1+\psi_1\overline{\psi}_1)^2}\frac{\partial \psi_1}{\partial \bar{z}}-i\frac{\partial y}{\partial \bar{z}}\right]=i(1-\overline{\psi}_1^2)\frac{\partial y}{\partial \bar{z}}$$

whence we obtain

$$\Theta\frac{\partial y}{\partial \bar{z}}=\frac{i(1+\overline{\psi}_1^2)}{(1+\psi_1\overline{\psi}_1)^2}\frac{\partial \psi_1}{\partial \bar{z}},$$

using (9) follows the second formula of (12). By the similar way we have also the first formula of (12). The last formula of (12) follows from the next formula:

$$\frac{1}{2}(\alpha - i\beta)\left(\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right) = \lambda^2 \frac{\psi_1}{(1+\psi_1\overline{\psi}_1)^2}. \tag{14}$$

We shall prove at first this equation. By mean the definition of ψ_1 we get

$$\frac{1}{2}(\alpha - i\beta)\left(\frac{\partial x}{\partial z} + i\frac{\partial y}{\partial z}\right) - \lambda^2 \frac{\psi_1}{(1+\psi_1\overline{\psi}_1)^2} =$$

$$\frac{1}{4}\{(\alpha - i\beta)\ (x_u + ix_v + iy_u - y_v) - \lambda^2\ (1-c)\ (a+ib)\}. \tag{15}$$

The real part of the above formula is equal to

$$\frac{1}{4}\{\alpha(x_u - y_v) + \beta\,(x_v + y_u) - \lambda^2\ (1-c)\ a\} =$$

$$\frac{1}{4}\left\{\alpha x_u - \alpha y_v + \beta x_v + \beta y_u - \left(1 - \frac{1}{\lambda^2}(x_u\, y_v - x_v\, y_u)\right)(\beta y_u - \alpha y_v)\right\} =$$

$$\frac{1}{4\lambda^2}\{\alpha x_u \lambda^2 + \beta x_v \lambda^2 + (x_u\, y_v \beta y_u - x_u y_v^2 \alpha - x_v\, y_u^2 \beta + x_v\, y_u\, \alpha y_v)\} = 0.$$

The last equal held using the relations of (3). By the similar way we can see that the imaginary part of (15) is also zero. This prove the formula.

Under the condition that $\dfrac{\partial \psi_1}{\partial \bar{z}} \neq 0$, we have

$$\frac{1}{2}(\alpha + i\beta)\left|\frac{\partial x}{\partial \bar{z}} + i\frac{\partial y}{\partial \bar{z}}\right|^2 = \frac{\lambda^2\overline{\psi}_1\left(\dfrac{\partial x}{\partial \bar{z}} + i\dfrac{\partial y}{\partial \bar{z}}\right)}{(1+\psi_1\overline{\psi}_1)^2}.$$

Using (14), we obtain

$$\frac{\Theta}{2}(\alpha + i\beta) = \frac{-2\overline{\psi}_1}{(1+\psi_1\overline{\psi}_1)^2}\frac{\partial \psi_1}{\partial \bar{z}}.$$

But we know, see remark (1), $\dfrac{\partial \alpha}{\partial u} + \dfrac{\partial \beta}{\partial v} = 0$; then, there exist a differential function ξ such that $\xi_u = -\beta$ and $\xi_v = \alpha$. Hence,

$$\frac{\Theta}{2}(\xi_u + i\xi_v) = \frac{-2i\overline{\psi}_1}{(1+\psi_1\overline{\psi}_1)^2}\frac{\partial\psi_1}{\partial\bar{z}}.$$

This concludes the proof. ■

5. Integrability Condition

We shall show in this section that the Gauss map of a minimal immersion in $\mathcal{H}_3$ satisfies a second order differential equation which help us to find a integrability condition for the system (12).

Theorem 7. *Let $f: M \to \mathcal{H}_3$ be an isometric immersion of M into $\mathcal{H}_3$. Then f is minimal if the Gauss map ψ satisfy*

$$\frac{\partial^2\psi}{\partial\bar{z}\partial z} - \frac{2\overline{\psi}}{\psi\overline{\psi}-1}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\psi}{\partial z} = 0 \tag{16}$$

Proof. Firstly we derive the system (12) with respect a z. From the first equation of this system we have

$$\frac{\partial^2 x}{\partial z\partial\bar{z}} = \frac{2i(1-\overline{\psi}^2)}{(\psi\overline{\psi}-1)^2}\left[\frac{\partial^2\psi}{\partial z\partial\bar{z}} - \frac{2\overline{\psi}}{(\psi\overline{\psi}-1)}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\psi}{\partial z}\right] - \frac{4i(\psi-\overline{\psi})}{(\psi\overline{\psi}-1)^3}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\overline{\psi}}{\partial z}.$$

Notice that the second term of the right side is real and equal to $\frac{\lambda^2 bc}{4} = \frac{-(\alpha y_u + \beta y_v)}{4}$ and by the first equation of (4) it is equal to $\frac{\partial^2 x}{\partial z\partial\bar{z}}$.

By the similar way, from the second equation of (12), we have

$$\frac{\partial^2 y}{\partial z\partial\bar{z}} = \frac{2i(1+\overline{\psi}^2)}{(\psi\overline{\psi}-1)^2}\left[\frac{\partial^2\psi}{\partial z\partial\bar{z}} - \frac{2\overline{\psi}}{(\psi\overline{\psi}-1)}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\psi}{\partial z}\right] - \frac{4(\psi+\overline{\psi})}{(\psi\overline{\psi}-1)^3}\frac{\partial\psi}{\partial\bar{z}}\frac{\partial\overline{\psi}}{\partial z}.$$

In this case the second term of the right side is real and equal to $\frac{-\lambda^2 ac}{4} = \frac{\alpha x_u + \beta x_v}{4}$ and it is equal to $\frac{\partial^2 y}{\partial z\partial\bar{z}}$. Finally, from the third equation of (12) we have:

$$\frac{\partial^2\xi}{\partial z\partial \bar z}=\frac{\overline{\psi}}{(\psi\overline{\psi}-1)^2}\left[\frac{\partial^2\psi}{\partial z\partial \bar z}-\frac{2\overline{\psi}}{(\psi\overline{\psi}-1)}\frac{\partial\psi}{\partial \bar z}\frac{\partial\psi}{\partial z}\right]+\frac{4}{(\psi\overline{\psi}-1)^3}\frac{\partial\psi}{\partial \bar z}\frac{\partial\overline{\psi}}{\partial z}.$$

The second term of the right side is equal to $\frac{\lambda^2 c}{4}$ and, by using the fact that $\xi_u=-\beta$ and $\xi_v=\alpha$, this is equal to $\frac{\partial^2\xi}{\partial z\partial \bar z}$. Then the Gauss map ψ satisfy (16) if f is a minimal immersion of M into $\mathcal{H}_3$. ■

Furthermore we can see that equation (16) is just the complete integrability condition for the system (12). Therefore we have the following

Theorem 8. Let M be a simply connected 2-dimensional smooth Riemannian manifold and ψ: $M \to S^2$ be a smooth mapping which satisfies the differential equation (16). Then ψ is a Gauss map of the following minimal surface of $\mathcal{H}_3$:

$$x=\Re\int_0^z \frac{2i(1-\psi_1^2)}{(\psi_1\overline{\psi}_1-1)^2}\,\overline{\frac{\partial\psi_1}{\partial \bar z}}\,dz+c_1$$

$$y=\Re\int_0^z \frac{2(1+\psi_1^2)}{(\psi_1\overline{\psi}_1-1)^2}\,\overline{\frac{\partial\psi_1}{\partial \bar z}}\,dz+c_2$$

$$\xi=\Re\int_0^z \frac{-4\psi_1}{(\psi_1\overline{\psi}_1-1)^2}\,\overline{\frac{\partial\psi_1}{\partial \bar z}}\,dz+c_3$$

Proof. This follows from Theorems (6) and (7). ■

We have found a correspondence from the set of solution of the differential equation (16) to set of minimal surfaces of $\mathcal{H}_3$. Now we shall study the uniqueness of the correspondence.

Theorem 9. Let $\psi(z)$ (resp. $\tilde{\psi}(z)$) be a smooth mapping satisfying (16) on a simply connected 2-dimensional manifold M. We define a minimal immersion $X(z)$ (resp. $\tilde{X}(z)$) by the above theorem. Then the two condition are equivalent:

1. There exist a holomorphic mapping $w = f(z)$ with $f'(z) \neq$ on M such that $\tilde{X}(f(z)) = X(z)$, $z \in M$.
2. There exist a holomorphic mapping $w = f(z)$ with $f'(z)) \neq$ on M such that $\tilde{\psi}(f(z)) = \psi(z)$, $z \in M$.

Proof. We can repeat the proof of Theorem 5 of [2]. ■

At last we shall give some examples:

Example 1. Let $\psi : \mathbb{C} \to \mathbb{C}$ be $\psi(z) = \bar{z}$. Then the minimal immersion obtained by this ψ is the horizontal plane:

$$X(z) = \left(\frac{2(z+\bar{z}}{z\bar{z}-1}, \frac{2i(z-\bar{z}}{z\bar{z}-1}, cte. \right), \quad z \in \mathbb{C} - \{S^1\}.$$

More generally, if we set $\chi(w) = \overline{g(w)}$ where $g(w)$ is holomorphic function with $g'(w) \neq 0$ on a simply connected D, we have that the minimal immersion made by this $\chi(w)$ is a horizontal plane. In fact, this follows Theorem (9) and by noting that $\psi(g(w)) = \chi(w)$. Therefore: *The Gauss map of a minimal surface of $\mathcal{H}_3$ is antiholomorphic if the minimal surface is a plane.*

6. References

[1] Figueroa, Ch. "*The Gauss map of minimal surface in the Heisenberg group*", Campinas: UNICAMP-IMECC, 1996. Relatório Técnico Nº 71.

[2] Kenmotsu, K. "*Weierstrass formula for Surfaces of Prescribed Mean Curvature*", Math. Ann. 245, (1979), 89-99.

[3] Ripoll, J. "*On Hypersurfaces of Lie groups*", Illinois Journal of Mathematics vol. 35, 1 (1991), 47-55.

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