*Pro Mathematica Vol. XIII, Nos. 25-26, 1999*

# DISCRETE ANALOGUE OF CAUCHY'S INTEGRAL FORMULA

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## *1. Introduction*

*In an earlier paper [3], in 1993, a class of functions named (p, q)-analytic functions and defined on a suitable geometric lattice was introduced. In yet another paper [4] we made a study on complex integrals of discrete functions. The present paper is a continuation of [4] and here a discrete analogue of Cauchy's integral formula has been established.*

## 2. Notations, Definitions and The Lattice

In order to introduce the concept of $(p, q)$-analytic function the following geometric lattice was considered in [3]:

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$$K = \{ (p^m x_0, q^n y_0) ; \ m, \ n \in \mathbb{Z}, \ \textit{the set of integers}, \ 0 < p < 1,$$

$$0 < q < 1, \ (x_0, y_0) \ \textit{fixed}, \ x_0 > 0, \ y_0 > 0 \}, \qquad (2.1)$$

where the complex number $z$ is used synonymously with its components $(x, y)$. Thus if $z \in K$ then $z = (x, y) = (p^m x_0, q^n y_0)$.

We also recall here, for convenience, some of the definitions given in [3].

**Definition 2.1.** The *'discrete plane'* $Q'$ with respect to some fixed point $z' = (x', y')$ in the first quadrant, is defined by the set of lattice points,

$$Q' = \{ (p^m x', \ q^n y') ; m, \ n \in \mathbb{Z}, \ \textit{the set of integers} \}.$$

**Definition 2.2.** Two lattice points $z_i, \ z_{i+1} \in Q'$ are said to be *'adjacent'* if $z_{i+1}$ is one of $(px_i, y_i)$, $(p^{-1} x_i, y_i)$, $(x_i, q y_i)$ or $(x_i, q^{-1} y_i)$.

**Definition 2.3.** A *'discrete curve'* $C$ in $Q'$ connecting $z_0$ to $z_n$ is denoted by the sequence

$$C \equiv < z_0, z_1, \ldots, z_n > ,$$

where $z_i, \ z_{i+1}$ ; $i = 0, 1, \ldots, (n-1)$, are adjacent points of $Q'$.

If the points are distinct ($z_i \neq z_j$ ; $i \neq j$) then the discrete curve $C$ is said to be *'simple'*.

**Definition 2.4**. A *'discrete closed cruve'* $C$ $Q'$ is given by the sequence $< z_0, z_1, z_2, \ldots, z_n >$ where $< z_0, z_1, \ldots, z_{n-1} >$ is simple and $z_0 = z_n$.

Denote by $\overline{C}$ the continuous closed curve formed by joining adjacent points of the discrete closed curve $C$. Then $\overline{C}$ encloses certain points of $Q'$, denoted by Int. $(C)$.

**Definition 2.5.** A *'finite discrete domain'* $D$ is defined as

$$D = \{ z \in Q' ; \quad z \in C \cup Int \ (C) \}.$$

In general a '*discrete domain*' $D$ is defined as a union (infinite or otherwise) of finite discrete domains.

$\partial(D)$ denotes the discrete closed curve around the finite discrete domain $D$.

$$\text{i.e. } \partial(D) = D - Int(D).$$

**Definition 2.6.** A '*basic set*' with respect to $z \in Q$ is defined as

$$S(z) = \{(x, y), (px, y), (px, qy), (x, qy)\},$$

and the discrete closed curve around $S(z)$ is denoted by

$$\partial(S) = < (x, y), (px, y), (px, qy), (x, qy), (x, y) > \tag{2.2}$$

The order of points of $\partial(S)$ as in (2.2) is said to be positive direction. The reverse sequence is denoted by $-\partial(S)$.

It is evident from the above definitions that a discrete domain $D$ is composed of a union of basic sets.

**Definition 2.7.** Function defined on the points of a discrete domain $D$ are said to be '*discrete functions*'.

**Definition 2.8.** The p-difference and q-difference operator $D_{p,x}$ and $D_{q,y}$ are defined as follows:

$$D_{p,x}[f(z)] = \frac{f(z) - f(px, y)}{(1-p)x} \tag{2.3}$$

$$D_{q,y}[f(z)] = \frac{f(z) - f(x, qy)}{(1-q)\, iy} \tag{2.4}$$

where $f$ is a discrete function.

**Definition of (*p, q*) – Analytic Functions.** The two operators (2.3) and (2.4) involve a 'basic triad' of points denoted by

$$T(z) = \{ (x, y), (px, y), (x, qy)\} \tag{2.5}$$

Let $D$ be a discrete domain. Then a discrete function $f$ is said to be '*(p,q)- analytic*' at $z \in D$ if

$$D_{p,x}[f(z)] = D_{q,y}[f(z)] \tag{2.6}$$

If in addition (2.6) holds for every $z \in D$ such that $T(z) \subseteq D$ then $f$ is said to be $(p, q)$ – analytic in $D$. (2.7)

For simplicity if (2.6) or (2.7) holds, the common operator $D$ is used where

$$D \equiv D_{p,x} \equiv D_{q,y} \tag{2.8}$$

Further, the operator $R_{p,q}$ is defined as

$$R_{p,q}[f(z)]=\{(1-p)x-i(1-q)y\}f(x,y)-(1-p)xf(x,qy)+i(1-q)yf(px,y), \tag{2.9}$$

where $f : K \to \mathbb{C}$ (the field of complex numbers).

$R_{p,q}\, f(z)$ is called $(p, q)$ – residue of the function at $z$.

From (2.7) it is easily seen that $f$ is $(p, q)$-analytic in a discrete domain $D$ iff

$$R_{p,q}[f(z)] = 0, \tag{2.10}$$

for every $z \in D$ with $T(z) \subseteq D$.

Since a discrete domain $D$ is the union of basic sets $S$ so if the discrete domain $D$ is given by

$$D = \bigcup_{i=1}^{N} S(z_i),$$

then the '*subdomain* $D_N$' is defined by

$$D_N = \{z_i \; ; \; i = 1, 2, \ldots, N\} \tag{2.11}$$

## 3. $(r, s)$ – Analytic Functions

In order to develop a discrete analogue of Cauchy's integral formula it is also necessary to introduce the concept of $(r, s)$-analytic function.

If $r = p^{-1}$ and $s = q^{-1}$ then $r > 1$ and $s > 1$. Let the operators $\theta_{r,x}$, $\theta_{s,y}$ be defined in a manner similar to the operators $D_{p,x}$, $D_{q,y}$.

$$\theta_{r,x}[f(z)] = \frac{f(z) - f(rx, y)}{(1-r)x} ,$$

$$\theta_{r,y}[f(z)] = \frac{f(z) - f(x, sy)}{(1-s)iy} .$$

A discrete function *f*, defined on *Q'* is said to be '(*r*, *s*)-*analytic*' at $z$ if

$$\theta_{r,x}[f(z)] = \theta_{s,y}[f(z)] \tag{3.1}$$

and the common operator is denoted by $\theta$.

Now equation (3.1) is equivalent to $B_{p,q}[g(z)] = 0$, where the operator $B_{p,q}$ is defined as

$$\begin{aligned} B_{p,q}[g(z)] &= \{(1-r)x - (1-s)iy\}\; g(z) - (1-r)x\; g(x, sy) + (1-s)iy\; g(rx, y) \\ &= \{(1-p^{-1})x - (1-q^{-1})\, iy\}\, g(z) - (1-p^{-1})xg\,(x, q^{-1}y) \\ &\quad + (1-q^{-1})iy\; g(p^{-1}x, y) \end{aligned} \tag{3.2}$$

## 4. Cauchy's integral formula

The following definition of a '*conjoint line integral*' is the (*p*, *q*)–function analogue of the one introduced by Isaacs [2].

If $C = < z_0, z_1, \dots, z_n >$ is a discrete curve in *D*, and if $f$ and $g$ are two discrete functions, then the conjoint line integral along $C$ is defined as

$$\int_{z_0}^{z_n} (f \oplus g)(t)\, d(t; p, q) = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} (f \oplus g)(t)\, d(t; p, q)$$

where

$$\int_{z_j}^{z_{j+1}} (f\oplus g)(t)d(t;p,q)=\begin{cases}(z_{j+1}-z_j)f(z_j)g(z_{j+1}); \text{ for} \\ z_{j+1}=(px_j,y_j) \text{ or } (x_j,qy_j) \\ (z_{j+1}-z_j)f(z_{j+1})g(z_j); \text{ for} \\ z_{j+1}=(p^{-1}x_j,y_j)\text{or}(x_j,q^{-1}y_j)\end{cases} \quad (4.1)$$

The following two theorems are $(p,q)$-analogues of monodiffric results given by Kurowski [5] and Berzseuyi [1]. The proofs are essentially the same and so are omitted.

**Theorem 4.1.** If $f$ is $(p, q)$-analytic and $g$ is $(r, s)$-analytic in $D$ then,

$$\int_C (f\oplus g)(t)\ d(t;\ p,q)=0$$

where $C$ is any closed curve in $D$.

**Theorem 4.2.** If $D$ is a finite discrete domain and if $f$ and $g$ are discrete functions defined on $D$, then

$$\int_{\partial(D)} (f\oplus g)(t)\ d(t;\ p,q) = \sum_{t\in D_N} [f(t)B_{p,q}\ g(p\rho,\ q\sigma)]$$

$$= \sum_{t\in D_N} [f(t)B_{p,q}\ g(p\rho,q\sigma) - g(p\rho,q\sigma)\ R_{p,q}\ f(t)]$$

where $D_N$ is given by (2.11) and $t=\rho+i\sigma$ so that $f(t)\equiv f(\rho,\sigma)$.

The latter theorem is the $(p, q)$ – analogue of Green's Identity.

If $f$ is $(p, q)$ – analytic, then since $R_{p,q}\ f=0$ the following holds.

**Corollary 4.1.** If $f$ is $(p, q)$ – analytic on some finite discrete domain $D$ then,

$$\int_{\partial(D)} (f\oplus g)(t)\ d(t;\ p,q) = \sum_{t\in D_N} f(t)\,B_{p,q}\ g(p\rho,q\sigma).$$

A discrete function $G_a$ is called a '*singularity function*' if it satisfies

$$B_{p,q}\,[G_a(t)] = \begin{cases} 1;\ t = a, & a = a_1 + ia_2 \\ 0;\ t \neq a; & a,\ t \in Q' \end{cases} \tag{4.2}$$

If such a function can be found then Corollary (4.1) reduces to

$$\int_{\partial(D)} (f \oplus G_a)(t)\ d(t;\, p, q) = f(p^{-1} a_1,\ q^{-1} a_2),$$

an analogue of Cauchy's integral formula.

## 5. References


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