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SOME PROPERTIES OF THE BEURLING FUNCTION

Julio Alcántara-Bode

If $\rho(x) = x - [x]$ *is the fractionary part function, we study some properties of the Beurling function*

$$J(\alpha) = \int_0^1 \rho\left(\frac{1}{x}\right)\rho\left(\frac{\alpha}{x}\right) dx,$$

$$\alpha \in [0, 1]$$

In 1955 A. Beurling proved the following theorem.

Theorem. *If*

$$M = \{ f \,/\, f(x) = \sum_{k=1}^{N} a_k \rho\left(\frac{\theta_k}{x}\right), \quad \sum_{k=1}^{N} a_k \theta_k = 0, \quad 0 < \theta_k \le 1, \quad a_k \in \mathbb{C},$$

$$1 \le k \le N, \quad N \ge 2 \}$$

✏ *Profesor de la Sección Matemáticas, Departamento de Ciencias, PUCP.*

then the Riemann Hypothesis (R. H.) holds iff $\bar{M} = L^2(0,1)$. *Moreover* $\bar{M} = L^2(0,1)$ *iff* $1 \in \bar{M}$.

If μ is the Möbius function, $\theta, x \in]0,1]$, pointwise it holds that

$$\sum_{n=1}^{\infty} \mu(n) \left\{ \rho\left(\frac{\theta}{nx}\right) - \frac{1}{n}\rho\left(\frac{\theta}{x}\right) \right\} = -\chi_{]0,\theta]}(x)$$

where χ_A denotes the characteristic function of the set A.

The partial sums of this series are in M, but the series does not converge strongly in $L^2(0,1)$ (R. Heath-Brown, private communication); but to establish R. H. weak convergence would suffice.

To check the condition $1 \in \bar{M}$ one has to minimize the norm

$$\left\| 1 + \sum_{j=1}^{N} a_j \rho\left(\frac{\alpha_j}{\cdot}\right) \right\|^2 = 1 + 2\sum_{j=1}^{N} a_j \int_0^1 \rho\left(\frac{\alpha_j}{x}\right) dx + \sum_{j=1}^{N} a_j^2 \int_0^1 \rho\left(\frac{\alpha_j}{x}\right)^2 dx$$

$$+ 2\sum_{i<j} a_i a_j \int_0^1 \rho\left(\frac{\alpha_i}{x}\right) \rho\left(\frac{\alpha_j}{x}\right) dx$$

where $\sum_{j=1}^{N} a_j \alpha_j = 0$.

Beurling proved that

$$\int_0^1 \rho\left(\frac{\theta}{x}\right) x^r \, dx = \frac{\theta}{r} - \frac{\zeta(r+1)}{r+1} \theta^{r+1}, \theta \in]0,1], \mathrm{Re}\ r > -1,$$

where ζ is the Riemann zeta function. Taking the limit $r \to 0$ one gets

$$\int_0^1 \rho\left(\frac{\theta}{x}\right) dx = -\theta \ln\theta + (1-\gamma)\theta$$

where γ is Euler constant.

From Stirling formula one gets

$$\int_0^1 \rho\left(\frac{\theta}{x}\right)^2 dx = [\ln(2\pi)-\gamma]\,\theta-\theta^2 .$$

It only remains to evaluate

$$\int_0^1 \rho\left(\frac{\alpha_i}{x}\right)\rho\left(\frac{\alpha_j}{x}\right)dx = \alpha_i \int_0^1 \rho\left(\frac{1}{x}\right)\rho\left(\frac{\alpha_j/\alpha_i}{x}\right) + (1-\alpha_i)\,\alpha_j ,$$

where $\alpha_j \leq \alpha_i$.

Therefore if $1 \geq \alpha_1 > \alpha_2 > \ldots > \alpha_N > 0$, then

$$\left\| 1+\sum_{j=1}^{N} a_j\,\rho\left(\frac{\alpha_j}{\cdot}\right)\right\|^2 = 1-2\sum_{j=1}^{N} a_j\alpha_j \ln\alpha_j + \sum_{j=1}^{N} a_j^2\,\alpha_j\,[\ln(2\pi)-\gamma]$$

$$+\,2\sum_{i<j} a_j a_i \left\{\alpha_i\, J\left(\frac{\alpha_j}{\alpha_i}\right) + \alpha_j\right\}$$

where $J(\beta) = \int_0^1 \rho\left(\frac{1}{x}\right)\rho\left(\frac{\beta}{x}\right)$, $\beta\in[0,1]$.

The next result, proven by Skölem and Bang in 1957, is important to study some properties of the function J.

Theorem. *If* $\alpha\in\,]0,1]$ *and* $Q_\alpha = \left\{\left[\frac{n}{\alpha}\right]\ n\in\mathbb{N}\right\}$ *then*

i) $Q_\alpha \cap Q_{1-\alpha} = \phi$ *iff* $\alpha\notin\mathbb{Q}$

ii) $Q_\alpha \cup Q_{1-\alpha} = \mathbb{N}$ *iff* $\alpha\notin\mathbb{Q}$.

Morover if $\alpha = \frac{p}{q}\in\mathbb{Q}$, $p, q\in\mathbb{N}$, $(p,q)=1$, then

i) $Q_\alpha \cap Q_{1-\alpha} = \{j\,q\ /\ j\in\mathbb{N}\}$

ii) $Q_\alpha \cup Q_{1-\alpha} = \mathbb{N}\setminus\{j\,q-1\ /\ j\in\mathbb{N}\}$

using these results of Skölem and Bang one can show that if $\alpha \in [0, 1]$, $\alpha \notin \mathbb{Q}$, then

$$\rho\left(\frac{\alpha}{x}\right) + \rho\left(\frac{1-\alpha}{x}\right) - \rho\left(\frac{1}{x}\right) = \sum_{m=1}^{\infty} \chi_{\left]\frac{\alpha}{m}, \left[\frac{m}{\alpha}\right]^{-1}\right]}(x)$$

$$+ \sum_{m=1}^{\infty} \chi_{\left]\frac{1-\alpha}{m}, \left[\frac{m}{1-\alpha}\right]^{-1}\right]}(x) . \tag{1}$$

If $\alpha = \frac{p}{q} \in \mathbb{Q}$, then

$$\rho\left(\frac{\alpha}{x}\right) + \rho\left(\frac{1-\alpha}{x}\right) - \rho\left(\frac{1}{x}\right) = \sum_{m=1}^{\infty} \chi_{\left]\frac{\alpha}{m}, \left[\frac{m}{\alpha}\right]^{-1}\right]}(x)$$

$$+ \sum_{m=1}^{\infty} \chi_{\left]\frac{1-\alpha}{m}, \left[\frac{m}{1-\alpha}\right]^{-1}\right]}(x) + \sum_{r=1}^{\infty} \chi_{\left]\frac{1}{qr}, \frac{1}{qr-1}\right]}(x) . \tag{2}$$

If in (1) and (2) the term $-\rho\left(\frac{1}{x}\right)$ is moved to the right hand side, we square and integrate over x between 0 and 1 we get

$$J(\beta) = K(\beta) \quad \text{if} \quad \beta \in [0, 1], \ \beta \notin \mathbb{Q} \tag{3}$$

and

$$J(\beta) = K(\beta) + \frac{p+q}{q}\left\{\ln \Gamma\left(1 - \frac{1}{p+q}\right) - \frac{\gamma}{p+q}\right\} \tag{4}$$

if $\beta = \frac{p}{q} \in \mathbb{Q} \cap [0, 1]$

where Γ is the gamma function and

$$K(\beta) = \frac{\ln(1+\beta)}{2} + \frac{\beta}{2}\ln\left(\frac{1+\beta}{\beta}\right) - \beta$$

$$- (1+\beta)\sum_{m=1}^{\infty}\left\{\ln\left(1-\frac{\rho(m\beta)}{m(1+\beta)}\right) + \frac{\rho(m\beta)}{m(1+\beta)}\right\}$$

$$- (1+\beta)\sum_{m=1}^{\infty}\left\{\ln\left(1-\frac{\rho\left(\frac{m}{\beta}\right)\beta}{m(1+\beta)}\right) + \frac{\rho\left(\frac{m}{\beta}\right)\beta}{m(1+\beta)}\right\}.$$

If we square (1) and (2), and integrate over x between 0 and 1 we obtain

$$-\frac{\alpha\ln\alpha}{2} - \frac{(1-\alpha)\ln(1-\alpha)}{2} = J(1) - J(\alpha) - J(1-\alpha) + \alpha + (1-\alpha)\,J\left(\frac{\alpha}{1-\alpha}\right)$$

$$\forall\,\alpha \in \left[0,\ \frac{1}{2}\right] \tag{5}$$

that is a functional equation that relates the values of J at 3 points.

J also obeys a functional equation that relates the values at five points. To clarify the results (3), (4) and (5) we integrate equations (1) and (2) between 0 and 1 to obtain

$$-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha) = \sum_{m=1}^{\infty}\left\{\frac{1}{\left[\frac{m}{\alpha}\right]} + \frac{1}{\left[\frac{m}{1-\alpha}\right]} - \frac{1}{m}\right\} \tag{6}$$

and

$$-\alpha\ln\alpha - (1-\alpha)\ln(1-\alpha) = \sum_{m=1}^{\infty}\left\{\frac{1}{\left[\frac{m}{\alpha}\right]} + \frac{1}{\left[\frac{m}{1-\alpha}\right]} - \frac{1}{m}\right\}$$

$$+ \sum_{r=1}^{\infty}\frac{1}{qr(qr-1)} \tag{7}$$

respectively, (6) and (7) have been used in deriving (3) and (4).

Now if g: $\mathbb{N} \to \mathbb{C}$ is such that $\sum_{n=1}^{\infty} |g(n)| < \infty$ we get from the results of Skölem and Bang that

$$\sum_{n=1}^{\infty} \left\{ g\left(\left[\frac{n}{\alpha}\right]\right) + g\left(\left[\frac{n}{1-\alpha}\right]\right) - g(n) \right\} = 0 \qquad (8)$$

if $\alpha \in [0, 1]$, $\alpha \notin \mathbb{Q}$, and

$$\sum_{n=1}^{\infty} \left\{ g\left(\left[\frac{n}{\alpha}\right]\right) + g\left(\left[\frac{n}{1-\alpha}\right]\right) - g(n) \right\} + \sum_{r=1}^{\infty} \{ g(rq-1) - g(rq) \} = 0 \qquad (9)$$

if $\alpha = \dfrac{p}{q} \in [0,1] \cap \mathbb{Q}$.

Now if f: $\mathbb{N} \to \mathbb{C}$ is such that $\sum_{n=1}^{\infty} \left| f(n) - \frac{1}{n} \right| < \infty$ we get from (6), (7), (8) and (9) that

$$\sum_{n=1}^{\infty} \left\{ f\left(\left[\frac{n}{\alpha}\right]\right) + f\left(\left[\frac{n}{1-\alpha}\right]\right) - f(n) \right\} = -\alpha \ln \alpha - (1-\alpha) \ln (1-\alpha) \qquad (10)$$

or

$$\sum_{n=1}^{\infty} \left\{ f\left(\left[\frac{n}{\alpha}\right]\right) + f\left(\left[\frac{n}{1-\alpha}\right]\right) - f(n) \right\} = -\alpha \ln \alpha - (1-\alpha) \ln (1-\alpha)$$

$$- \sum_{r=1}^{\infty} \{ f(qr-1) - f(qr) \} \qquad (11)$$

depending whether $\alpha \in [0, 1]$ is irrational or not. It can be shown that of we replace (3) or (4) in (5), the resulting equations reduces to (10) or (11) for a convenient f.

The function J is continous, but $\nexists\ J'(\alpha)\ \ \forall\ \alpha \in [0,1] \cap \mathbb{Q}$. More precisely we have

$$\lim_{l\to\infty} \frac{J\left(\frac{p}{q}+\frac{1}{lq}\right) - J\left(\frac{p}{q}\right)}{\frac{1}{lq}\ln l} = -\frac{1}{2}\frac{q}{p}\ , \quad l \in \mathbb{N},\ \frac{p}{q} \in\ ]0,1[\ \cap \mathbb{Q},$$

$$\lim_{h\to 0^+} \frac{J(h)-J(0)}{h\ln\frac{1}{h}} = \frac{1}{2}, \quad \lim_{h\to 0^+} \frac{J(1)-J(1-h)}{h\ln\frac{1}{h}} = \frac{1}{2}.$$

We don't have at present information about $J'(\alpha)$ for $\alpha \in [0,1]$, $\alpha \notin \mathbb{Q}$.

Talk given at the ICM, Berlin, 1998.

References

[1] Alcántara-Bode, J. (1999). "*Génesis de la Hipótesis de Riemann*". Pro-Mathematica, vol XIII, N° 25-26, 5-10.

[2] Alcántara-Bode, J. (1996). "*Reorderings of some divergent series*". Pro-Mathematica, vol X, N° 19-20, 5-8.

Julio Alcántara-Bode
jalcant@pucp.edu.pe