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# ON $g\alpha$–COMPACT SPACES

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***Abstract***

*The purpose of this paper is to introduce and discuss the concept of $g\alpha$-compactness for topological spaces. An example is considered to show that it is strictly stronger than that of compactness.*

**Keywords and phrases**: Topological spaces, $\alpha$-open sets, $g\alpha$-closed sets, compactness, $g\alpha$-compactness.

## 1. Introduction

O. Njastad [9] and N. Levine [5] introduced and investigated the notion of $\alpha$-open sets (originally called $\alpha$-sets) and generalized closed sets (written in short as $g$-closed sets) of a topological space respectively. Recently in a significant contribution to the theory of generalized closed sets H. Maki, R. Devi and K. Balachandran [6] defined the concept of generalized $\alpha$-closed

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sets (shortly $g\alpha$-closed sets) as a generalization of $\alpha$-closed sets. These sets were also considered by various authors (e.g. [1], [2], [3], [4], [11]). Using this the concepts of $g\alpha$-continuous mappings [7], separation axioms ([3], [4], [11]) and connectedness in topological spaces were introduced. In this paper we introduce $g\alpha$-compactness and study it. This notion comes out to be strictly stronger than that of compactness. The significance of the present study lies in the fact that the sufficient condition in Theorem 2.3 holds for $g\alpha$-compact spaces but in general it fails for compact spaces.

Throughout this paper $(X,\tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$ respectively) denote nonempty topological spaces in which no separation axioms are assumed unless explicitely, stated. For a subset A of a space $(X, \tau)$ the closure of $A$ the interior of $A$ and the complement of $A$, are denoted by $Cl(A)$, $Int(A)$ and $A^c$ respectively. A subset $A$ of a space $(X, \tau)$ is said to be $\alpha$-open [9] if $A \subset Int(Cl(Int(A)))$. The complement of $\alpha$-open sets are called $\alpha$-closed sets. Alternatively a subset $A \subset (X, \tau)$ is called $\alpha$-closed if $A \supset Cl(Int(Cl(A)))$. The family of all $\alpha$-open sets of $(X, \tau)$ is denoted by $\alpha(X)$ which is a topology on $X$ such that $\alpha(X) \supset \tau$ [9].

A subset $A$ of $(X, \tau)$ is called:

(i) generalized closed (written as $g$-closed) [5] if $Cl(A) \subset U$ whenever $A \subset U$ and $U$ is open in $(X, \tau)$. Every closed sets is $g$-closed but not conversely [5].

(ii) generalized $\alpha$-closed (written as $g\alpha$-closed) [6] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and $U$ is $\alpha$-open in $(X, \tau)$. Every $\alpha$-closed set is $g\alpha$-closed but not conversely and the concepts of $g$-closed sets and $g\alpha$-closed sets are independent [6].

The complement of an $g\alpha$-closed set is called $g\alpha$-open. The family of all $g\alpha$-open sets will be denoted by $g\alpha(X)$.

A mapping $f$: $(X,\tau) \to (Y,\sigma)$ is said to be $g\alpha$-continuous (resp. $g\alpha$-irresolute) if the inverse image of every open (resp. $g\alpha$-open) set of $Y$ is $g\alpha$-open in $X$.

## 2. gα-compactness

**Definition 1.** *A collection* $\{A_i : i \in I\}$ *of* $g\alpha$*-open sets in a topological space* $(X, \tau)$ *is called* $g\alpha$*-open cover of a set* $H$ *of* $X$ *if* $H \subset \bigcup\{A_i : i \in I\}$ *holds.*

**Definition 2.** *A topological space* $(X, \tau)$ *is said to be* $g\alpha$*-compact if every* $g\alpha$*-open cover of* $X$ *has finite subcover.*

**Definition 3.** *A subset* $H$ *of a topological space* $(X, \tau)$ *is said to be* $g\alpha$*-compact relative to* $X$*, if for every collection* $\{A_i : i \in I\}$ *of* $g\alpha$*-open sets of* $X$ *such that* $H \subset \bigcup\{A_1 : i \in I\}$ *there exists a finite subset* $I_0$ *of* $I$ *such that* $H \subset \bigcup\{A_1 : i \in I_0\}$.

**Remark 2.1.** *Since every open set is an* $g\alpha$*-open, it follows that every* $g\alpha$*-compact space is compact. But the converse may be false. The following example shows this.*

**Example 2.2.** *Let* $X = \{x\} \bigcup \{x_i : i \in I\}$ *where the indexed set* $I$ *is uncountable. Let* $\tau = \{\phi, \{x\}, X\}$ *be the topology on* $X$*. Evidently,* $X$ *is a compact space. However, it is not a* $g\alpha$*-compact space because* $\{\{x, x_i\} : i \in I\}$ *is a* $g\alpha$*-covering of* $X$ *but it has no finite subcover.*

The following result can be easily verified. Its proof is straightforward.

**Theorem 2.3.** *A topological space* $(X, \tau)$ *is* $g\alpha$*-compact if and only if every family of* $g\alpha$*-closed subsets of* $X$ *which has the finite intersection property has a nonempty intersection.*

**Remark 2.4.** *The sufficient condition in the above theorem fails for compact spaces. For, in the example 2.2 ([8], Example 3.1)* $\{X - \{x, x_i\} : i \in I\}$ *is a family of* $g\alpha$*-closeds in a compact space* $X$ *whose intersection is empty but each of its finite subfamilies always has non-empty intersection.*

We will prove that $g\alpha$-compactness of a topological space is equivalent to compactness of a larger space, namely that having the collection of all $g\alpha$-open subsets of the given space as a subbase.

**Theorem 2.5.** *Let* $(X, \tau)$ *be a topological space and* $\tau_\xi$ *be a topology on* $X$ *which has* $g\alpha(X)$ *as a subbase. Then* $(X, \tau)$ is $g\alpha$*-compact if and only if* $(X, \tau_\xi)$ *is compact.*

***Proof.*** If $(X, \tau_\xi)$ is compact then $(X, \tau)$ is $g\alpha$-compact, since $g\alpha(X) \subset \tau_\xi$. The converse is a consequence of the famous Alexander's subbase Theorem ([10], p.18). □

**Theorem 2.6.** *Let $F$ be a $\tau_\xi$-closed subset of a $g\alpha$-compact space $X$. Then $F$ is also $g\alpha$-compact.*

***Proof.*** *Let $F$ be* any $\tau_\xi$-closed subset in $X$ and $\{U_{\beta_i} : \beta_i \in I\}$ be a $\tau_\xi$-open cover of $F$. Since $F^c$ is $\tau_\xi$-open, $\{U_{\beta_i} : \beta_i \in I\} \cup F^c$ is a $\tau_\xi$-open cover of $X$. Since $X$ is $\tau_\xi$-compact by Theorem 2.5 there exists a finite subset $I_0 \subset$ I such that $X = \bigcup\{U_{\beta_i} : \beta_i \in I_0\} \cup F^c$. But $F$ and $F^c$ are disjoint; hence $F \subset \bigcup\{U_{\beta_i} : \beta_i \in I_0\}$. Therefore $F$ is $g\alpha$-compact relative to $X$ and this completes the proof. □

**Theorem 2.7.** *A topological space* $(X, \tau)$ is *$g\alpha$-compact if and only if every family of $\tau_\xi$-closed subset of $X$ with finite intersection property has nonempty intersection.*

***Proof.*** *Necessity.* Let $X$ be $g\alpha$-compact. Let $H = \{V_{\beta_i} : \beta_i \in I\}$ be any family of $\tau_\xi$-closed subset of $X$ with finite intersection property. Suppose $\bigcap\{V_{\beta_i} : \beta_i \in I\} = \phi$. Then $\{V^c_{\beta_i} : \beta_i \in I\}$ is a $\tau_\xi$-open cover of $X$. Hence it must contain a finite subcover $\{V^c_{\beta_{i_j}} : j = 1, 2, \ldots, n\}$ for $X$. This implies that $\bigcap\{V_{\beta_{i_j}} : j = 1, 2, \ldots, n\} = \phi$ which contradicts the assumption that $H$ has finite intersection property.

*Sufficiency.* Let $H = \{V_{\beta_i} : \beta_i \in I\}$ be a cover of $X$ by $\tau_\xi$-open sets and consider the family $\Psi = \{V^c_{\beta_i} : \beta_i \in I\}$ of $\tau_\xi$-closed sets. Since $H$ is a cover of $X$, it follows that the intersection of all members of $\Psi$ is empty. Hence $\Psi$ does not have the finite intersection property, in other words, there is a finite number of $\tau_\xi$-open sets $V_{\beta_1}, \ldots, V_{\beta_n}$ in $H$ such that $V_{\beta_1}^c \cap \ldots \cap V_{\beta_n}^c = \phi$. This implies that $\{ V_{\beta_1}, \ldots, V_{\beta_n} \}$ is a finite subcover of $X$ in $H$. Hence using Theorem 2.5 $X$ is $g\alpha$-compact. □

**Definition 4.** *Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $\tau_\xi$ be a topology on X which has $g\alpha(X)$ as a subbase. A mapping $f: (X, \tau) \to (Y, \sigma)$ is called $g\alpha(\xi)$-continuous if $f: (X, \tau_\xi) \to (Y, \sigma)$ is continuous.*

**Definition 5.** *Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. Let $\tau_\xi$ and $\sigma_\xi$ be respectively the topology on X and Y which has $g\alpha(X)$ and $g\alpha(Y)$ as subbase. A mapping $f: (X, \tau) \to (Y, \sigma)$ is called $g\alpha^*(\xi)$-continuous if*

$$f: (X, \tau_\xi) \to (Y, \sigma_\xi)$$

*is continuous.*

**Theorem 2.8.** *Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $\tau_\xi$ be the topology on X which has $g\alpha(X)$ as a subbase. If $f: (X, \tau) \to (Y, \sigma)$ is $g\alpha$-continuous, then f is $g\alpha(\xi)$-continuous.*

***Proof.*** Let $f$ be $g\alpha$-continuous and let $V \in \sigma$. Then $f^{-1}(V) \in g\alpha(X)$ and so $f^{-1}(V) \in \tau_\xi$. Thus $f$ is $g\alpha(\xi)$-continuous. □

**Theorem 2.9.** *Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces. Let $\tau_\xi$ and $\sigma_\xi$ be respectively the topologies on X and Y which have $g\alpha(X)$ and $g\alpha(Y)$ as subbases. If $f: (X, \tau) \to (Y, \sigma)$ is $g\alpha$-irresolute then f is $g\alpha^*(\xi)$-continuous.*

***Proof.*** Let $f$ be $g\alpha$-irresolute and let $V \in \sigma_\xi$. Then $V = \bigcup_i (\bigcap_{j=1}^{n} U_{i_{n_j}})$, where $U_{i_{n_j}} \in g\alpha(Y)$ and

$$f^{-1}(V) = f^{-1}(\bigcup_i (\bigcap_{j=1}^{n} U_{i_{n_j}})) = \bigcup_i (\bigcap_{j=1}^{n} f^{-1}(U_{i_{n_j}})).$$

Since $f$ is $g\alpha$-irresolute, $f^{-1}(U_{i_{n_j}}) \in g\alpha(X)$. This implies that $f^{-1}(V) \in \tau_\xi$ and thus $f$ is $g\alpha^*(\xi)$-continuous. □

**Theorem 2.10.**

*(i)* *A $g\alpha(\xi)$-continuous image of a $g\alpha$-compact space is compact.*

*(ii)* *Let $(X, \tau)$ and $(Y, \sigma)$ be topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be $g\alpha^*(\xi)$-continuous. If a subset $F$ of $X$ is $g\alpha$-compact relative to $X$, then the image $f(F)$ is $g\alpha$-compact relative to $Y$.*

***Proof.***

(i): Let $f : X \to Y$ be a $g\alpha(\xi)$-continuous mapping from a $g\alpha$-compact $X$ onto a topological space $Y$. Let $\{V_{\beta_i} : \beta_i \in I\}$ be an open cover of $Y$. Then $\{f^{-1}(V_{\beta_i}) : \beta_i \in I\}$ is $\tau_\xi$-open cover of $X$. Since $X$ is $g\alpha$-compact by Theorem 2.5 $X$ is $\tau_\xi$-compact, and it has a finite subcover say $\{f^{-1}(V_{\beta_1}), \ldots, f^{-1}(V_{\beta_n})\}$. Since $f$ is onto $\{V_{\beta_1}, \ldots, V_{\beta_n}\}$ is a cover of $Y$ and so $Y$ is compact.

(ii): Let $\{V_{\beta_i} : \beta_i \in I\}$ be a cover of $f(F)$ by $\tau_\xi$-open sets in $Y$. Since $F$ is $g\alpha$-compact relative to $X$ by Theorem 2.6, $F$ is $\tau_\xi$-compact. So there exists a finite subset $I_0 \subset I$ such that $F \subset \bigcup\{f^{-1}(V_{\beta_i}) : \beta_i \in I_0\}$ and so $f(F) \subset \bigcup\{V_{\beta_i} : \beta_i \in I_0\}$. Hence $f(F)$ is $\tau_\xi$-compact relative to $Y$. Thus $f(F)$ is $g\alpha$-compact relative to $Y$. □

**Corollary 2.11.** *A $g\alpha^*(\xi)$-continuous image of a $g\alpha$-compact space is $g\alpha$-compact.*

**Theorem 2.12.** *Let $f : (X, \alpha) \to (Y, \sigma)$ be an $g\alpha$-irresolute mapping of $X$ onto $Y$ if $X$ is $g\alpha$-compact then $Y$ is $g\alpha$-compact.*

***Proof.*** By Theorem 2.9 and Corollary 2.11. □

**Theorem 2.13.** *Let F and G be subsets of a topological space $(X, \tau)$ such that F is gα-compact relative to X and G is $\tau_\xi$-closed in X. Then $F \cap G$ is gα-compact relative to X.*

***Proof.*** Let $\{V_{\beta_i} : \beta_i \in I\}$ be a cover of $F \cap G$ by $\tau_\xi$-open subsets of $X$. Since $G^c$ is a $\tau_\xi$-open set, $\{V_{\beta_i} : \beta_i \in I\} \cup G^c$ is a cover of $F$. Since $F$ is gα-compact, it is $\tau_\xi$-compact relative to $X$. Hence there exists a finite subset $I_0 \subset I$ such that

$$F \subset (\bigcup\{V_{\beta_i} : \beta_i \in I_0\}) \cup G^c .$$

Therefore

$$F \cap G \subset \bigcup\{V_{\beta_i} : \beta_i \in I_0\} .$$

Hence $F \cap G$ is $\tau_\xi$-compact. Therefore $F \cap G$ is gα-compact and this completes the proof. □

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