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DISCRETE ANALYTIC CONTINUATION OF A (p, q)-ANALYTIC FUNCTION

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Abstract

In this paper a method is devised for the continuation into the discrete plane Q' of functions defined on the positive half-axes and the properties of continuation operator discussed.

1. Introduction

In 1993 the first author [7] introduced the concept of (p, q)-analyticity by considering fuctions defined on the following geometric lattice:

$$K = \{(p^m x_0, q^n y_0);\ m, n, \in Z,\ \textit{the set of integers},\ 0 < p < 1,\ 0 < q < 1,\ (x_0, y_0)\ \textit{fixed},\ x_0 > 0,\ y_0 > 0\}. \tag{1.1}$$

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In what follows $z \in K$, $z = (x, y) = (p^m x_0, q^n y_0)$.

We recall here some of the definitions given in [7].

Definition 1. *The 'discrete plane' Q' with respect to some fixed point $z' = (x', y')$ in the first quadrant, is defined by the set of lattice points,*

$$Q' = \{ (p^m x', q^n y') ; m, n \in Z \text{ the set of integers} \}.$$

Definition 2. *Two lattice points $z_i, z_{i+1} \in Q'$ are said to be 'adjacent' if z_{i+1} is one of (px_i, y_i), $(p^{-1} x_i, y_i)$, (x_i, qy_i) or $(x_i, q^{-1} y_i)$.*

Definition 3. *A 'discrete curve' C in Q' connecting z_0 to z_n* is denoted by the sequence

$$C \equiv < z_0, z_1, \dots, z_n >,$$

where z_i, z_{i+1}; $i = 0, 1, \dots, (n-1)$ are adjacent points of Q'.

If the points are distinct ($z_i \neq z_j$; $i \neq j$) then the discrete curve C is said to be 'simple'.

Definition 4. *A 'discrete closed curve' C in Q' is given by the sequence $< z_0, z_1, z_2, \dots, z_n >$ where $< z_0, z_1, \dots, z_{n-1} >$ is simple and $z_0 = z_n$.*

Denote by $\overline{C}$ the continuous closed curve formed by joining adjacent points of the discrete closed curve C. Then $\overline{C}$ encloses certain points of Q', denoted by $Int(C)$.

Definition 5. *A 'finite discrete domain' B is defined as*

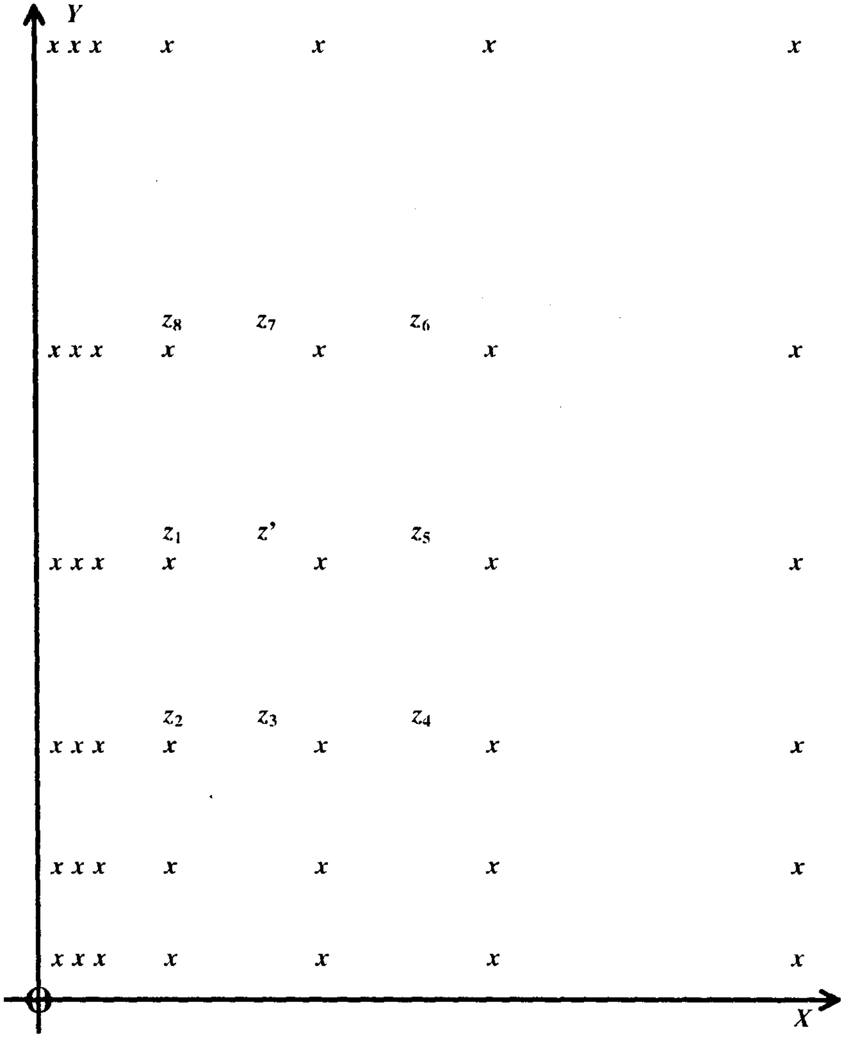
$$B = \{ z \in Q' ; \; z \quad C \cup Int(C) \}.$$

Definition 6. *A 'basic set' respect to $z \in Q'$ is defined as*

$$S(z) = \{ (x, y), (px, y), (px, qy), (x, qy) \},$$

and the discrete closed curve around $S(z)$ is denoted by

$$\partial (s) = < (x, y), (px, y), (px, qy), (x, qy), (x, y) > \qquad (1.2)$$



THE LATTICE

$z' = (x', y')$

POINTS $z_1, z_2, \ldots, z_8$ ARE GIVEN BY

$z_1 = (px', y'), z_2 = (px', qy'), z_3 = (x', qy'), z_4 = (p^{-1}x', qy')$

$z_5 = (p^{-1}x', y'), z_6 = (p^{-1}x', q^{-1}y'), z_7 = (x', q^{-1}y'), z_8 = (px', q^{-1}y')$

Figure 1.

Definition 7. *Functions defined on the points of a discrete domain B are said to be 'discrete functions'.*

Definition 8. *The p-difference and q-difference operators* $D_{p,x}$ *and* $D_{q,y}$ *are defined as follows:*

$$D_{p,x}[f(z)] = \frac{f(z) - f(px, y)}{(1-p)x} \tag{1.3}$$

$$D_{q,y}[f(z)] = \frac{f(z) - f(x, qy)}{(1-q)iy} \tag{1.4}$$

where f *is a discrete function.*

We now recall the definition of (p, q)-analytic function as introduced in [7]. The two operators (1.3) and (1.4) involve a 'basic triad' of points denoted by

$$T(z) = \{(x, y), (px, y), (x, qy)\}. \tag{1.5}$$

Definition 9. *Let B be a discrete domain. Then a discrete function f is said to be '(p, q)-analytic' at* $z \in B$ *if*

$$D_{p,x}[f(z)] = D_{q,y}[f(z)]. \tag{1.6}$$

If in addition (1.6) holds for every $z \in B$ such that

$T(z) \subseteq B$ then f is said to be (p, q)-analytic in B. (1.7)

For simplicity if (1.6) or (1.7) holds, the common operator D_q is used where

$$D_q \quad D_{p,x} \quad D_{q,y}. \tag{1.8}$$

Definition 10. *The operator* $R_{p,q}$ *is defined as*

$$R_{p,q}[f(z)] = \{(1-p)x - i(1-q)y\} f(x, y) - (1-p)x f(x, qy) + i(1-q)y f(px, y) \tag{1.9}$$

where $f : K \to C$*, the field of complex numbers.*

$R_{p,q} f(z)$ is called (p, q)-residue of the function at z.

From (1.7) it is easily seen that f is (p, q)-analytic in a discrete domain D iff

$$R_{p,q}[f(z)] = 0. \tag{1.10}$$

Definition 11. *Since a discrete domain B is the union of basic sets S so if the discrete domain B is given by*

$$B = \bigcup_{i=1}^{N} S(z_i),$$

then the 'subdomain' B_N is defined by

$$B_N = \{z_i \; ; \; i = 1, 2, \ldots, N\}. \tag{1.11}$$

In the present paper the discrete plane Q' is extended to include points on the positive half-axes. If a (p, q)-analytic function is defined on a subset of Q' then it has a unique extension as a (p, q)-analytic function to certain other points of the discrete plane. An outline of results of this type is given and a method is devised for the continuation into Q' of functions defined on the axes.

2. Boundary Conditions

a) The (p, q)-difference operator $R_{p,q}$, defined in (1.9), involves a basic triad óf points,

$$T(z) = \{z, (px, y), (x, qy)\}.$$

From (1.10) it follows that given the value of a (p, q)-analytic function f at any two points of $T(z)$, then it is uniquely determined at the third point. In fact

$$\begin{aligned} f(z) &= \frac{(1-p)x\, f(x, qy) - i(1-q)y\, f(px, y)}{(1-p)x - i(1-q)y} \\ f(px, y) &= \frac{(1-p)x\, f(x, qy) - \{(1-p)x - i(1-q)y\}\, f(z)}{i(1-q)y} \\ f(x, qy) &= \frac{\{(1-p)x - i(1-q)y\}\, f(z) + i(1-q)y\, f(px, y)}{(1-p)x}. \end{aligned} \tag{2.1}$$

b) From the above it is easily verified that if a (p, q)-analytic function f is defined at the horizontal set of points $\{(p^m x, y);\ m \in Z$ then it can be uniquely continued as a (p, q)- analytic function to all points of Q' below this set, (i.e. all points of the form

$$\{(p^m x,\ q^n\ y);\quad m \in Z;\quad n = 0, 1, 2, \ldots\}).$$

c) Similarly if the function f is defined on the vertical set $\{(x,\ q^n y); n \in Z\}$, then f has a unique continuation as a (p, q)-analytic function to all points of Q' to the left of this set, (i.e. all points of the form $\{(p^m x,\ q^n y);\ n \in Z;\ m = 0, 1, 2, \ldots\}$).

d) If f is defined on the sets $\{(p^m x,\ y);\ m \in Z$ and $\{(x, q^n y); n = 0, -1, -2, \ldots\}$ (fig. 2) then it has a unique continuation as a (p, q)-analytic function to all points of Q'. The result for the region to the left and below these sets follows from boundary conditions (b) and (c). The value of f at the point C (fig. 2) is given by the values of f at A, B by boundary condition (a). Similarly the function f is determined uniquely at all other lattice points in the region R_1 (points of the form $\{(p^m x, q^n y);\ m = 0, -1, -2, \ldots;\ n = 0, -1, -2, \ldots\}$).

e) If f is defined on the sets, $\{(p^m x, y);\quad m = 0,\ 1,\ 2, \ldots\}$ and $\{(x, q^n y);\ n = 0, 1, 2, \ldots\}$ (fig. 3) then by repeated application of boundary condition (a) the function has unique continuation into the rectangular region $\{(p^m x, q^n y);\ m = 0, 1, 2, \ldots;\ n = 0, 1, 2, \ldots\}$ denoted by R_2 in fig. 3.

The above boundary conditions are the (p, q)-analogues of results in monodiffric theory outlined by Isaacs [5] and Berzsenyi [1].

The following theorem is similar to a result of Berzsenyi and the proof, being equivalent, is omitted.

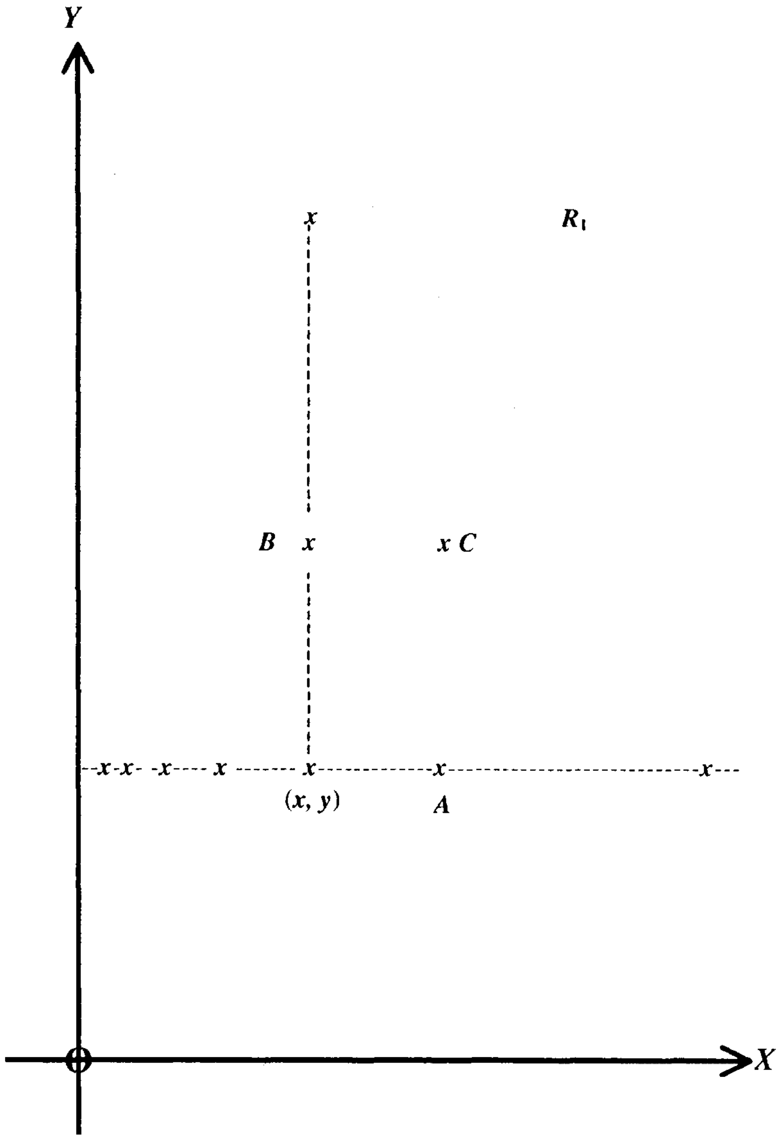


Figure 2.

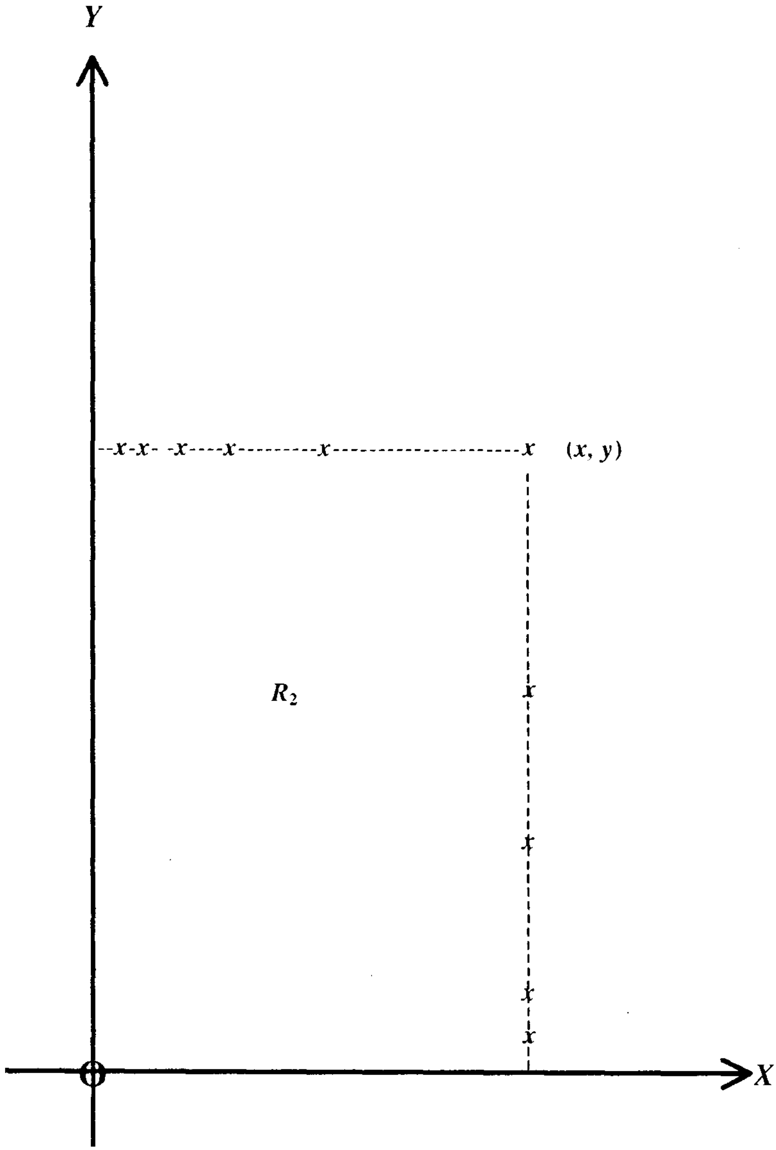


Figure 3.

Theorem 1. *Let B be a finite discrete domain and f be a (p, q)-analytic function defined on the boundary points $\partial(B)$. Then there exists a unique (p, q)-analytic function g, defined on D such that $f = g$ on $\partial(B)$.*

In fact more than this is true. A function defined on the boundary $\partial(B)$ can also be continued to certain points outside B as follows:

The discrete domain B consists of a union of basic sets, i.e. a union of lattice points of the form

$$B = \{\, (p^i x,\ q^j y)\ ;\ i \in I,\ \ j \in J \}$$

where I, J are sets of integers determined by B. Let

$$m_B = \min_{i \in I} \ i$$

$$n_B = \min_{j \in J} \ j$$

$$M_B = \max_{\substack{i \in I \\ j \in J}} \ i + j$$

$$\ell_1 = \{ (p^{m_B} x, q^j y);\ j = n_B,\ n_B + 1, \ldots, (M_B - m_B) \}$$

$$\ell_2 = \{ (p^i x, q^{n_B} y);\ i = m_B,\ m_B + 1, \ldots, (M_B - n_B) \}$$

$$\ell_3 = \{ (p^i x, q^j y);\ i + j = M_B \quad \text{where} \quad i = m_B,\ m_B + 1, \ldots, M_B - n_B \ \text{ and } \ j = n_B,\ n_B + 1, \ldots, M_B - m_B \}.$$

Figure 4 illustrates the above notation. The boundary $\partial(B)$ is indicated by the solid line. ℓ_1 is given by the horizontal set of points between A and B, ℓ_2 by the vertical points between B and C and ℓ_3 is given by the diagonal-like set of points between A and C.

If G represents the subset of Q' bounded by and including ℓ_1, ℓ_2, ℓ_3 then the following holds:

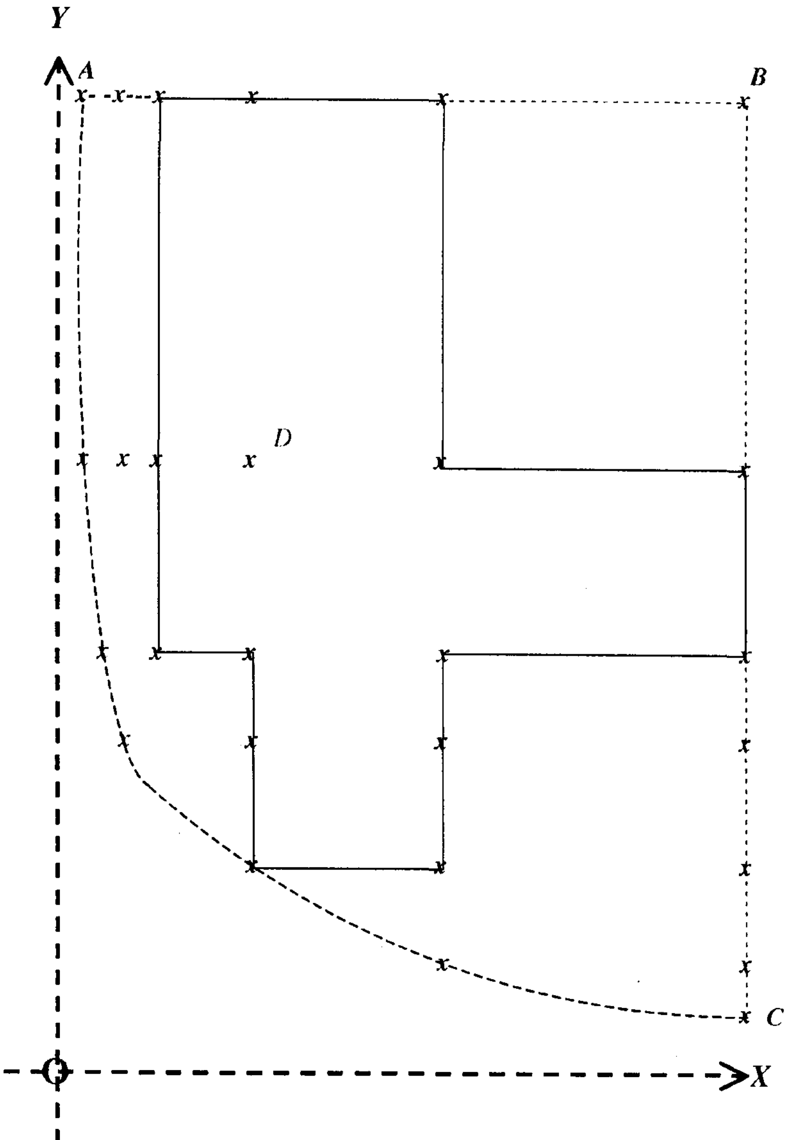


Figure 4.

Theorem 2. *If a (p, q)-analytic function f is defined on the boundary $\partial(B)$ of some finite discrete domain B, then there exists a unique (p, q)-analytic function g defined on G such that $f = g$ on $\partial(B)$.*

The result follows for points of B by theorem 1. By repeated application of boundary condition (a) the theorem can easily be shown to be true for all other points of G.

3. Continuation from the Axes

The discrete plane Q' consists of horizontal and vertical sets of points tending towards the axes (fig. 1). It proves useful to consider discrete functions also defined on the axes. Consequently the discrete plane is extended as follows:

Let

$$X \equiv \{(p^m x', 0);\ m \in Z\}$$

$$Y \equiv \{(0, q^n y');\ n \in Z\}$$

where (x', y') is the fixed point from which the lattice Q' is defined.

The 'extended discrete plane' $\bar{Q}$ is then defined as

$$\bar{Q} = Q' \cup X \cup Y. \tag{3.1}$$

The 'discrete rectangular domain' R' is defined by

$$R' = \{(p^m x', q^n y');\ m = 0, 1, 2, \ldots;\ n = 0, 1, 2, \ldots\} \tag{3.2}$$

If X^+, Y^+ are defined by

$$\begin{aligned} X^+ &\equiv \{(p^m x', 0);\ m = 0, 1, 2, \ldots\} \\ Y^+ &\equiv \{(0, q^n y');\ n = 0, 1, 2, \ldots\} \end{aligned} \tag{3.3}$$

then the 'extended rectangular domain' $\bar{R}$ is defined as

$$\bar{R} \equiv R' \cup X^+ \cup Y^+. \tag{3.4}$$

Discrete function can now be defined on X, Y. The values on the axes, of a discrete function f defined on R', are defined to be

$$f(x,0) = \lim_{n \to \infty} f(x, q^n y')$$

$$f(0,y) = \lim_{m \to \infty} f(p^m x', y); \; (x,y) \in R' .$$

Alternatively this can be expressed as

$$f(x,0) = \lim_{y \to 0} f(x,y); \quad (x,y) \in R' \tag{3.5}$$

$$f(0,y) = \lim_{x \to 0} f(x,y); \; (x,y) \in R' \tag{3.6}$$

where $\lim\limits_{y \to 0}$ has the same meaning as $\lim\limits_{n \to \infty} q^n y'$ and similarly for $\lim\limits_{x \to 0}$.

The definition of (p, q)-analyticity is now extended to functions defined on the axes.

A function f is said to be (p, q)-analytic on X^+ if the limit in (3.5) exists for each x such that $(x,0) \in X^+$, and if

$$\lim_{y \to 0} D_y f(x,y) = D_x f(x,0); \; (x,y) \in R' . \tag{3.7}$$

Similarly f is (p, q)-analytic on Y^+ if the limit in (3.6) exists and

$$\lim_{x \to 0} D_x f(x,y) = D_y f(0,y); \; (x,y) \in R' . \tag{3.8}$$

If f is (p, q)-analytic in R' and if (3.7), (3.8) hold, then f is said to be (p, q)-analytic in $\overline{R}$. (3.9)

This definition can of course be extended to all of Q' and Q but for present purposes the above suffices.

The p-difference or q-difference operator of order j are defined by

$$D^j_{p,x} [f(z)] = D_{p,x} [D^{j-1}_{p,x} f(z)]; \quad D^0_{p,x} [f(z)] = f(z), \quad j = 0,1,2, \ldots$$

and

$$D^j_{q,y}[f(z)] = D_{q,y}\,[D^{j-1}_{q,y}\; f(z)];\quad D^0_{q,y}[f(z)] = f(z),\quad j = 0, 1, 2, \ldots$$

Analytic functions of a continuous complex variable have derivatives of all orders. A corresponding result is true for (p, q)-analytic functions defined on R , and is now considered.

Lemma 1. If f is (p, q)-analytic in $\overline{R}$ then for each $z \in \overline{R}$, $D^j_{p,x}[f(z)]$ and $D^j_{q,y}[f(z)]$ exist and

$$D^j_{p,x}[f(z)] = D^j_{q,y}[f(z)],\ j = 0, 1, 2, \ldots .$$

Proof. If $z \in \overline{R}$, then $z \in R'$, X^+ or Y^+.

(i) Let $z \in R'$, and f be a (p, q)- analytic function in R'. From the definition of the subdomain B_N it follows that if $B = R'$ then $B_N = R'$ also.

Hence by theorem 1 of [7] $D_{p,x}[f(z)]$ and $D_{q,y}[f(z)]$ are (p, q)-analytic in R'. From which it follows then that.

$$D_{p,x}\,[D_{p,x}\,f(z)] = D_{q,y}\,[D_{p,x}\,f(z)]$$

and

$$D_{p,x}\,[D_{q,y}\,f(z)] = D_{q,y}\,[D_{q,y}\,f(z)].$$

It is readily shown that the operation of the p-difference operator $D_{p,x}$ on the q-difference operator $D_{q,y}$ is commutative and so,

$$D^2_{p,x}[f(z)] = D^2_{q,y}[f(z)] \quad D^2[f(z)].$$

By theorem 1 of [7] again, $D^2 f(z)$ is (p, q)-analytic in R' and so by induction it follows that

$$D^j_{p,x}[f(z)] \quad \text{and} \quad D^j_{q,y}[f(z)]$$

are (p, q)-analytic in R' and satisfy

$$D^j_{p,x}[f(z)] = D^j_{q,y}[f(z)] \qquad D^j[f(z)].$$

(ii) If $z \in X^+$ and f is (p, q)-analytic in X^+, then by (3.7) $f(x, 0)$ exists for $(x, 0)$ X^+ and,

$$D_{p,x} f(x,0) = \lim_{y \to 0} D_{q,y} f(x,y), \ (x,y) \in R'.$$

It can be shown (see for example Hahn [2]) that

$$\begin{aligned} &(-1)^j (1-p)^j \, x^j \, p^{j(j-1)/2} \, D^j_{p,x}[f(x,0)] \\ &= \sum_{k=0}^{j} \binom{j}{k}_p (-1)^k \, p^{k(k-1)/2} \, f(p^{j-k} x, 0). \end{aligned} \tag{3.10}$$

The points $(p^{j-k}x, 0)$; $k = 0, 1, \ldots, j$; belong to X^+ and hence by (3.7) it follows that $f\ (p^{j-k}x, 0)$ exists for $k = 0, 1, \ldots, j$. The above formula verifies the existence of $D^j_{p,x} f(x,0)$.

Now,

$$\lim_{y \to 0} D^2_{q,y} f(z) = \lim_{y \to 0} D_{q,y}[D_{q,y} f(z)]$$

and since $D_{q,y} f(z)$ is (p, q)-analytic for $z \in R'$ by (i) above

$$\lim_{y \to 0} D^2_{q,y} f(z) = \lim_{y \to 0} D_{p,x}[D_{q,y} f(z)]$$

$$= \lim_{y \to 0} \frac{D_{q,y} f(z) - D_{q,y} f(px, y)}{(1-p)x}$$

$$= \frac{\lim_{y \to 0} D_{q,y} f(z) - \lim_{y \to 0} D_{q,y} f(px, y)}{(1-p)x}.$$

Hence, since f is (p, q)-analytic on X^+, by (3.7) it follows that,

$$\lim_{y\to 0} D_{q,y}^2 f(z) = \frac{D_{p,x} f(x,0) - D_{p,px} f(px,0)}{(1-p)x}$$

$$= D_{p,x} [D_{p,x} f(x,0)]$$

$$= D_{p,x} f(x,0).$$

Similarly, by induction it can be shown that

$$\lim_{y\to 0} D_{q,y}^j [f(z)] = D_{p,x}^j f(x,0) \ ; \ j = 0, 1, 2, \ldots,$$

and hence that $D_{p,x}^j \ f$ is (p, q)-analytic on X^+.

Similarly $D_{q,y}^j \ f$ is (p, q)-analytic on Y^+. This completes the proof of the lemma.

A method is now derived by which functions defined on the axes can be continued into the discrete plane as (p, q)-analytic functions.

If f is (p, q)-analytic in $\bar{R}$, then

$$D_{p,x} [f(z)] = D_{q,y} [f(z)]; \ z \in R' \tag{3.11}$$

$$= \frac{f(z) - f(x, qy)}{(1-q)iy}$$

Hence,

$$f(x, qy) = f(x, y) - (1-q)\, iy \ D_{p,x} [f(x, y)].$$

The operator $D_{p,x}$ is now treated as a constant k, and a formal symbolic method is used.

From the above,

$$f(x, qy) = (1 - (1-q)\, iy \ k) \ f(x, y).$$

It follows that

$$f(x, q^2 y) = (1 - (1-q)\, iy \ k) \ (1 - q\,(1-q)\, iy \ k) \ f(x, y)$$

and in general,

$$f(x, q^n y) = (1-(1-q)\, iy\; k)_n \quad f(x, y).$$

Taking the limit as $n \to \infty$, it follows that since f is (p, q)-analytic in $\overline{R}$, then

$$\lim_{n \to \infty} f(x, q^n y) = f(x, 0) \text{ exists, and so,}$$

$$f(x, 0) = (1-(1-q)\, iy\; k)_\infty\, f(x, y).$$

Hence (formally),

$$f(x, y) = \frac{1}{(1-(1-q)iy\;\; k)_\infty} \; f(x, 0),$$

and by the definition of q-analogue $e_q(x)$ of exponential function, we get

$$f(x, y) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j\, k^j\; f(x, 0).$$

Replacing k^j by $D^j_{p,x}$,

$$f(x, y) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j\; D^j_{p,x}\; [f(x,\; 0)], \tag{3.12}$$

where $D^j_{p,x}\; [f(x,\; 0)]$ exists by lemma 1.

The method used in attempting to solve equation (3.11) have to course been formal symbolic ones, similar to the procedures used by Boole [Treatise on the calculus of finite differences, Macmillan (1880)]. It remains to be verified that $f(x,\; y)$, as given by (3.12), in fact represents a solution of (3.11) when the series converges.

Theorem 3. *If*

$$f(x, y) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j\; D^j_{p,x}\; [f(x,\; 0)]$$

is convergent in some rectangular, discrete domain R', then f is (p, q)-analytic in R', and hence satisfies equation (3.11).

Proof. If $z \in R'$ then with $f(x, y)$ given as above,

$$D_{p,x}[f(x,y)] = D_{p,x} \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [f(x,0)] .$$

By Corollary 2 of [7], $I_{p,x} \sum_0^{\infty} = \sum_0^{\infty} D_{p,x}$, and so

$$D_{p,x}[f(x,y)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^{j+1} [f(x,0)] . \qquad (3.13)$$

Now,

$$D_{q,y}[f(x,y)] = D_{q,y} \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [f(x,0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} D_{q,y} [(iy)^j] D_{p,x}^j [f(x,0)] .$$

From the definition of $I_{q,y}$ it follows that

$$D_{q,y}[(iy)^j] = \begin{cases} 0 & ; \quad j = 0 \\ \dfrac{(1-q^j)}{(1-q)} (iy)^{j-1} ; & j = 1,2,\ldots \end{cases}$$

and so

$$D_{q,y}[f(x,y)] = \sum_{j=0}^{\infty} \frac{(1-q)^{j-1}}{(1-q)_{j-1}} (iy)^{j-1} D_{p,x}^j [f(x,0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^{j+1} [f(x,0)]$$

$$= D_{p,x}[f(x,y)] \qquad \text{(from (4.2.13))}.$$

This proves the theorem and justifies the formal symbolic methods.

In fact more than the above is true. The series representation of f is also (p, q)-analytic on X^+ which can be verified as follows.

Corollary 1. If the series in (3.12) is convergent in R', then

$$\lim_{y\to 0} f(x, y) = f(x, 0), \qquad \text{and}$$

$$\lim_{y\to 0} D_{q,y}[f(x, y)] = D_{p,x} f(x, 0).$$

Proof. From (3.12),

$$f(x, y) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [f(x,0)]$$

and hence

$$\lim_{y\to 0} f(x, y) = f(x, 0).$$

Similarly,

$$\lim_{y\to 0} D_{q,y} f(x, y) = \lim_{y\to 0} \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^{j+1} [f(x,0)]$$

$$= D_{p,x} f(x, 0).$$

This completes the proof.

From the above it is clear that functions defined on the X-axis can be continued by means of (3.12) (under certain convergence conditions) into (p, q)-analytic functions defined in the discrete plane Q'.

If the operator C_y is denoted by

$$C_y \equiv \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j \tag{3.14}$$

then the function

$$f(z) = C_y\ [f(x,\ 0)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j\ [f(x,\ 0)] \tag{3.15}$$

is called the '$(p,\ q)$-analytic continuation' of $f(x,\ 0)$, into a $(p,\ q)$-analytic function f defined at the point $(x,\ y) \in Q'$. Similarly, it can be shown that

$$C_x\ [f(0,\ y)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} x^j D_{q,y}^j\ [f(0,\ y)] \tag{3.16}$$

represents the $(p,\ q)$-analytic continuation from the y-axis.

4. Properties of the Continuation Operator C

The continuation operator C_y is said to exist if the series representation (3.14) converges.

(a) If k is a scalar constant, then

$$C_y(k) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j\ [k].$$

But $$D_{p,x}^j[k] = \begin{cases} k & ;\ j = 0 \\ 0 & ;\ j = 1,2,\dots, \end{cases}$$

and so

$$C_y(k) = k.$$

(b) Since D is a linear operator,

$$C_y[k\ f(x,\ 0)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j\ [k\ f(x,0)]$$

$$= k \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j\ [f(x,0)],$$

and so

$$C_y [k f(x, 0)] = k C_y [f(x, 0)].$$

(c) If $C_y [f(x, 0)]$, $C_y [g(x, 0)]$ exist then,

$$C_y [f(x, 0)] + C_y [g(x, 0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [f(x,0)] + \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [g(x,0)]$$

and since the two series converge,

$$C_y [f(x, 0)] + C_y [g(x, 0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j [D_{p,x}^j f(x,0) + D_{p,x}^j g(x,0)].$$

But $D_{p,x}$ is a linear operator and hence,

$$C_y [f(x, 0)] + C_y [g(x, 0)] = C_y [f(x, 0) + g(x, 0)].$$

But (b) and (c) it is clear that C_y is a linear operator.

(d) If $f_n(x,0) \rightarrow f(x,0)$ pointwise and the series representation of $C_y [f_n(x,0)]$ is uniformly convergent in n then

$$\lim_{n \to \infty} C_y [f_n(x,0)]$$

$$= \lim_{n \to \infty} \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{p,x}^j [f_n(x,0)]$$

and so by uniform convergence

$$\lim_{n \to \infty} C_y [f_n(x,0)] = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j \lim_{n \to \infty} D_{p,x}^j [f_n(x,0)].$$

It is readily shown that theorem 3 of [7] extends to all orders of p-difference and q-difference operators so that

$$\lim_{n\to\infty} D^j [f_n (x,0)] = D^j [f(x,0)] .$$

Hence

$$\lim_{n\to\infty} C_y [f_n (x,0)] = C_y [f(x, 0)].$$

(e) It is interesting to note that since

$$\lim_{q\to 1} \frac{(1-q)^j}{(1-q)_j} = \frac{1}{j}$$

and assuming f to be analytic in the classical sense, it follows that,

$$\lim_{q\to 1} D^j_{p,x} f(x,0) = f^{(j)} (x, 0).$$

Hence

$$\lim_{q\to 1} C_y [f(x, 0)] = \sum_{j=0}^{\infty} \frac{(iy)^j}{j!} f^{(j)} (x,0) ,$$

which is the Maclaurin series representation of an analytic function about a point the X-axis.

The continuation operator C may be regarded therefore as the (p, q)-analogue of a Maclaurin series.

5. References

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