Pro Mathematica Vol. XV, Nos. 29-30

A NOTE ON A POLYNOMIAL SET GENERATED BY G$(2xt - t^2)$ FOR THE CHOICE $G(u) = {}_0F_1(--;\alpha;u)$

Mumtaz ***Ahmad Khan***

Abstract

The present paper is a study of a class of polynomial set defined by means of a generating function of the form $G(2xt - t^2)$ for the choice $G(u) = {}_0F_1(--;\alpha;u)$. The paper contains some interesting results in the form of recurrence relations, generating functions, finite series of product of polynomials, hypergeometric form, relationship with Shively's pseudo Laguerre and other polynomials, integral representation, fractional integral and Laplace transform of the polynomial.

Department of Applied Mathematics, India

1 Introduction

The Hermite polynomials $H_n(x)$, the Legendre polynomials $P_n(x)$ and the Gegenbauer polynomials $c_n^v(x)$ are respectively defined by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{1.1}$$

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{1.2}$$

$$\text{and } (1-2xt+t^2)^{-v} = \sum_{n=0}^{\infty} C_n^v(x)t^n \tag{1.3}$$

A careful inspection of the L.H.S. of (1.1), (1.2) and (1.3) reveals the fact that the above polynomials are generated by the function $G(2xt-t^2)$ for the choices $G(u) = e^u, G(u) = (1-u)^{-\frac{1}{2}}$ and $G(u) = (1-u)^{-\nu}$ respectively.

The present paper deals with a class of polynomial set generated by $G(2xt-t^2)$ for yet another choice of $G(u) = {}_0F_1(--;\alpha;u)$.

We need the following results in this paper:

$$F\left[\begin{matrix}\gamma, & \gamma+\frac{1}{2}; & \\ & & z\\ & 2\gamma; & \end{matrix}\right] = (1-z)^{-\frac{1}{2}}\left[\frac{2}{1+\sqrt{1-z}}\right]^{2\gamma-1} \tag{1.4}$$

$$F\left[\begin{matrix}\gamma, & \gamma-\frac{1}{2}; & \\ & & z\\ & 2\gamma; & \end{matrix}\right] = \left[\frac{2}{1+\sqrt{1-z}}\right]^{2\gamma-1} \tag{1.5}$$

$$\begin{aligned}{}_0F_1(--;a;x)\,{}_0F_1(--;b;x) &= {}_2F_3(\frac{1}{2}a+\frac{1}{2}b,\ \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2};\\ &\quad a, b,\ a+b-1;\ 4x)\end{aligned} \tag{1.6}$$

$$\begin{aligned}{}_0F_1(--;a;x)\,{}_0F_1(--;a;-x) &= {}_0F_3(--;a,\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};\\ &\quad -\frac{1}{4}x^2)\end{aligned} \tag{1.7}$$

KUMMER'S FIRST FORMULA. If b is neither zero nor a negative integer,

$$ {}_1F_1(a;b;z) = e^z \; {}_1F_1(b-a;\; b;\; -z) \tag{1.8} $$

In this paper we also need the following theorem [2,p.132]:

Theorem. *From*

$$ G(2xt - t^2) = \sum_{n=0}^{\infty} g_n(x)t^n $$

it follows that $g_0'(x) = 0$, *and for* $n \geq 1$,

$$ xg_n'(x) \;-\; n\; g_n(x) = g_{n-1}'(x). \tag{1.9} $$

2 Definition of $M_n^{(\alpha)}(x)$

We define the class of polynomial set $M_n^{(\alpha)}(x)$ *generated by* $G(2xt - t^2)$ *for the choice* $G(u) = {}_0F_1(--;\alpha;u)$ *by means of the following generating relation:*

$$ {}_0F_1(--;\; \alpha;\; 2xt - t^2) = \sum_{n=0}^{\infty} M_n^{(\alpha)}(x)t^n \tag{2.1} $$

Since

$$ \begin{aligned} {}_0F_1(--;\; \alpha;\; 2xt - t^2) &= \sum_{n=0}^{\infty} \frac{(2xt - t^2)^n}{n!\,(\alpha)_n} \\ &= \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!\,(\alpha)_n}(1 - \frac{t}{2x})^n \\ &= \sum_{n=0}^{\infty} \frac{(2xt)^n}{n\,(\alpha)_n} \sum_{k=0}^{n} \frac{n!\,(-1)^k}{k!\,(n-k)!}(\frac{t}{2x})^k \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2x)^{n-2k}\, t^n}{k!\,(n-2k)!\,(\alpha)_{n-k}} \end{aligned} $$

it follows from (2.1) that

$$M_n^{(\alpha)}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!\,(\alpha)_{n-k}}. \tag{2.2}$$

Examination or equation (2.2) shows that $M_n^{(\alpha)}(x)$ is a polynomial of degree precisely n in x and that

$$M_n^{(\alpha)}(x) = \frac{2^n\, x^n}{n!\,(\alpha)_n} + \pi_{n-2}\,(x), \tag{2.3}$$

in which $\pi_{n-2}(x)$ is a polynomial of degree $(n-2)$ in x. From (2.2) it follows that $M_n^{(\alpha)}(x)$ is an even function of x for even n, and odd function of x for odd n:

$$M_n^{(\alpha)}(-x) = (-1)^n M_n^{(\alpha)}(x). \tag{2.4}$$

From (2.2) it follows readily that

$$M_{2n}^{(\alpha)}(0) = \tfrac{(-1)^n}{n!(\alpha)_n} \quad ; \quad M_{2n+1}^{(\alpha)}(0) = 0;$$

$$\left[\tfrac{d}{dx} M_{2n+1}^{(\alpha)}(x)\right]_{x=0} = \tfrac{(-1)^n}{n!(\alpha)_{n+1}} \quad ; \quad \left[\tfrac{d}{dx} M_{2n}^{(\alpha)}(x)\right]_{x=0} = 0.$$

3 Recurrence Relations

From (1.9) and (2.1), we obtain the following differential recurrence relation:

$$x\, D\, M_n^{(\alpha)}(x) = D\, M_{n-1}^{(\alpha)}(x) + n\, M_n^{(\alpha)}(x), \quad D \equiv \frac{d}{dx} \tag{3.1}$$

Further, we have

$$
\begin{aligned}
DM_n^{(\alpha)}(x) &= \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k 2^{n-2k}(n-2k)x^{n-1-2k}}{k!\,(n-2k)!\,(\alpha)_{n-k}} \\
&= \frac{n}{x}\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k 2^{n-2k} x^{n-2k}}{k!\,(n-2k)!(\alpha)_{n-k}} - \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^k 2^{n+1-2k} x^{n-1-2k}}{(k-1)!\,(n-2k)!(\alpha)_{n-k}} \\
&= \frac{n}{x}\, M_n^{(\alpha)}(x) - \sum_{k=0}^{[\frac{n}{2}]-1} \frac{(-1)^{k+1}\, 2^{n+1-2k-2}\, x^{n-1-2k-2}}{k!\,(n-2-2k)!\,(\alpha)_{n-k-1}} \\
&= \frac{n}{x} M_n^{(\alpha)}(x) + \frac{2}{\alpha x} \sum_{k=0}^{[\frac{n-2}{2}]} \frac{(-1)^k\, 2^{n-2-2k}\, x^{n-2-2k}}{k!\,(n-2-2k)!\,(\alpha+1)_{n-2-k}} \\
&= \frac{n}{x}\, M_n^{(\alpha)}(x) + \frac{2}{\alpha x}\, M_{n-2}^{(\alpha+1)}(x).
\end{aligned}
$$

Thus we arrive at

$$
x\, D\, M_n^{(\alpha)}(x) = n\, M_n^{(\alpha)}(x) + \frac{2}{\alpha}\, M_{n-2}^{(\alpha+1)}(x). \tag{3.2}
$$

From (3.1) and (3.2) we obtain the following pure recurrence relation:

$$
n\, \alpha\, M_n^{(\alpha)}(x) = 2x\, M_{n-1}^{(\alpha+1)}(x) - 2M_{n-2}^{(\alpha+1)}(x). \tag{3.3}
$$

4 Additional Generating Functions

Consider

$$
\sum_{n=0}^{\infty} (\alpha)_n\, M_n^{(\alpha)}(x)\, t^n
$$

$$
= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \frac{(\alpha)_n\, (-1)^k\, 2^{n-2k}\, x^{n-2k}\, t^n}{k!\, (n-2k)!\, (\alpha)_{n-k}}
$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\alpha)_{n+2k}\,(-1)^k\;2^n\;x^n\;t^{n+2k}}{k!\;n!\;(\alpha)_{n+k}}$$

$$= \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(\alpha+n)_{2k}\,(-1)^k\;t^{2k}}{k!\;(\alpha+n)_k}$$

$$= \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!}\; {}_2F_1\left[\begin{matrix} \frac{\alpha+n}{2}, & \frac{\alpha+n+1}{2}; & \\ & & -4t^2 \\ & \alpha+n; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!}(1+4t^2)^{-\frac{1}{2}} \left[\frac{2}{1+\sqrt{1+4t^2}}\right]^{\alpha+n+1} \qquad \text{by (1.4)}$$

$$= (1+4t^2)^{-\frac{1}{2}} \left[\frac{2}{1+\sqrt{1+4t^2}}\right]^{\alpha-1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{4xt}{1+\sqrt{1+4t^2}}\right)^n$$

Thus we arrive at the generating function

$$(1+4t^2)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{1+4t^2}}\right)^{\alpha-1} e^{4xt/(1+\sqrt{1+4t^2})}$$

$$= \sum_{n=0}^{\infty} (\alpha)_n\; M_n^{(\alpha)}(x)\; t^n. \tag{4.1}$$

Next consider,

$$\sum_{n=0}^{\infty} (\alpha-1)_n\; M_n^{(\alpha)}(x)\; t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \frac{(\alpha-1)_n\;(-1)^k\;(2x)^{n-2k}\;t^n}{k!\;(n-2k)!\;(\alpha)_{n-k}}.$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha-1)_n (2xt)^n}{n!\;(\alpha)_n} \sum_{k=0}^{\infty} \frac{(\alpha+n-1)_{2k}\,(-1)^k\,t^{2k}}{k!\;(\alpha+n)_k}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha-1)_n\,(2xt)^n}{n!\,(\alpha)_n}\ {}_2F_1\left[\begin{array}{c} \frac{\alpha+n}{2},\quad \frac{\alpha+n}{2}-\frac{1}{2}; \\ \alpha+n; \end{array} -4t^2\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha-1)_n\,(2xt)^n}{n!\,(\alpha)_n} \left[\frac{2}{1+\sqrt{1+4t^2}}\right]^{\alpha+n-1} \qquad \text{by (1.5)}$$

$$= \left[\frac{2}{1+\sqrt{1+4t^2}}\right]^{\alpha-1} {}_1F_1\left[\begin{array}{c} \alpha-1; \\ \alpha; \end{array} \frac{4xt}{1+\sqrt{1+4t^2}}\right].$$

We now use Kummer's formula (1.8) and arrive at the following generating function

$$\left[\frac{2}{1+\sqrt{1+4t^2}}\right]^{\alpha-1} e^{4xt/(1+\sqrt{1+4t^2})}\ {}_1F_1\left[\begin{array}{c} 1; \\ \alpha; \end{array} \frac{-4xt}{1+\sqrt{1+4t^2}}\right]$$
$$= \sum_{n=0}^{\infty} (\alpha-1)_n\ M_n^{(\alpha)}\ (x)\ t^n.$$

Using the definition of Bessel's function, yet another generating relation is as given below:

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x)\, t^n = \Gamma(\alpha)(i\sqrt{2xt-t^2})^{1-\alpha}\ J_{\alpha-1}(2i\sqrt{2xt-t^2}) \qquad (4.2)$$

5 Finite Series of Product of Polynomials

In view of (1.6), we have

$${}_0F_1\left[\begin{array}{c} --; \\ 2\alpha; \end{array} 2xt-t^2\right] {}_0F_1\left[\begin{array}{c} --; \\ 2\beta; \end{array} 2xt-t^2\right]$$
$$= {}_2F_3\left|\begin{array}{cc} \alpha+\beta, & \alpha+\beta-\frac{1}{2}; \\ 2\alpha, 2\beta; & 2\alpha+2\beta-1; \end{array} 4(2xt-t^2)\right| \qquad (5.1)$$

Using (2.1) in (5.1), we obtain

$$\sum_{n=0}^{\infty} M_n^{(2\alpha)}(x)\, t^n \sum_{k=0}^{\infty} M_k^{(2\beta)}(x)\, t^k$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n\, (\alpha+\beta-\frac{1}{2})_n\, 4^n (2xt-t^2)^n}{n!\, (2\alpha)_n\, (2\beta)_n\, (2\alpha+2\beta-1)_n}$$

or

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} M_{n-k}^{(2\alpha)}(x)\; M_k^{(2\beta)}(x)\; t^n$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\alpha+\beta)_n\; (\alpha+\beta-\frac{1}{2})_n\; 4^n (2x)^{n-k}\, (-1)^k\; t^{n+k}}{k!\; (n-k)!\, (2\alpha)_n\, (2\beta)_n\, (2\alpha+2\beta-1)_n}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \frac{(\alpha+\beta)_{n-k}(\alpha+\beta-\frac{1}{2})_{n-k}\; 4^{n-k}(2x)^{n-2k}(-1)^k\; t^n}{k!\,(n-2k)!\,(2\alpha)_{n-k}(2\beta)_{n-k}(2\alpha+2\beta-1)_{n-k}}$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n(\alpha+\beta-\frac{1}{2})_n(8xt)^n}{n!\,(2\alpha)_n(2\beta)_n(2\alpha+2\beta-1)_n}$$
$$\sum_{k=0}^{[\frac{n}{2}]} \frac{(-n)_{2k}(1-2\alpha-n)_k(1-2\beta-n)_k(2-2\alpha-2\beta-n)_k}{k!\; (1-\alpha-\beta-n)_k\; (\frac{3}{2}-\alpha-\beta-n)_k\; (16x^2)^k}$$

Equating the coefficient of t^n on both sides, we get

$$\sum_{k=0}^{n} M_{n-k}^{(2\alpha)}(x) M_k^{(2\beta)}(x) = \frac{(\alpha+\beta)_n\, (\alpha+\beta-\frac{1}{2})_n\, (8x)^n}{n!\,(2\alpha)_n\, (2\beta)_n\, (2\alpha+2\beta-1)_n}$$
$${}_5F_2\left[\begin{array}{l} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2},\; 1-2\alpha-n, 1-2\beta-n,\; 2-2\alpha-2\beta-n; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{1}{4x^2} \\ 1-\alpha-\beta-n,\; \frac{3}{2}-\alpha-\beta-n; \end{array}\right] \tag{5.2}$$

Using (2.1) in (5.2), we obtain

$$\sum_{n=0}^{\infty} M_n^{(2\alpha)}(x)\, t^n \sum_{k=0}^{\infty} M_k^{(2\alpha)}(ix) i^k\, t^k = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2xt-t^2)^{2n}}{n!\,(2\alpha)_n\,(\alpha)_n\,(\alpha+\frac{1}{2})_n 4^n}$$

Where $i = \sqrt{-1}$.

$$\begin{aligned}
\text{or} \quad & \sum_{n=0}^{\infty}\sum_{k=0}^{n} M_{n-k}^{(2\alpha)}(x) M_k^{(2\alpha)}(ix) i^k\, t^n \\
= & \sum_{n=0}^{\infty}\sum_{k=0}^{2n} \frac{(-1)^{n+k}\,(2n)!\,(2xt)^{2n-k}\, t^{2k}}{n!\,(2\alpha)_n(\alpha)_n\,(\alpha+\frac{1}{2})_n\, 2^{2n} k!\,(2n-k)!} \\
= & \sum_{n=0}^{\infty}\sum_{k=0}^{2n} \frac{(-1)^{n+k}\,(\frac{1}{2})_n\,(2x)^{2n-k}\, t^{2n+k}}{k!(2n-k)!\,(2\alpha)_n\,(\alpha)_n\,(\alpha+\frac{1}{2})_n} \\
= & \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^n(\frac{1}{2})_n(2x)^{2n-2r}\, t^{2n+2r}}{(2r)!\,(2n-2r)!\,(2\alpha)_n\,(\alpha)_n\,(\alpha+\frac{1}{2})_n} \\
+ & \sum_{n=0}^{\infty}\sum_{r=0}^{n-1} \frac{(-1)^{n+1}(\frac{1}{2})_n(2x)^{2n-1-2r}\, t^{2n+2r+1}}{(2r+1)!\,(2n-1-2r)!\,(2\alpha)_n(\alpha)_n(\alpha+\frac{1}{2})_n} \\
= & \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^{n-r}\,(\frac{1}{2})_{n-r}\,(2x)^{2n-4r}\, t^{2n}}{(2r)!\,(2n-4r)!\,(2\alpha)_{n-r}(\alpha)_{n-r}(\alpha+\frac{1}{2})_{n-r}} \\
+ & \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n-1}{2}]} \frac{(-1)^{n-r+1}(\frac{1}{2})_{n-r}\,(2x)^{2n-1-4r}\, t^{2n+1}}{(2r+1)!\,(2n-1-4r)!\,(2\alpha)_{n-r}\,(\alpha)_{n-r}\,(\alpha+\frac{1}{2})_{n-r}}
\end{aligned} \tag{5.3}$$

Equating the coefficients of t^{2n} on both sides of (5.3), we get

$$\sum_{k=0}^{2n} M_{2n-k}^{(2\alpha)}(x)\, M_k^{(2\alpha)}(ix)\, i^k = \frac{(-1)^n\, x^{2n}}{n!\,(2\alpha)_n\,(\alpha)_n\,(\alpha+\frac{1}{2})_n}$$

$${}_7F_2\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{4}, -\frac{n}{2}+\frac{1}{2}, -\frac{n}{2}+\frac{3}{4}, 1-2\alpha-n, 1-\alpha-n, \quad \frac{1}{2}-\alpha-n; \\ \frac{1}{2}, \frac{1}{2}-n; \end{matrix} \; -\frac{4}{x^4}\right] \tag{5.4}$$

Equating the coefficient of t^{2n+1} on both sides of (5.3), we get

$$\sum_{k=0}^{2n+1} M_{2n+1-k}^{(2\alpha)}(x)\, M_k^{(2\alpha)}(ix)\, i^k = \frac{(-1)^{n+1}\,(\frac{1}{2})_n\,(2x)^{2n-1}}{(2n-1)!\,(2\alpha)_n\,(\alpha)_n\,(\alpha+\frac{1}{2})_n} \times$$

$$\times\ {}_7F_2\left[\begin{matrix} -\frac{n}{2}+\frac{1}{4}, -\frac{n}{2}+\frac{1}{2}, -\frac{n}{2}+\frac{3}{4}, -\frac{n}{2}+1, 1-2\alpha-n, 1-\alpha-n, \quad \frac{1}{2}-\alpha-n; \\ \frac{3}{2}, \frac{1}{2}-n; \end{matrix}\ -\frac{4}{x^4}\right] \tag{5.5}$$

6 Hipergeometric Form of $M_n^{(\alpha)}(x)$

From (2.2), we obtain the following hypergeometric form of $M_n^{(\alpha)}(x)$:

$$M_n^{(\alpha)}(x) = \frac{2^n\, x^n}{n!\,(\alpha)_n}\ {}_3F_0\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \quad 1-\alpha-n; \\ ----; \end{matrix}\ \frac{1}{x^2}\right] \tag{6.1}$$

7 Relationship with Shively's Pseudo-Laguerre and other Polynomials

Shively [3] studied the pseudo-Laguerre set

$$R_n(a;x) = \frac{(a)_{2n}}{n!\ (a)_n}\ {}_1F_1(-n;\ a+n;\ x), \tag{7.1}$$

which are related to the proper simple Laguerre polynomial

$$L_n(x) = {}_1F_1(-n;\ 1;\ x)$$

by

$$R_n(a;\, x) = \frac{1}{(a-1)_n} \sum_{k=0}^{n} \frac{(a-1)_{n+k}\, L_{n-k}(x)}{k!}. \tag{7.2}$$

Shively obtained Toscano's following generating relation:

$$e^{2t}\, {}_0F_1\left(-; \frac{1}{2} + \frac{1}{2}a;\ t^2 - xt\right) = \sum_{n=0}^{\infty} \frac{R_n(a,\, x)\, t^n}{\left(\frac{1}{2}+\frac{1}{2}a\right)_n}. \tag{7.3}$$

Replacing a by $2\alpha - 1$, t by $i\,t$ and x by $2i\,x$ in (7.3), we obtain

$${}_0F_1(-;\, \alpha;\, 2xt - t^2) = e^{-2t} \sum_{n=0}^{\infty} \frac{R_n(2\alpha-1, 2i\,x)\, i^n\, t^n}{(\alpha)_n},\ i = \sqrt{-1}$$

or

$$\begin{aligned}\sum_{n=0}^{\infty} M_n^{(\alpha)}(x)\, t^n &= \sum_{n=0}^{\infty} \frac{(-2t)^n}{n!} \sum_{k=0}^{\infty} \frac{R_k(2\alpha-1,\, 2i\,x) i^k\, t^k}{(\alpha)_k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-2)^{n-k}\, i^k\, R_k(2\alpha-1,\, 2i\ x)\, t^n}{(n-k)!\, (\alpha)_k}.\end{aligned}$$

Equating the coefficients of t^n on both sides we get the following relationship between $M_n^{(\alpha)}(x)$ and Shively's pseudo-Laguerre polynomials

$$M_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-2)^{n-k}\, i^k\, R_k(2\alpha-1,\, 2i\ x)}{(n-k)!\, (\alpha)_k}. \tag{7.4}$$

In view of (7.2) the polynomials $M_n^{(\alpha)}(x)$ has following relationship with proper simple Laguerre polynomials:

$$M_n^{(\alpha)}(x) = \underset{k=0}{\overset{n}{M}} \sum_{r=0}^{k} \frac{(-2)^{n-k}\, i^k (2\alpha-2)_{k+r}\, L_{k-r}\,(2i\,x)}{(n-k)!\ r!\ (\alpha)_k\, (2\alpha-2)_k}. \tag{7.5}$$

Generalized Laguerre or Sonine polynomials $L_n^{(\alpha)}(x)$ have the following generating function:

$$e^t \, {}_0F_1(-;\ 1+\alpha;\ -xt) = \sum_{n=0}^{\infty} \frac{L_n^{(x)}(x)\ t^n}{(1+\alpha)_n}. \tag{7.6}$$

We can rewrite (7.6) as

$$\begin{aligned} {}_0F_1(-;\ \alpha;\ -2xt) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n\, L_k^{(\alpha-1)}\,(2x)\, t^{n+k}}{n!\ (\alpha)_k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{n-k}\, L_k^{(\alpha-1)}(2x)\, t^n}{(n-k)!\ (\alpha)_k}. \end{aligned}$$

$\therefore$

$$\begin{aligned} {}_0F_1(-;\alpha;2xt-t^2) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{n-k}\, L_k^{(\alpha-1)}\,(2x)(\frac{t^2}{2x}-t)^n}{(n-k)!\ (\alpha)_k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{k}\, L_k^{(\alpha-1)}\,(2x)\, t^n}{(n-k)!\ (\alpha)_k}(1-\frac{t}{2x})^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n} \frac{(-1)^{k+r}\, n!\, L_k^{(\alpha-1)}\,(2x)\, t^{n+r}}{(n-k)!\,(n-r)!\,r!\,(\alpha)_k\,(2x)^r} \end{aligned}$$

$$\sum_{n=0}^{\infty} M_n^{(\alpha)}(x)\, t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^{k+r}(n-r)!\, L_k^{(\alpha-1)}(2x)\, t^n}{(n-r-k)!\,(n-2r)!\,r!\,(\alpha)_k\,(2x)^r}.$$

Equating the coefficient of t^n, we get the following relationship between $M_n^{(\alpha)}(x)$ and generalized Laguerre polynomials

$$M_n^{(\alpha)}(x) = \sum_{k=0}^{n}\sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^{k+r}(n-r)!\, L_k^{(\alpha-1)}\,(2x)}{(n-r-k)!\,(n-2r)!\ r!\ (\alpha)_k\,(2x)^r}, \tag{7.7}$$

Legendre polynomials $P_n(x)$ have the following generating relation:

$$e^{xt}\ {}_0F_1(-;1;\frac{1}{4}t^2(x^2-1)) = \sum_{n=0}^{\infty} \frac{P_n\ (x)\ t^n}{n!}. \qquad (7.8)$$

We can rewrite (7.8) as

$$\begin{aligned} {}_0F_1(-;\ 1;\ \frac{1}{4}t^2\,(x^2-1)) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-x)^n P_k(x) t^{n+k}}{n!\ k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-x)^{n-k}\ P_k(x)\ t^n}{(n-k)!\ k!} \end{aligned}$$

Replacing x by $\sqrt{1+8x}$ and t by $\sqrt{t-\frac{t^2}{2x}}$, we get

$${}_0F_1(-;\ 1;\ 2xt-t^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-x)^{n-k}\ P_k(x)}{(n-k)!\ k!}(t-\frac{t^2}{2x})^{\frac{n}{2}}$$

$$\begin{aligned} \sum_{n=0}^{\infty} M_n^{(1)}(x)\,t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-x)^{n-k}\ P_k(x)\,t^{\frac{n}{2}}(1-\frac{t}{2x})^{\frac{n}{2}}}{(n-k)!\ k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^{n-k}\ x^{n-k-r}\ (-\frac{n}{2})_r\ P_k(x)\,t^{\frac{n}{2}+r}}{(n-k)!\ k!\ r!\ 2^r} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]}\sum_{k=0}^{n} \frac{(-1)^{n-k}\ x^{n-k-r}\ (-\frac{n}{2})_r\ P_k(x)\ t^{\frac{n}{2}+r}}{(n-k)!\ k!\ r!\ 2^r} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{[\frac{n}{2}]}\sum_{k=0}^{n} \frac{(-1)^{n-k}\ x^{n-k-3r}(1)_{\frac{n}{2}}\ (-1)^r\ P_k(x)\,t^{\frac{n}{2}+r}}{(n-k)!\ k!\ r!\ (1)_{\frac{n}{2}-r}\ 2^r} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{4}]} \sum_{k=0}^{n-2r} \frac{(-1)^{n+r-k}\, x^{n-k-3r}\, (1)_{\frac{n}{2}-r}\, P_k(x)\, t^{\frac{n}{2}}}{(n-k-2r)!\; k!\; r!\; (1)_{\frac{n}{2}-2r}\, 2^r}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{[\frac{n}{4}]} \sum_{k=0}^{n-2r} \frac{(-\frac{n}{2}+\frac{1}{2})_r (-1)^{n+r-k} x^{n-k-3r} P_k(x) 2^r t^{\frac{n}{2}}}{(n-k-2r)!\; k!\; r!}$$

Equating the coefficients of t^n on both sides, we get

$$M_n^{(1)}(x) = \sum_{r=0}^{[\frac{n}{2}]} \sum_{k=0}^{2n-2r} \frac{(-n+\frac{1}{2})_r\, (-1)^{r-k}\, x^{2n-k-3r}\, 2^r\, P_k(x)}{(2n-k-2r)!\; k!\; r!} \tag{7.9}$$

8 Integral Representation

Using the definition of Beta function it is easy to derive the following integral representation for $M_n^{(\alpha)}(x)$ (see Rainville [2]. p.18) by using the form (2.2) of $M_n^{(\alpha)}(x)$:

$$\int_0^t x^{\beta}(t-x)^{\gamma-1}\, M_n^{(\alpha)}(x)\, dx = \frac{2^n\, \Gamma(1+\beta+n)\, \Gamma(\gamma)\, t^{\beta+\gamma+n}}{n!\; \Gamma(1+\beta+\gamma+n)\, (\alpha)_n} \times$$

$$\times\ {}_5F_2 \left[\begin{array}{c} -\frac{n}{2},\quad -\frac{n}{2}+\frac{1}{2}, 1-\alpha-n,\quad \frac{1}{2}(-\beta-\gamma-n),\quad \frac{1}{2}(1-\beta-\gamma-n); \\ \frac{1}{2}(-\beta-n),\quad \frac{1}{2}(1-\beta-n); \end{array} \ \frac{1}{t^2} \right] \tag{8.1}$$

9 Fractional Integral

Let L denote the linear space of (equivalent classes of) complex-valued functions $f(x)$ which are Lebesgue-integrable on $[0, \alpha], \alpha < \infty$.

For $f(x) \in L$ and complex number μ with $R\ell(\mu) > 0$, the Riemann-Liouville fractional integral of order μ is defined as (see Prabhakar [1], p. 72).

$$I^{\mu}\left[f(x)\right] = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)dt \text{ for almost all } x \in [0,\alpha]. \quad (9.1)$$

Using the operator I^{μ} on the form (2.2) of $M_n^{(\alpha)}(x)$, we obtain

$$I^{\gamma}\left[x^{\beta} M_n^{(\alpha)}(x)\right] = \frac{2^n x^{\beta+\gamma+n}\,\Gamma(1+\beta+n)\Gamma(\gamma)}{n!\;\Gamma(1+\beta+\gamma+n)\;(\alpha)_n} \times$$

$$\times\ {}_4F_1\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, 1-\alpha-n, & -\beta-\gamma-n; \\ & -\beta-n; \end{matrix}\ \frac{2}{x}\right]. \quad (9.2)$$

10 Laplace Transform

In the usual notation the Laplace transform is given by

$$L\{f(t):s\} = \int_0^{\infty} e^{-st} f(t)dt,\ R\ell(s-a) > 0, \quad (10.1)$$

where $f \in L(0,R)$ for every $R > 0$ and $f(t) = 0\,(e^{at}), t \to \infty$. Using (10.1) on the form (2.2) of $M_n^{(\alpha)}(x)$, we obtain

$$L\{t^{\beta}\ M_n^{(\alpha)}\ (xt):s\} = \frac{2^n\ x^n}{n!\ (\alpha)_n\ s^{1+\beta+n}}$$

$${}_3F_2\left[\begin{matrix} -\frac{n}{2}, & -\frac{n}{2}+\frac{1}{2}, 1-\alpha-n; \\ & \frac{1}{2}(-\beta-n), \frac{1}{2}(1-\beta-n); \end{matrix}\ \frac{s^2}{4x^2}\right]. \quad (10.2)$$

References

[1] Prabhakar, T.R. (1969). *Two singular integral equations involving confluent hypergeometric functions"*. Proc. Camb. Phil. Soc., vol. 66, pp. 71-89.

[2] Rainville, E.D. (1971). *Special Functions"*. Macmillan, New York, 1960; Reprinted by Chelsea Publ. Co., Bronx, New York.

[3] Shively, R.L. (1953). *On pseudo Laguerre polynomials"*. Michigan thesis.

Mumtaz Ahmad Khan
Department of Applied Mathematics
Faculty of Engineering,
Aligarh Muslim University,
Aligarh-202002, (U.P.), India