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WEAK AND PERIODICAL SOLUTIONS OF THE NAVIER-STOKES EQUATION IN NONCYLINDRICAL DOMAINS

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Abstract

We consider the Navier Stokes equation in noncylindrical domain and prove the existence of weak and periodical solutions.

Key Words and Phrases: Navier-Stokes, Noncylindrical domains, Weak and periodical solutions.

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1 Introduction

Let $T > 0$ be a real number and $\{\Omega_t\}_{0\leq t\leq T}$ a family of bounded open sets of $\mathbf{R}^n$ with boundary Γ_t. Let us consider the noncylindrical domain of $\mathbf{R}^{n+1}$, given by $\widehat{Q} = \bigcup\limits_{0<t<T} \Omega_t \times \{t\}$, with lateral boundary $\widehat{\sum} = \bigcup\limits_{0<t<T} \Gamma_t \times \{t\}$ regular.

We have the Navier-Stokes problem:

$$
(I) \left| \begin{array}{ll}
u' - \Delta u + \sum\limits_{i=1}^{n} u_i \dfrac{\partial u}{\partial x_i} = f - \nabla p & in\ \widehat{Q} \\
div\, u = 0 & in\ \widehat{Q} \\
u = 0 & on\ \widehat{\sum} \\
u(x,0) = u_0(x) & x \in \Omega_0
\end{array} \right.
$$

where $u(x,t) = (u_1(x,t), \ldots, u_n(x,t)), (x,t) \in \widehat{Q}$,

$\Delta u = (\Delta u_1, \ldots, \Delta u_n), \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$.

In the last thirty years a lot of papers concerning the existence of solutions to the Navier-Stokes equation in noncylindrical domain have been written. Among these papers, it is worth mentioning the articles of J. L. Lions [17], H. Fujita and N. Sauer [7], [8], H. Morimoto [23], in which the penality method is used, the results of R. Salvi [25], [26], [27], L.A. Medeiros and J. Limaco Ferrel [15] based on elliptic regularization, the paper of O.A. Ladyzhenskaya [13], M. Otani and Y. Yamada [24] obtained with the Rothe's method and with the subdifferential operator theory respectively, and the works of J.O. Sather [28], D.N. Bock [2], A. Inoue and M. Wakimoto [11], T. Miyakawa and Y. Teramoto [22] derived with tools of Differential Geometry and the paper of M. Milla Miranda and J. Limaco Ferrel [16] used change of variable for a cilyndrical domain and Galerkin's method.

Let $K : [0,T] \to \mathbf{R}^{n^2}$, a function, where $K(t)$ is a $n \times n$ matrix. Let Ω be an open bounded set of $\mathbf{R}^n$, which, without loss of generality, can be consider containing the origin of $\mathbf{R}^n$. We assume that the boundary Γ of Ω is smooth, and consider the sets

$$\Omega_t = \{x = K(t)y; \quad y \in \Omega\} \tag{1}$$

In this work we study the existence of weak solutions to Problem (I), and also study the existence of periodical solution to (I). For that, by a suitable change of variable which is more general than that one used in [16], we transform the non cylindrical problem (I) in another problem defined in the cylinder $Q = \Omega \times]0, T[$.

2 Notation and Hypotheses

We make following hypothesis on $K(t)$.

(H1) $K \in C^1$, where $K(t)$ is an invertible matrix.

(H2) $K^{-1} \in C^1$, where $K^{-1}(t)$ is an invertible matrix of $K(t)$.

Consider the notation

$$K(t) = (\alpha_{ij}(t)), \quad \text{and} \quad K^{-1}(t) = (\beta_{ij}(t)) \tag{2}$$

as well as the convention of summation of repeated indices; that is $\alpha_i \beta_i = \sum_{i=1}^{n} \alpha_i \beta_i$.

By $< .,. >$ we will represent the duality pairing between X' and X, X' being the dual of the space X.

In order to state the main results we introduce some spaces. Let $\mathcal{V}_t$ be the space

$$\mathcal{V}_t = \{\varphi \in (D(\Omega_t))^n; \; div\, \varphi = 0\}$$

and $V_s(\Omega_t)$ be the closure of $\surd_t$ in the space $(H^s(\Omega_t))^n$, where s is a non negative real number.

We use the particular notation

$$V_1(\Omega_t) = V(\Omega_t) \quad \text{and} \quad V_0(\Omega_t) = H(\Omega_t).$$

The inner product of these spaces is denoted, respectively by $(u, z)_{H(\Omega_t)}$ and $((u, z))_{V(\Omega_t)}$. Then for $u = (u_1, \dots u_n)$ and $z = (z_1, \dots z_n)$, we have

$$(u, z)_{H(\Omega_t)} = \int_{\Omega_t} u_i(x) z_i(x) dx, ((u, z))_{V(\Omega_t)} = \int_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial z_i(x)}{\partial x_j} dx.$$

Note that $V_s(\Omega_t)$ is continuously embedded in $(H_0^1(\Omega))^n$ for $s \geq \frac{n}{2}$ and $V(\Omega_t) = \{u \in (H_0^1(\Omega_t))^n,\ div\, u = 0\}$.

In a similar way we introduce the spaces $V_s(\Omega)$, where $\surd$ has a form

$$\surd = \{\psi \in (D(\Omega))^n;\ div\, \psi = 0\}.$$

We consider, the particular notations
$V_1(\Omega) = V$, $V_0 = H$ and $(v, w)_H = (v, w)$, $((v, w))_V = ((v, w))$, $|v|_H = |v|$, $\|v\|_V = \|v\|$.

In order to state the variational formulation of Problem (I) we introduce the following bilinear and trilinear forms respectively:

$$\hat{a}(t; u, z) = \int_{\Omega_t} \frac{\partial u_i(x)}{\partial x_j} \frac{\partial z_i(x)}{\partial x_j} dx, \tag{3}$$

$$\hat{b}(t; u, z, \xi) = \int_{\Omega_t} u_i(x) \frac{\partial z_j(x)}{\partial x_i} \xi_j(x) dx, \tag{4}$$

We define the weak solution for the problem (I) in the following form

$$(II)\quad\left|\begin{array}{l} u \in L^2(0,T;V(\Omega_t)) \cap L^\infty(0,T;H(\Omega_t)) \\ -\int_0^T (u,\varepsilon'_H)_{(\Omega_t)}dt + \int_0^T \hat{a}(t;u,\varepsilon)dt+ \\ +\int_0^T \hat{b}(t;u,u,\varepsilon)dt = \int_0^T (f,\varepsilon)_{H(\Omega_t)}dt \\ u(0) = u_0 \\ \forall_\varepsilon \in L^2(0,T;V(\Omega_t) \bigcap (L^n(\Omega_t))^n),\ \varepsilon' \in L^2(0,T;H(\Omega_t)), \\ \varepsilon(0) = \varepsilon(T) = 0 \end{array}\right.$$

In order to transform the noncylindrical problem (II) in a problem defined in the cylinder Q, we introduce the functions:

$$u(x,t) = K(t)v(K^{-1}(t)x,t) \quad , \quad f(x,t) = K(t)g(K^{-1}(t)x,t) \tag{5}$$

$$p(x,t) = K(t)q(K^{-1}(t)x,t) \quad , \quad u_0(x) = K(0)v_0(K^{-1}(0)x). \tag{6}$$

$$a(t;v,w) = \int_\Omega \delta(t)a_{jk}(t)\frac{\partial v_i(y)}{\partial y_j}\ \frac{\partial w_i(y)}{\partial y_k}\,dy, \tag{7}$$

$$b(t;v,w,\psi) = \int_\Omega \delta(t)v_i(y)\frac{\partial w_j(y)}{\partial y_i}\ \psi_j(y)dy, \tag{8}$$

$$c(t;v,w) = \int_\Omega \delta(t)\beta'_{lr}(t)\alpha_{rj}(t)y_j\ \frac{\partial v_i(y)}{\partial y_l}\ w_i(y)dy, \tag{9}$$

$$\begin{aligned} d(t;v,w) = & \int_\Omega \delta(t)\ [\alpha_{rl}(t)\,\beta'_{lr}(t)\,v_k(y)\,w_k(y) \\ & - \alpha_{ir}(t)\,\beta'_{ki}(t)\,v_r(y)\,w_k(y)]\,dy, \end{aligned} \tag{10}$$

with $a_{jk}(t) = \beta_{ji}(t)\beta_{ki}(t)$ and $\delta(t) = |\det K(t)|$, where $\det M$ means the determinant of a $n \times n$ matrix M.

Then from (5) to (9) and the problem (II) we obtain the definition of weak solution for the cylinder problem:

$$
(III)\quad \left|\begin{array}{l}
v \in L^2(0,T;V) \cap L^\infty(0,T;H) \\
-\int_0^T(\delta(t)v,\psi')dt + \int_0^T a(t;v,\psi)dt \int_0^T b(t;v,v,\psi)dt \\
+\int_0^T c(t;v,\psi)dt + \int_0^T d(t;v,\psi)dt = \int_0^T(\delta(t)g,\psi)dt \\
v(0) = v_0 \\
\forall_\psi \in L^2(0,T;V \cap (L^n(\Omega_t))^n), \psi' \in L^2(0,T;H), \psi(0) = \psi(T) = 0.
\end{array}\right.
$$

3 Main Results

Theorem 3.1. *Assume that the hypothesis* $(H1)$ *and* $(H2)$ *are satisfied. If* $f \in L^2(0,T;H(\Omega_t))$ *and* $u_0 \in H(\Omega_0)$*, then there exists* $u : \hat{Q} \to \mathbf{R}^n$*, solution of Problem* (II)*.*

Theorem 3.2. *If* $g \in L^2(0,T;H)$ *and* $v_0 \in H$*, then there exists* $v : Q \to \mathbf{R}$*, solution of Problem* (III)*.*

The following lemmas will be utilized to prove the theorems given.

Lemma 3.3. *Consider the bilinear form* $a(t;v,w)$ *defined by* (7) *and the operator* $A(t)$ *defined by* $A(t)v = -\frac{\partial}{\partial y_l}(a_{lr}(t)\frac{\partial v}{\partial y_r})$, $v \in (H_0^1(\Omega))^n$. *Then there exist positive constants* a_0, a_1, a_2*, such that*

(i) $< A(t)v, w > = a(t;v,v)\forall v, w \in V$

(ii) $a(t;v,v) \geq a_0\delta(t)||v||^2,\ a(t;v,v) \geq a_1\delta(t)|v|^2,\ v \in V$

(iii) $|a(t;v,w)| \leq a_2\delta(t)||v||\,||w||,\ \forall v, w \in V.$

The proof is given in [16].

Lemma 3.4. *Let* $b(t;v,w,\psi)$*, be the trilinear form,* $c(t,v,w), d(t,v,w)$ *the bilinear forms defined, respectively by* (8), (9),)(10). *Then there exist positive constants* $b_i, c_i, d_i, i = 1,\ 2,$ *such that* *(i)* $b(t,v,v,w) \;=\; -b(t;v,w,v)\forall v \in V, w \in V_s(\Omega), s = \frac{n}{2}$

(ii) $|b(t;v,w,\psi)| \le b_0\, \delta(t)\|v\|\,\|w\|\,\|\psi\|_{(L^n(\Omega))^n}\forall v,\, w \in V,$
$\psi \in V \cap (L^n(\Omega))^n$

(iii) *For each $v \in V$, the linear form $w \to b(t;v,v,w)$ is continuous in $V_s(\Omega)$, $s = \frac{n}{2}$ and $b(t;v,v,w) \;=\; < B(t)v, w >_{V_s',\, V_s}$ where $B(t)v \in V_s'(\Omega)$ and $\|B(t)v\|_{V_s'} \le b_1\|v\|_{[L^p(\Omega)]^n}$ with,*

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{2n} \qquad (p < q, \quad \frac{1}{q} = \frac{1}{2} - \frac{1}{n}) \tag{11}$$

(iv) $|c(t;v,w)| \le c_0\ \delta(t)\|v\|\,|w|, \forall v \in V, w \in H$

(v) *For each $v \in V$, the linear form $w \to c(t;v,w)$ is continuous in H and $c(t;v,w) = (C(t)v,w)$, where $C(t)v \in H$ and $|C(t)v| \le c_1\|v\|$*

(vi) $|d(t;v,w)| \le d_0\, \delta(t)\, \|v\|\, |w|,\ \forall v,\, w \in H$

(vii) *For each $v \in V$, the linear form $w \to d(t;v,w)$ is continuous in H and $d(t;v,w) = (D(t)v,w)$, where $D(t)v \in H$ and $|D(t)v| \le d_1\,\|v\|$.*

The proof with some modifications is analogues to the proof given in [16].

Lemma 3.5. *Let $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Then $v \in L^4(0,T;(L^p(\Omega))^n)$, where p is given by (11) and*

$$\|v_i(t)\|_{L^p(\Omega)} \le c\|v_i(t)\|^{\frac{1}{2}}_{H_0'(\Omega)}\,|v_i(t)|^{\frac{1}{2}}_{L^2(\Omega)},\ v = (v_1,\ldots,v_n)$$

for some positive constant c.

The proof of Lemma 3.5 appear in [17].

Now, we consider another hypotheses over $K(t)$, in order to state results about the existence of periodical solutions

(H3) $K(0) = K(T)$
(H4) Assume that there exists a positive constant α, such that

$$a_1 - \frac{|\delta'(t)|}{\delta(t)} - c_0 - d_0 > \alpha,$$

where $\delta(t) = |\det K(t)|$ and a_1, c_0, d_0 are constants given in the Lemma 3.4.

Theorem 3.6. *Assume that hypothesis $(H1) - (H4)$ are satisfied. If $f \in L^2(0,T;H(\Omega_t))$ then there exists $u : \widehat{Q} \to \mathbf{R}^n$ solution of Problem (II) such that $u(0) = u(T)$.*

Theorem 3.7. *If $g \in L^2(0,T;H)$, then there exists $v : Q \to \mathbf{R}^n$, solution of Problem (III) such that $v(0) = v(T)$.*

4 Proofs

Proof of Theorem 2. Let (w_j) be a special basis of $V_s(\Omega)$, with $s = \frac{n}{2}$. We consider the approximated problem :

$$(AP)\begin{cases} ((\delta(t)v_m)', w_j) + a(t; v_m, w_j) + b(t; v_m, v_m, w_j) + c(t; v_m, w_j) \\ +d(t; v_m, w_j) = (\delta(t)g, w_j),\ j = 1,\dots m, v_m(t) \in \surd_m = [w_1, \dots w_m]. \\ v_m(0) = v_{0m}, v_{0m} \to v_0 \ in\ H. \end{cases}$$

First Estimate. Considering $w_j = v_m(t)$ in (AP), and Lemma 3.4 (i) we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}(\delta(t)|v_m|^2) - \tfrac{1}{2}\delta'(t)|v_m|^2 + a(t; v_m, v_m) + c(t; v_m, v_m) + d(t; v_m, v_m)$$

$$= (\delta(t)g, v_m).$$

Applying Lemmas 3.3 and 3.4, we have

$$\tfrac{1}{2}\tfrac{d}{dt}(\delta(t)|v_m|^2) + a_0\delta(t)\,||v_m||^2 \le \left(\tfrac{1}{2}|\delta'(t)| + c_0 + d_0 + \tfrac{1}{2}\right)\delta(t)|v_m|^2$$

$$+ \tfrac{1}{2}|\delta(t)g|^2.$$

Hence there exists a positive constant c, such that

$$\frac{d}{dt}(\delta(t)|v_m|^2) + \|v_m\|^2 \le c\,\delta(t)|v_m|^2 + c|g|^2.$$

Integrating in $[0,t[$, we obtain

$$\delta(t)\,|v_m|^2 + \int_0^t \|v_m\|^2 ds \le c\int_0^t \delta(s)|v_m|^2 ds + c\int_0^t |g(s)|^2 ds + \delta(0)|v_{0m}|^2.$$

Then Gronwall's inequality implies that,

$$\begin{aligned} (\sqrt{\delta}v_m) &\quad \text{is bounded in} \quad L^\infty(0,T;H) \text{ and} \\ v_m &\quad \text{is bounded in} \quad L^2(0,T;V) \end{aligned} \tag{12}$$

Second Estimate. Let P_m the ortogonal projection $P_m : H \to \sqrt{}_m$ given by $P_m\varphi = \sum_{j=1}^{m}(\varphi, w_j)w_j$. By the special choice of (w_μ), we have $\|P_m\|_{L(V_s,V_s)} \le 1$ and, by transposition, $\|P^*_m\|_{L(V_s',V_s')} \le 1$. We note that $P_m v_m' = v_m'$.

Multiplying the approximate equation (AP) by w_j, adding from $j = 1$ to $j = m$, and using the notation from Lemmas 3.3 and 3.4, we obtain

$$(\delta(t), v_m)' = -P^*_m A(t)v_m - P^*_m B(t)v_m - P^*_m C(t)v_m + P^*_m D(t)v_m. \tag{13}$$

Using again Lemmas 3.4 and 3.5, we have that each term of the second member of (13) is bounded in $L^2(0,T;V_s'(\Omega))$. This implies that

$$(\delta v_m)' \quad \text{is boundary in} \quad L^2(0,T;V_s'(\Omega)). \tag{14}$$

From (12) and (14) there exists a subsequence of (v_m) still denoted by (v_m) and a function v, such that,

$$v_m \rightharpoonup v \quad \text{in} \quad L^2(0,T;V) \tag{15}$$

$$\delta v_m \overset{*}{\rightharpoonup} \delta v \quad \text{in} \quad L^2(0,T;H) \tag{16}$$

$$(\delta v_m)' \rightharpoonup (\delta v)' \quad \text{in} \quad L^2(0,T;V_s') \tag{17}$$

$$\delta v_m \to \delta v \quad \text{in} \quad L^2(0,T;H) \quad \text{strong and (a.e) in} \quad Q. \tag{18}$$

From Lemma 3.6 and (12), we get that

$$(\delta v_{mi}\, v_{mj}) \quad \text{is bounded in} \quad L^2(0,T;L^{\frac{p}{2}}(\Omega)) \tag{19}$$

From (18) and (19), we obtain

$$\delta v_{mi}\, v_{mj} \rightharpoonup \delta v_i v_j \quad \text{in} \quad L^2(0,T;L^{\frac{p}{2}}(\Omega)) \tag{20}$$

By (20) and Lemma 3.4 (i), we have

$$b(t;v_m,v_m,w_j) \to b(t;v,v,w_j) \quad \text{in} \quad L^2(0,T) \tag{21}$$

Using (15) - (21), we can pass to the limit in the approximate equation (AP), obtaining then a solution v of the problem (III).

Proof of Theorem 1. Let $u(x,t) = K(t)\ v(K^{-1}(t)x,t)$, where v is a weak solution of Problem (II).

Let $\varepsilon(x,t) = \psi(K^{-1}(t)\,x,\ t)$, where $\psi \in L^2(0,T;(L^n(\Omega))^n)$, $\psi' \in L^2(0,T;H)$, $\psi(0) = \psi(T) = 0$. Then

$\varepsilon \in L^2(0,T;V(\Omega_t)) \cap (L^n(\Omega_t))^n)$, $\varepsilon' \in L^2(0,T;H(\Omega_t))$, $\varepsilon(0) = \varepsilon(T) = 0.$

Since, $x = K(t)y$, $y = K^{-1}(t)x$, $x_r = \alpha_{rj}(t)y_j$, $y_l = \beta_{lr}(t)x_r$, then,

$$\frac{\partial\varepsilon_i(x,t)}{\partial t} = \beta'_{ki}(t)\psi_k(y,t) + \beta_{ki}(t)\frac{\partial\psi_k(y,t)}{\partial t} + \beta_{ki}(t)\beta'_{ls}(t)\alpha_{sj}(t)y_j\frac{\partial\psi_k(y,t)}{\partial y_l}.$$

Therefore,

$$
\begin{aligned}
-\int_{\Omega_t} u_i(x,t)\frac{\partial \varepsilon_i(x,t)}{\partial t}dx = -\int_{\Omega} \delta(t)v_i(y,t)\frac{\partial \Psi_i(y,t)}{\partial t}dy + \\
+\int_{\Omega} \delta(t)\alpha_{sj}(t)\beta'_{ls}(t)y_j\frac{\partial v_i(y,t)}{\partial y_l}\Psi_i(y,t)dy \\
-\int_{\Omega} \delta(t)\alpha_{ir}(t)\beta'_{ki}(t)v_r(y,t)\Psi_k(y,t)dy \\
+\int_{\Omega} \delta(t)\alpha_{sl}(t)\beta'_{ls}(t)v_k(y,t)\Psi_k(y,t)dy
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\hat{a}(t;u,\varepsilon) &= \int_{\Omega_t} \frac{\partial u_i(x,t)}{\partial x_j}\,\frac{\partial \varepsilon_i(x,t)}{\partial x_j}dx \\
&= \int_{\Omega} \delta(t)a_{jk}(t)\frac{\partial v_i(y,t)}{\partial y_j}\,\frac{\partial \psi_l(y,t)}{\partial y_k}dy
\end{aligned} \tag{23}
$$

where $a_{jk}(t) = \beta_{ji}(t)\beta_{ki}(t)$.

$$
\begin{aligned}
\hat{b}(t;u,u,\varepsilon) &= \int_{\Omega_t} u_i(x,t)\frac{\partial u_j(x,t)}{\partial x_i}\,\varepsilon_j(x,t)dx \\
&= \int_{\Omega} \delta(t)v_i(y,t)\frac{\partial v_j(y,t)}{\partial y_i}\,\psi_j(y,t)dy
\end{aligned} \tag{24}
$$

$$
\int_{\Omega} f^T(x,t)\varepsilon(x,t)dx = \int_{\Omega} \delta(t)g(y,t)^T\psi(y,t)dy \tag{25}
$$

where $g(y,t) = k^{-1}(t)f(K(t),t)$.

Integrating (22) - (25) in $[0,T]$ and using the definitions (7) - (10), we conclude that u is a weak solution of the problem (I). □

Proof of Theorem 3. Let (w_j) be a special basis of $V_s(\Omega)$, where we fix $s = \frac{n}{2}$. We consider the approximate problem:

$$
(IV)\left|\begin{array}{l}((\delta(t)v_m)', w_j) + a(t; v_m, w_j) + b(t; v_m, v_m, w_j) + c(t; v_m, w_j) \\ \\ +d(t; w_m, w_j) = (\delta(t)g, w_j), j = 1, ...m, v_m(t) \in \surd_m = [w_1, ...w_m]. \\ \\ v_m(0) = v_0, \qquad v_0 \text{ arbitrary}\end{array}\right.
$$

We prove that there exists R > 0, independent of m, such that

$$
|v_0| \leq \mathrm{R} \Rightarrow |v_m(T)| \leq \mathrm{R}.
$$

In fact, if we consider $w_j = v_m$ in (IV), we have

$$
\begin{aligned}
&\tfrac{1}{2}\delta(t)\tfrac{d}{dt}|v_m|^2 \;+\; a(t; v_m, v_m) \;\; + c(t; v_m, v_m) + d(t; v_m, v_m) \\
&= \; \delta'(t)|v_m|^2 \;+\; (\delta(t)g, v_m). \qquad (26)
\end{aligned}
$$

From (26) and Lemmas 3.3 and 3.4, we have

$$
\frac{1}{2}\delta(t)\frac{d}{dt}|v_m|^2 + a_1\delta(t)|v_m|^2 \leq \left(\frac{|\delta'(t)|}{\delta(t)} + c_0 + d_0\right)\delta(t)|v_m|^2 + (\delta(t)g, v_m).
$$

From $(H3)$ and the Schwartz inequality, we have

$$
\frac{d}{dt}|v_m|^2 + \alpha|v_m|^2 \leq \frac{1}{2}|g|^2 \qquad \text{then}
$$

$$
\frac{d}{dt}(e^{\alpha t}|v_m|^2) \leq \frac{1}{\alpha}e^{\alpha t}|g|^2, \qquad \text{or}
$$

$$
e^{\alpha t}|v_m(T)|^2 \leq |v_0|^2 + \frac{1}{\alpha}\int_0^T e^{\alpha t}|g|^2 dt \leq |v_0|^2 + c \leq \mathrm{R}^2 + c.
$$

Hence,

$$
|v_m(T)|^2 \leq \frac{\mathrm{R}^2 + c}{e^{\alpha T}} \leq \mathrm{R}^2. \quad \text{Therefore} \quad \mathrm{R}^2 \geq \frac{c}{e^{\alpha T} - 1}.
$$

Then, the maping $v_0 \to v_m(T)$ apply B_{R} in B_{R}, where B_{R} is the disc of radius R, centered in the origin contained in the V_m space with the topology $|\cdot|_{H(\Omega)}$. Therefore, there exists $u_{0m} \in B_{\mathrm{R}}$, such that $v_m(t) = v_{0m}$.

Let v_m be the solution of approximate problem (IV) such that $v_m(0) = v_{0m}$. Since v_{om} is bounded in H, then we have:

$$\begin{array}{lll} v_m & \text{is bounded in} & L^2(0,T;V) \\ \delta v_m & \text{is bounded in} & L^\infty(0,T;H) \\ (\delta v_m)' & \text{is bounded in} & L^2(0,T;V_s'(\Omega)). \end{array}$$

Therefore, we can extract a subsequence (v_m), such that

$$\begin{array}{lll} v_m \rightharpoonup v & \text{weak in} & L^2(0,T;V) \\ \delta v_m \overset{*}{\rightharpoonup} \delta v & \text{weak star in} & L^2(0,T;H) \\ (\delta v_m)' \rightharpoonup (\delta v)' & \text{weak in} & L^2(0,T;V_s') \\ \delta v_m \to \delta v & \text{in} & L^2(0,T;H) \text{ and (a.e) in } Q \end{array}$$

Then $v_m(0) \to v(0)$, $\quad v_m(T) \to v(T)$ in V_s'. Since $v_m(0) = v_m(T)$, then $v(0) = v(T)$. □

Proof of Theorem 4. Let $u(x,t) = k(t)v(K^{-1}(t)x,t)$, where v is a solution of Problem (II), such that

$$v(0) = v(T). \tag{27}$$

We know that u is solution of (I). Then from (H_3) and (27), we have $u(0) = u(T)$. □

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